Kumamoto J. Sci., Ser. A, Vol. 6, Nos. 3/4, pp. 105-129 (1964) 105

ON THE RELIABILITY THEORY OF THE SYSTEM WITH GROUP REDUNDANCY

Masanori KODAMA

Department of Mathematical Institute, General Education, Kumamoto University
(Received September 21, 1964)

§ 1. Introduction.

The purpose of the present paper is to show a generalization of a theorem related to the reliability of a system previously discussed by the author [7]. It is assume that the system we discuss is supported by several spare-equipments and an failed equipment can be switched on a spare as soon as the equipment fails. In such a case, a reliability function is defined and the mean time between failure (MTBF), the variance, and the relative improvement are extensively obtained. A problem concerning a mulitistage allocation of the reliability is also discussed as an application.

Suppose that a system consists of $m+n$ equipments independent to each other. We assume that the failure of the system means the failure of more than $n+1$ equipments and the $m+n-i$ equipments are operating at $t=0$. In other words, when m group with n redundancies are given, m equipments are used actually as functional unit (main operating) and n equipments are used as paralled or stand-by redundancy. The above mentioned system is called the system with group redundancy.

We shall treat in this paper two types of models. In model I, we assume that as soon as an equipment failed, the failure is detected, where in model II the failure is detected only when the equipment is during the main operating, as the system is not equipped with the detecting function on those which are in stand-by operating. The model I will be treated in § 3 and the model II in § 4.

Throughout this paper we assume that reliability of the function of detecting and switching is one. We also assume that the time until failure and the time needed for repair are both subject to known exponential distribution functions and all of those are statistically independent.

§ 2. Structure of reliability problem.

To characterize the model of reliability problem used in this paper, we may begin first discussing the distribution of times to failure for the equipment. We assume that the equipment is one of two possible states-good or failed.

The probability $R(t)$ that the equipment is good at time t is called the reliability function for the equipment. We assume that reliability $R(t)$ has the properties $R(0)=1$, $R(\infty)=0$, $R(t)$ is twice differentiable, and $R'(t)<0$.

Suppose a complex equipment consisting of more than one stochastically failing components, and assume the equipment is good only if all components are good, then the reliability function $R(t)$ of the system may be given by

(2.1) $$R(t)=\prod_j r_j(t).$$

where $r_j(t)$ is the reliability function of the j-th part, and on the other hand as for the maintenance we assume that equipment is always in one of two possible states-repaired and not repaired.

The probability $M(t)$ that the equipment is repaired until time t after break down is called the maintainability function of the equipment. We assume that maintainability function $M(t)$ has the properties $M(0)=0$, $M(\infty)=1$, $M(t)$ is twice differentiable, and $M'(t)>0$.

We repeatedly use the easily verified relations

(2.2) $$R(t)=e^{-\int_0^t \lambda(x)dx},$$

(2.3) $$M(t)=1-e^{-\int_0^t \mu(x)dx}.$$

where

(2.4) $$\lambda(x)=\frac{-R'(x)}{R(x)} \text{ for } R(x)>0.$$

(2.5) $$\mu(x)=\frac{M'(x)}{1-M(x)} \text{ for } M(x)<1.$$

and $\lambda(x)$ is called the failure rate. For the exponential failure and repair distribution, we have

(2.6) $$R(t)=e^{-\lambda t}.$$

(2.7) $$M(t)=1-e^{-\mu t}.$$

where λ and μ are positive constants.

§ 3. Model I.

Let E_i denotes the state that i equipments in the system of $m+n$ equipments are failure, and E_{n+1} means the failure of the system itself. The failure and repairment are transitions from one state to another. The probability of transition from E_i to E_{i+1} and E_{i-1} during time interval $(t, t+h)$ are assume to $\lambda_i h+0(h)$ and $\mu_i h+0(h)$, and the probability that during $(t, t+h)$ more than one change occure is $0(h)$ and the probability of no change is $1-(\lambda_i+\mu_i)h+0(h)$, where $\lambda_i>0$, $\mu_i\geqq 0$. This process $\{X(t), t\geqq 0\}$ is nothing but a birth and death process. Let

(3.1) $$F_{ij}(t)=P\{X(\tau)=j \text{ for some } \tau,\ 0<\tau\leqq t\,|\,X(0)=i\},\ i\neq j,$$

(3.2) $$F_{ii}(t)=\mathrm{P}\{X(\tau_1)\neq i,\ X(\tau_2)=i \text{ for some } \tau_1,\ \tau_2,\ 0<\tau_1<\tau_2\leqq t\,|\,X(0)=i\}.$$

Hence, $F_{ii}(t)$ and $F_{ii}(t)$ are the first passage time and recurrence time distributions, respectively. In words, $F_{ij}(t)$, $i\neq j$ is the probability that the system starting from the state E_i, visits E_j at some time before t, and $F_{ii}(t)$ is the probability that the system

starting from E_i, leaves E_i and then returns to E_i some time before t.
Let

$$ {}_j p_{ik}(t) = P\{X(t)=k,\ X(s)\neq j \text{ for all } s,\ 0<s<t \mid X(0)=i\}. \tag{3.3} $$

It follows from the stochastic continuity of $\{X(t),\ t\geqq 0\}$ that ${}_jp_{ik}(t)\equiv 0$ if $i=j$ or $k=j$. These probabilities are well defined on account of separability of the process. We observe that

$$ F_{ij}(t) = 1 - \sum_k {}_jp_{ik}(t), \qquad i\neq j, \tag{3.4} $$

For fixed j, the matrix $({}_jp_{ik})$ is a substochastic transition matrix; [1]

$$ \sum_k {}_jp_{ik}(t)\,{}_jp_{kl}(s) = {}_jp_{il}(t+s). \tag{3.5} $$

It is unnecessary to exclude j from the summation since corresponding term vanishes. It follows at once that

$$ F_{ij}(s+t) - F_{ij}(s) = \sum_k {}_jp_{ik}(s) F_{kj}(t), \qquad i\neq j, \tag{3.6} $$

Hence an application of lemma 2 in [1] shows that $F_{ij}(t)$ has a continuous derivative $f_{ij}(t)$ satisfying

$$ f_{ij}(s+t) = \sum_k {}_jp_{ik}(s) f_{kj}(t), \qquad t\in(0,\infty). \tag{3.7} $$

It is clear that $F_{ii}(t)$ has a continuous derivative $f_{ii}(t)$ on $(0,\infty)$, since the recurrence-time distribution is

$$ F_{ii}(t) = \left(\frac{\mu_i}{\lambda_i+\mu_i}\right) F_i * F_{i-1i}(t) + \left(\frac{\lambda_i}{\lambda_i+\mu_i}\right) F_i * F_{i+1i}(t). \tag{3.8} $$

where

$$ F_i(t) = e^{-(\lambda_i+\mu_i)t}. $$

and $*$ denotes a convolution.
It is clear that

$$ P\{X(s+\tau)=i \text{ for all } \tau,\ 0\leqq\tau\leqq t \mid X(s)=i\} = e^{-(\lambda_i+\mu_i)t}. \tag{3.9} $$

we define the reliability function. Let us put

$$ {}^{(1)}R_i(t) = P\{X(\tau)\in(0,1,\cdots,n) \text{ for all } \tau,\ 0\leqq\tau\leqq t \mid X(0)=i\}. \tag{3.10} $$

$$ {}^{(2)}R_i(t) = P\{X(\tau)\in(0,1,\cdots,i) \text{ for all } \tau,\ 0\leqq\tau\leqq t \mid X(0)=i\}, \tag{3.11} $$

$$ {}^{(3)}R_i(t) = P\{X(\tau)\in(i,i+1,\cdots,n) \text{ for all } \tau,\ 0\leqq\tau\leqq t \mid X(0)=i\}. \tag{3.12} $$

In other words ${}^{(1)}R_i(t)$, ${}^{(2)}R_i(t)$ and ${}^{(3)}R_i(t)$ denotes the probability that the system starting from the E_i remain within the states $E_k(k=0,1,\cdots,n)$, $E_l(l=0,1,\cdots,i)$ and $E_j(j=i,i+1,\cdots,n)$ duning time t, respectively. The probability ${}^{(1)}R_i(t)$ is called the reliability function of the system for the model I. By a standard enumeration of path it is found that

$$(3.13)\quad \begin{cases} {}^{(1)}R_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_i+\mu_i)\tau(2)}R_{i-1}(t-\tau)d\tau+\int_0^t d\eta\int_0^\eta \mu_i e^{-(\lambda_i+\mu_i)\tau}f_{i-1i}(\eta-\tau)^{(1)}R_i(t-\eta)d\tau \\ \quad +\int_0^t \lambda_i e^{-(\lambda_i+\mu_i)\tau(3)}R_{i+1}(t-\tau)d\tau+\int_0^t d\eta\int_0^\eta \lambda_i e^{-(\lambda_i+\mu_i)\tau}f_{i+1i}(\eta-\tau)^{(1)}R_i(t-\eta)d\tau \\ \quad ={}^{(1)}m_i(t)+\int_0^t {}^{(1)}R_i(t-\eta)f_{ii}(\eta)d\eta, \qquad i=1,2,\cdots,n-1, \\ {}^{(1)}R_0(t)={}^{(3)}R_0(t), \\ {}^{(1)}R_n(t)={}^{(2)}R_n(t). \end{cases}$$

$$(3.14)\quad \begin{cases} {}^{(2)}R_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_i+\mu_i)\tau(2)}R_{i-1}(t-\tau)d\tau+\int_0^t d\eta\int_0^\eta \mu_i e^{-(\lambda_i+\mu_i)\tau}f_{i-1i}(\eta-\tau)^{(3)}R_i(t-\eta)d\tau \\ \quad ={}^{(2)}m_i(t)+\int_0^t {}^{(2)}R_i(t)^{(2)}f_{ii}(\eta)d\eta, \qquad i=1,2,\cdots,n, \\ {}^{(2)}R_0(t)=e^{-\lambda_0 t}. \end{cases}$$

$$(3.15)\quad \begin{cases} {}^{(3)}R_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \lambda e_i^{-(\lambda_i+\mu_i)\tau(3)}R_{i+1}(t-\tau)d\tau+\int_0^t d\eta\int_0^\eta \lambda_i e^{-(\lambda_i+\mu_i)\tau}f_{i+1i}(\eta-\tau)^{(3)}R_i(t-\eta)d\tau \\ \quad ={}^{(3)}m_i(t)+\int_0^t {}^{(3)}R_i(t-\eta)^{(3)}f_{ii}(\eta)d\eta, \qquad i=0,1,\cdots,n-1, \\ {}^{(3)}R_n(t)=e^{-(\lambda_n+\mu_n)t}. \end{cases}$$

$$(3.16)\quad \begin{cases} f_{ii+1}(t)=\lambda_i e^{-(\lambda_i+\mu_i)t}+\int_0^t d\eta\int_0^\eta \mu_i e^{-(\lambda_i+\mu_i)\tau}f_{i-1i}(\eta-\tau)f_{ii+1}(t-\eta)d\tau, \\ \quad =\lambda_i e^{-(\lambda_i+\mu_i)t}+\int_0^t f_{ii+1}(t-\eta)^{(2)}f_{ii}(\eta)d\eta, \qquad i=1,2,\cdots,n, \\ f_{01}(t)=\lambda_0 e^{-\lambda_0 t}. \end{cases}$$

$$(3.17)\quad \begin{cases} f_{ii-1}(t)=\mu_i e^{-(\lambda_i+\mu_i)t}+\int_0^t d\eta\int_0^\eta \lambda_i e^{-(\lambda_i+\mu_i)\tau}f_{i+1i}(\eta-\tau)f_{ii-1}(t-\eta)d\tau, \\ \quad =\mu_i e^{-(\lambda_i+\mu_i)t}+\int_0^t f_{ii-1}(t-\eta)^{(3)}f_{ii}(\eta)d\eta, \qquad i=1,\cdots,n-1, \\ f_{nn-1}(t)=\mu_n e^{-(\lambda_n+\mu_n)t}. \end{cases}$$

where

$$(3.18)\quad \begin{cases} {}^{(1)}m_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_i+\mu_i)\tau(2)}R_{i-1}(t-\tau)d\tau+\int_0^t \lambda_i e^{-(\lambda_i+\mu_i)\tau(3)}R_{i+1}(t-\tau)d\tau, \\ {}^{(2)}m_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_i+\mu_i)\tau(2)}R_{i-1}(t-\tau)d\tau, \\ {}^{(3)}m_i(t)=e^{-(\lambda_i+\mu_i)t}+\int_0^t \lambda_i e^{-(\lambda_i+\mu_i)\tau(3)}R_{i+1}(t-\tau)d\tau, \\ {}^{(1)}f_{ii}(\eta)=\int_0^\eta e^{-(\lambda_i+\mu_i)}[\mu_i f_{i-1i}(\eta-\tau)+\lambda_i f_{i+1i}(\eta-\tau)]d\tau=\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(F_i*f_{i-1i}(\eta)) \\ \quad +\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(F_i*f_{i+1i}(\eta)), \\ {}^{(2)}f_{ii}(\eta)=\int_0^\eta \mu_i e^{-(\lambda_i+\mu_i)\tau}f_{i-1i}(\eta-\tau)d\tau=\left(\frac{\mu_i}{\lambda_i+\mu_i}\right)F*f_{i-1i}(\eta), \\ {}^{(3)}f_{ii}(\eta)=\int_0^\eta \lambda_i e^{-(\lambda_i+\mu_i)\tau}f_{i+1i}(\eta-\tau)d\tau=\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)F_i*f_{i+1i}(\eta), \\ F_i(t)=e^{-(\lambda_i+\mu_i)t}. \end{cases}$$

and $f_{ii-1}(t)dt$ and $f_{ii+1}(t)dt$ are the probability that after the system starting from E_i remains within states $E_k(k=i, i+1, \cdots, n)$, an transition occures from E_i to E_{i-1}, during time interval $(t, t+dt)$, and after the system starting from E_i remains within states E_l $(l=0, 1, \cdots, i)$, an transition occures from E_i to E_{i+1}, during $(t, t+dt)$, respectively.

The equation (3.13)~(3.17) are renewal equation. By a theorem 9 of [2] P 178, we deduce that ${}^{(l)}R_i(t)$ exists and ${}^{(l)}R_i(t)\geqq 0(l=1,2,3)$, since ${}^{(l)}m_i(t)\geqq 0(l=1,2,3)$, by induction on the number of the states. By induction on the number of the states, we see that ${}^{(l)}R_i(0)=1$, ${}^{(l)}R_i(t)$ is twice differentiable and ${}^{(l)}R_i'(t)<0$ $(l=1,2,3)$. Moreover, we know that ${}^{(l)}R_i(\infty)=0$, since $\int_0^\infty f_{ii}(t)dt<1$ and $\lim\limits_{t\to\infty} {}^{(l)}m_i(t)=0$ $(l=1,2,3)$ by induction.

The first term of the (3.13) is the probability that the system remain in E_i until time t, the second term is the probability that the first transition of the system is to E_{i-1} and this occured between $(\tau, \tau+d\tau)$ and then the system remain $(E_0, E_1, \cdots, E_{i-1})$ until t, the third term is that of the transitions $E_i\to E_{i-1}\to E_i$ in the time interval $(\tau, \tau+d\tau)$ and $(\eta, \eta+d\eta)$ and remaining within $(E_0, E_1, \cdots, E_n)$ afterward until t, the fourth term is the probability that the first transition of the system is to E_{i+1} and this occured between $(\tau, \tau+d\tau)$ and the system remain $(E_{i+1}, E_{i+2}, \cdots, E_n)$ until t, and the fifth term is that of the transition $E_i\to E_{i+1}\to E_i$ in the time interval $(\tau, \tau+d\tau)$ and $(\eta, \eta+d\eta)$ remaining within $(E_0, E_1, \cdots, E_n)$ afterward until t. The third term of (3.14) is that of the transition $E_i\to E_{i-1}\to E_i$ is in time interval $(\tau, \tau+d\tau)$ and $(\eta, \eta+d\eta)$ and remaining within $(E_0, E_1, \cdots, E_i)$ afterward until t. The third term of (3.15) is that of the transition $E_i\to E_{i+1}\to E_i$ is in time tnterval $(\tau, \tau+d\tau)$ and $(\eta, \eta+d\eta)$ and remaining within $(E_i, E_{i+1}, \cdots, E_n)$ afterward until t.

Let us put

$$(3.19)\quad \begin{cases} {}^{(l)}\varphi_i(s)=\int_0^\infty e^{-st}\,{}^{(l)}R_i(t)dt,\ {}^{(l)}\varphi_i^*(s)=\int_0^\infty e^{-st}\,{}^{(l)}R_i'(t)dt, & i=0,1,\cdots,n; l=1,2,3, \\ \psi_{ij}(s)=\int_0^\infty e^{-st}f_{ij}(t)dt. & j=i-1, i+1, i=1,2,\cdots,n-1. \end{cases}$$

By applying the Laplace transformation to Eqs (3.13)~(3.17), we obtain

$$(3.20)\quad \begin{cases} {}^{1}\varphi_i(s)=\dfrac{1+\mu_i{}^{(2)}\varphi_{i-1}(s)+\lambda_i{}^{(3)}\varphi_{i+1}(s)}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)-\lambda_i\psi_{i+1i}(s)},\quad {}^{1}\varphi_i(s)=\dfrac{\lambda_i(\psi_{i+1i}(s)+{}^{(3)}\varphi_{i+1}^*(s))}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)-\lambda_i\psi_{i+1i}(s)}, \\ \qquad\qquad i=1,2,\cdots,n-1, \\ {}^{(1)}\varphi_0(s)={}^{(3)}\varphi_0(s),\ {}^{(1)}\varphi_n(s)={}^{(2)}\varphi_n(s),\ {}^{(1)}\varphi_0^*(s)={}^{(3)}\varphi_0^*(s),\ {}^{(1)}\varphi_n^*(s)={}^{(2)}\varphi_n^*(s). \end{cases}$$

$$(3.21)\quad \begin{cases} {}^{(2)}\varphi_i(s)=\dfrac{1+\mu_i{}^{(2)}\varphi_{i-1}(s)}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)},\quad {}^{(2)}\varphi_i^*(s)=\dfrac{-\lambda_i}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)}, & i=1,2,\cdots,n, \\ {}^{(2)}\varphi_0(s)=\dfrac{1}{s+\lambda_0},\quad {}^{(2)}\varphi_0^*(s)=-\dfrac{\lambda_0}{s+\lambda_0}. \end{cases}$$

$$(3.22)\quad \begin{cases} {}^{(3)}\varphi_l(s)=\dfrac{1+\lambda_i{}^{(2)}\varphi_{i+1}(s)}{s+\lambda_i+\mu_i-\lambda_i\psi_{i+1i}(s)},\quad {}^{(3)}\varphi_i^*(s)=\dfrac{\lambda_i(\psi_{i+1i}(s)+{}^{(3)}\varphi_{i+1}^*(s))-\mu_i}{s+\lambda_i+\mu_i-\lambda_i\psi_{i+1i}(s)} \\ \qquad =\dfrac{-\lambda_i\lambda_{i+1}\cdots\lambda_n-\mu_i\mu_{i+1}\cdots\mu_nQ^{n-i}(s)}{\mu_i\mu_{i+1}\cdots\mu_nQ^{n-(i-1)}(s)} & i=0,1,\cdots,n-1, \\ {}^{(3)}\varphi_n(s)=\dfrac{1}{s+\lambda_n+\mu_n},\quad {}^{(3)}\varphi_n^*(s)=-\dfrac{\lambda_n+\mu_n}{s+\lambda_n+\mu_n}. \end{cases}$$

$$(3.23)\quad \begin{cases} \psi_{ii-1}(s)=\dfrac{\mu_i}{s+\lambda_i+\mu_i-\lambda_i\psi_{i+1i}(s)}=\dfrac{Q^{n-i}(s)}{Q^{n-(i-1)}(s)} & i=1,2,\cdots,n-1, \\ \psi_{nn-1}(s)=\dfrac{\mu_n}{s+\lambda_n+\mu_n}. \end{cases}$$

$$(3.24)\quad \begin{cases} \psi_{ii+1}(s)=\dfrac{\lambda_i}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)}=\dfrac{Q_i(s)}{Q_{i+1}(s)}, & i=1,2,\cdots,n, \\ \psi_{01}(s)=\dfrac{\lambda_0}{s+\lambda_0}. \end{cases}$$

where

$$(3.25)\quad \begin{cases} Q^0(s)=1,\ \mu_nQ^1(s)=s+\lambda_n+\mu_n, \\ \mu_iQ^{n-(i-1)}(s)=(s+\lambda_i+\mu_i)Q^{n-i}(s)-\lambda_iQ^{n-(i+1)}(s), & i=1,2,\cdots,n-1, \\ Q_0(s)=1,\ \lambda_0Q_1(s)=s+\lambda_0, \\ \lambda_iQ_{i+1}(s)=(s+\lambda_i+\mu_i)Q_i(s)-\mu_iQ_{i-1}(s), & i=1,2,\cdots,n. \end{cases}$$

From (3.21), (3.22), (3.23), (3.24), (3.25)

$$(3.26)\quad \begin{cases} {}^{(1)}\varphi_i^*(s)=\dfrac{\lambda_i(\psi_{i+1i}(s)+{}^{(3)}\varphi_{i+1}^*(s))}{s+\lambda_i+\mu_i-\mu_i\psi_{i-1i}(s)-\lambda_i\psi_{i+1i}(s)}=\dfrac{-\lambda_{i+1}\cdots\lambda_nQ_i(s)}{\mu_{i+1}\cdots\mu_n[Q_{i+1}(s)Q^{n-i}(s)-Q_i(s)Q^{n-(i+1)}(s)]} \\ =\dfrac{-\lambda_{i+2},\cdots\lambda_nQ_i(s)}{\mu_{i+2}\cdots\mu_n[Q^{n-(i+1)}(s)Q_{i+2}(s)-Q_{i+1}(s)Q^{n-(i+1)-1}(s)]}=-\dfrac{Q_i(s)}{Q_{n+1}(s)}, \quad i=0,1,\cdots,n. \end{cases}$$

It is clear from (3.25) that polinomial $Q_{n+1}(s)$, $Q_n(s)$, $\cdots Q_1(s)$, $Q_0(s)$ are modified Strums function. Hence (3.25) and modified Strums therom, the polinomial $Q_i(s)$ has distinct negative real roots Moreover from (3.25) it is clear that $(Q_{i+1}(s)-Q_i(s))/S$ is polinomial of degree i. Hence

$$(3.27)\quad {}^{(1)}\varphi_i(s)=\frac{Q_{n+1}(s)-Q_i(s)}{s}\cdot\frac{1}{Q_{n+1}(s)}=\frac{Q_{n+1}(s)-Q_i(s)}{s}\cdot\frac{\lambda_0\lambda_1\cdots\lambda_n}{\prod_{k=1}^{n+1}(s-s_k)}=\sum_{k=1}^{n+1}\frac{A_k}{s-s_k}$$

where

$$A_k=\frac{Q_{n+1}(s_k)-Q_i(s_k)}{s_k}\cdot\frac{1}{Q'_{n+1}(s_k)}, \qquad i=0,1,\cdots,n,$$

and $S_k(k=1,2,\cdots n+1)$ are $n+1$ distinct negative real roots of $Q_{n+1}(s)$. The important special case is when

$$(3.28)\quad \begin{cases} \lambda_j=(m+n-j)\lambda, & (j=0,1,\cdots,n) \\ \mu_l=l\mu, & (l=1,2,\cdots,n). \end{cases}$$

which means that the system consists of identical equipment independent to each other and the reliability of stand-by operating and main operating are equal. In this case we have

$$(3.29)\quad \begin{cases} Q_0(s)=1, \\ (m+n)\lambda Q_1(s)=s+(m+n)\lambda, \\ (m+n-i)\lambda Q_{i+1}(s)=Q_i(s)[s+(m+n-i)\lambda+i\mu]-i\mu Q_{i-1}(s). \end{cases}$$

By applying the inversion theorem to Eqs (3.27), we obtain the following theorem.

THEOREM 3.1 *The reliability function* ${}^{(1)}R_i(t)$ *of the model I is given by the following formula*

$$(3.30)\quad {}^1R_i(t)=\sum_{m=1}^{n+1} A_m e^{s_m t},$$

where

$$A_m=\frac{Q_{n+1}(s_m)-Q_i(s_m)}{s_m}\cdot\frac{1}{Q'_{n+1}(s_m)},$$

and $Q_i(s)$ *are given as* (3.25) *and given as* (3.29) *specially when* (3.28) *hold and* $S_i(i=1,2,\cdots,n)$ *are* $n+1$ *distinct negative real roots of* $Q_{n+1}(s)$.

Next we calculate the MTBF T_{i+1} of the system.

Let us put

$$T_{i+1}=\int_0^\infty tf_i(t)dt,\ (f_i(t)=F'_i(t)=(1-{}^{(1)}R'_i(t)),$$

then

$$(3.31)\quad T_{i+1}=\int_0^\infty tf_i(t)dt=\lim_{s\to 0}{}^{(1)}\varphi_i(s)$$

$$=\lim_{s\to 0}\left\{\frac{1}{\lambda_n}\left[\frac{\mu_n(Q_n(s)-Q_{n-1}(s)+\lambda_n(Q_n(s)-Q_i(s))}{s}+Q_n(s)\right]\right\}\cdot\frac{1}{Q_{n+1}(s)}$$

$$=\sum_{k=i}^{n} T_{k+1k}$$

$$=\sum_{k=i}^{n}\left(\frac{1}{\lambda_k}+\sum_{l=1}^{k}\prod_{j=k-l+1}^{k}\mu_j/\prod_{j=k-l}^{k}\lambda_j\right)$$

where

$$(3.32)\quad \begin{cases} T_{10}=\lim_{s\to 0}\dfrac{Q_1(s)-Q_0(s)}{s}=\dfrac{1}{\lambda_0}, \\ T_{i+1i}=\lim_{s\to 0}\dfrac{Q_{i+1}(s)-Q_i(s)}{s}=\dfrac{1}{\lambda_i}+\dfrac{\mu_i}{\lambda_i}T_{ii-1}, \end{cases}\qquad (i=1,2,\cdots,n).$$

From (3.25)

$$(3.33)\quad \lim_{s\to 0} Q_i(s)=1,\qquad (i=0,1,\cdots,n).$$

Specially when (3.29) hold then

$$(3.34)\quad T_{i+1}=\sum_{l=i}^{n}\left[(m+n-l-1)!\,l!\sum_{k=0}^{l}\frac{\mu^k}{(m+n-l+k)!\,(l-k)!\,\lambda^{k+1}}\right]\qquad (i=0,1,\cdots,n).$$

Our result may be summarized in the following theorem.

THEROM 3.2. *The MTBF T_{i+1} of the model I is given by*

(3.35) $$T_{i+1}=\sum_{k=i}^{n} T_{k+1k}=\sum_{k=i}^{n}\left(\frac{1}{\lambda_k}+\sum_{l=1}^{k}\prod_{j=k-l+1}^{k}\mu_j/\prod_{j=k-l}^{k}\lambda_j\right),$$

where

$$T_{10}=\frac{1}{\lambda_0},$$

$$T_{i+1i}=\frac{1}{\lambda_i}+\frac{\mu_i}{\lambda_i}T_{ii-1}.$$

and, specially when (3.28) *is true, is given by* (3.34).

THEROM 3.3. *The relative improvement $\triangle T_{h+1r+1}$ in MTBF of the system when number of spare increas from h to r is given by*

(3.36) $$\triangle T_{h+1r+1}=\frac{\sum_{k=i}^{r}\left(\frac{1}{\lambda_k}+\sum_{l=1}^{k}\prod_{j=k-l+1}^{k}\mu_j/\prod_{j=k-l}^{k}\lambda_j\right)}{\sum_{k=i}^{h}\left(\frac{1}{\lambda_k}+\sum_{l=1}^{k}\prod_{j=k-l+1}^{k}\mu_j/\prod_{j=k-l}^{k}\lambda_j\right)}$$

where λ_j is depend on the number of the equipments, and specially when (3.28) *hold, is given by*

(3.37) $$\triangle T_{h+1r+1}=\frac{\sum_{l=i}^{r}\left[(m+r-l-1)!\,l!\sum_{k=0}^{l}\frac{\mu^k}{(m+r-l+k)!\,(l-k)!\,\lambda^{k+1}}\right]}{\sum_{l=i}^{h}\left[(m+h-l-1)!\,l!\sum_{k=0}^{l}\frac{\mu^k}{(m+h-l+k)!\,(l-k)!\,\lambda^{k+1}}\right]}$$

In order to calculate the variance, Let us put

$$E_{i+1}(T^2)=\int_0^\infty t^2 f_i(t)dt,\quad (f_i(t)=F_i(t)=(1-{}^{(1)}R_i(t))'),$$

then

(3.38) $$E_{i+1}(T^2)=-\int_0^\infty t^{2\,(1)}R_i'(t)dt=-2\lim_{s\to 0}{}^{(1)}\varphi_i'(s)$$

$$=-2\lim_{s\to 0}\left\{\left(\frac{Q_{n+1}(s)-Q_i(s)}{s}\right)'\cdot\frac{1}{Q_{n+1}(s)}-\frac{Q_{n+1}(s)-Q_i(s)}{s}\cdot\frac{Q_{n+1}'(s)}{(Q_{n+1}(s))^2}\right\},$$
$$(i=0,1,\cdots,n)$$

Hence from $\lim_{s\to 0} Q_i(s)=1$, and $\lim_{s\to 0}\frac{Q_{n+1}(s)-Q_i(s)}{s}=T_{i+1}$ for $i=0,\cdots,n$, we have

(3.39) $$E_{i+1}(T^2)=-2(B_{i+1}^n-T_{i+1}A_{n+1}),\qquad (i=0,1,\cdots,u).$$

where

(3.40)
$$\begin{cases} A_0=0,\ A_1=\dfrac{1}{\lambda_0},\ A_{i+1}=\lim\limits_{s\to 0} Q'_{i+1}(s)=\dfrac{[1+\mu_i(A_i-A_{i-1})+\lambda_i A_i]}{\lambda_i}, & (i=1,2,\cdots,n), \\ B_1=\lim\limits_{s\to 0}\left(\dfrac{Q_1(s)-Q_0(s)}{s}\right)'=0,\ B_{i+1}=\lim\limits_{s\to 0}\left(\dfrac{Q_{i+1}(s)-Q_i(s)}{s}\right)'=\dfrac{\mu_i B_i+A_i}{\lambda_i}, & (i=1,2,\cdots,n), \\ B_1^n=\lim\limits_{s\to 0}\left(\dfrac{Q_{n+1}(s)-Q_0(s)}{s}\right)'=\sum\limits_{k=0}^{n}\left(\dfrac{\mu_k B_k+A_k}{\lambda_k}\right), & \\ B_{i+1}^n=\lim\limits_{s\to 0}\left(\dfrac{Q_{n+1}(s)-Q_i(s)}{s}\right)'=\sum\limits_{k=i}^{n}\left(\dfrac{\mu_k B_k+A_k}{\lambda_k}\right), & (i=1,2,\cdots,n) \end{cases}$$

Specially when (3.28) is true, we get

(3.34)
$$\begin{cases} A_0=0,\ A_1=\dfrac{1}{(m+n)\lambda},\ A_{i+1}=\dfrac{[1+i\mu(A_i-A_{i-1})+(m+n-i)\lambda A_i]}{(m+n-i)\lambda}, & (i=1,2,\cdots,n), \\ B_1=0,\ B_{i+1}=\dfrac{i\mu B_i+A_i}{(m+n-i)\lambda}, & \\ B_1^n=\sum\limits_{k=1}^{n}\dfrac{k\mu B_k+A_k}{(m+n-k)\lambda},\quad B_{i+1}^n=\sum\limits_{k=i}^{n}\dfrac{k\mu B_k+A_k}{(m+n-k)\lambda}, & (i=1,2,\cdots,n). \end{cases}$$

Thus we obtain the following theorem.

THEROM 3.4. *Variance V_{i+1} of the model I is given by the following formula*

(3.42) $$V_{i+1}=T_{i+1}(2A_{n+1}-T_{i+1})-2B_{i+1}^n$$

where A_{i+1}, B_{i+1}^n, T_{i+1}, are given as (3.40) *and* (3.34), *specially when* (3.28) *hold they are given as* (3.41).

EXAMPLE 1. *In the case when $m=n=1$, $i=1$, since then $\lambda_0=2\lambda$, $\lambda_1=\lambda$,* and $\mu_1=\mu$,

$$Q_0(s)=1,\ Q_1(s)=\frac{s+2\lambda}{2\lambda},$$

$$Q_2(s)=\frac{1}{2\lambda^2}s^2+\frac{3\lambda+\mu}{2\lambda^2}\cdot s+1,\quad \frac{Q_2(s)-Q_1(s)}{s}=\frac{s+2\lambda+\mu}{2\lambda^2},$$

The roots of $Q_2(s)$ are

$$s_1=\frac{-(3\lambda+\mu)+\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2},\quad s_2=\frac{-(3\lambda+\mu)-\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2},$$

Hence

$$^{(1)}R_1(t)=\frac{s_1+2\lambda+\mu}{s_1-s_2}\cdot e^{s_1 t}+\frac{s_2+2\lambda+\mu}{s_2-s_1}\cdot e^{s_2 t}$$
$$=e^{-\frac{(3\lambda+\mu)t}{2}}\left[\cosh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}\cdot t+\frac{\lambda+\mu}{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}\sinh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}t\right],$$

On the other hand,

$$T_2=\lim_{s\to 0}\varphi_1(s)=\frac{2\lambda+\mu}{2\lambda^2},$$

$$\triangle T_{12}=1+\frac{\mu}{2\lambda},$$

$$V_2=T_2(2A_2-T_2)-2B_2^1=\frac{\mu^2}{4\lambda^2}+\frac{3\mu}{2\lambda^3}+\frac{1}{\lambda^2}$$

EXAMPLE 2. *In the case* $m=n$, $i=0$, *since then* $\lambda_0=2\lambda$, $\lambda_1=\lambda$, *and* $\mu_1=\mu$,

$$Q_0(s)=1,\ Q_2(s)=\frac{1}{2\lambda^2}\cdot s^2+\frac{3\lambda+\mu}{2\lambda^2}\cdot s+1,$$

$$\frac{Q_2(s)-Q_0(s)}{s}=\frac{s+3\lambda+\mu}{2\lambda^2},$$

The roots of $Q_2(s)$ *are*

$$s_1=\frac{-(3\lambda+\mu)+\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2},\quad s_2=\frac{-(3\lambda+\mu)-\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2},$$

Hence

$$^{(1)}R_0(t)=\frac{s_1+3\lambda+\mu}{s_1-s_2}e^{s_1t}+\frac{s_2+3\lambda+\mu}{s_2-s_1}\cdot e^{s_2t}$$

$$=e^{-\frac{(3\lambda+\mu)t}{2}}\left[\cosh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}t+\frac{3\lambda+\mu}{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}\cdot\sinh\frac{\sqrt{\lambda^2+6\lambda\mu+\mu^2}}{2}t\right],$$

And also we get

$$T_1=\lim_{s\to 0}\varphi_0(s)=\frac{3\lambda+\mu}{2\lambda^2}$$

$$\triangle T_{12}=\frac{3\lambda+\mu}{2\lambda}$$

$$V_1=T_1(2A_2-T_1)-2B_1^1=\frac{\mu^2}{4\lambda^4}+\frac{3\mu}{2\lambda^3}+\frac{5}{4\lambda^2}$$

§4. Model II.

In this section we shall treat Model II. In this model there are two kinds of failures, as explained in the Introduction, repairable and non repairable one. Let us consider a system with $m+n$ equipments and let us assume the system is good only if at least m equipments are good.

Let us E_i^{m+n-j} denotes the state that i equipments in the system of $m+n$ equipments are in the repairable failure and $m+n-j$ equipments are operating, and then that the system reaches one of the states $E_h^{m-1}(h=1,2,\cdots,n+1)$ means the failure of the system itself. The transitions from one state to another are caused by either the occurance of failure or the accomplishment of repair. It is assumed that the probabilities of transition from E_i^{m+n-1} to $E_i^{m+n-(j+1)}$, $E_{i+1}^{m+n-(j+1)}$, and $E_{i-1}^{m+n-(j-1)}$ during time interval $(t,\ t+h)$ are given by $(\lambda_j-\lambda_n)h+0(h)$, $\lambda_n h+0(h)$ and $\mu_i h+0(h)$ respectively, and the probability that during $(t,\ t+h)$ more than one change occures is $0(h)$, and the probabitity of no change is $1-(\lambda_j+\mu_i)h+0(h)$ (Fig 1), where $\lambda_j>0$, $\mu_i\geqq 0$.

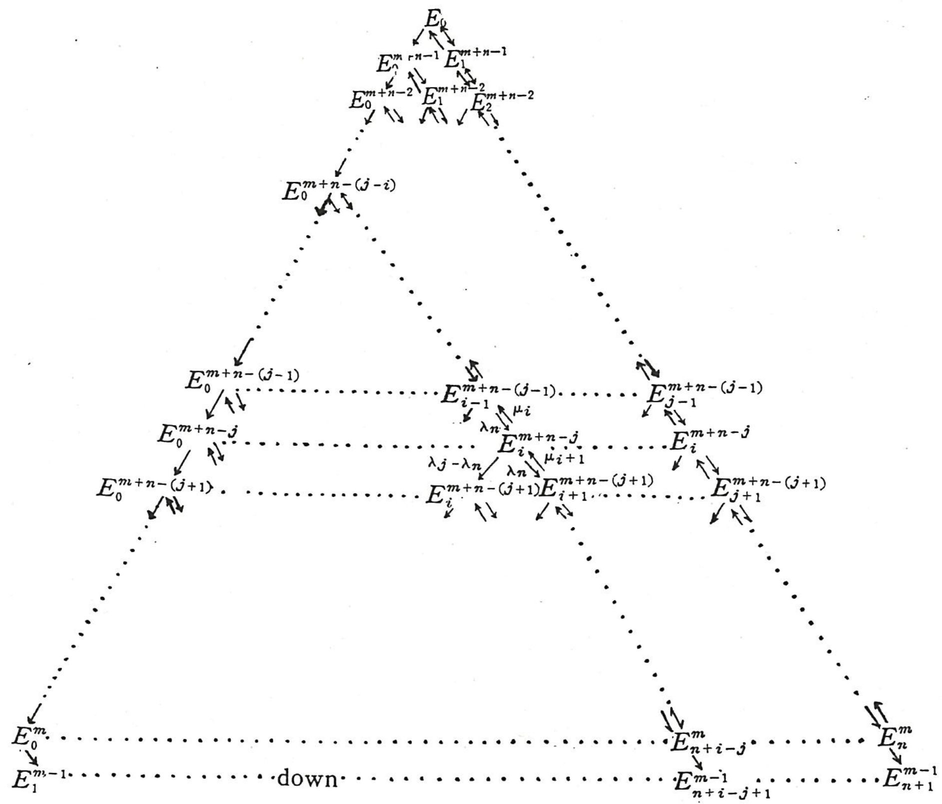

Fig. 1

Let

(4.1) $F_{ii}^{m+n-j}(t)=P\{X(\tau_1)\neq E_i^{m+n-j},\ X(\tau_2)=E_i^{m+n-j}$ for some $\tau_1,\tau_2,\ 0<\tau_1<\tau_2\leqq t\,|\,X(0)=E_i^{m+n-j}\}$,

(4.2) ${}^1F_{ik}^{*m+n-j}(t)=P\{\alpha_{ik}^j\leqq t\,|\,X(0)=E_i^{m+n-j}\}$,

(4.3) ${}^2G_{ik}^{m+n-j}(t)=P\{$An transition occures from E_k^{m+n-j} to S_k^j at some time $\tau(0<\tau\leqq t)$,

and $X(s)\notin S_{h_1}^{l_1}$ for all $s(0<s<\tau)\,|\,X(0)=E_i^{m+n-j}\}$,

(4.4) ${}^3G_{ik}^{m+n-j}(t)=P\{X(\tau)=E_{k-1}^{m+n-(j-1)}$ and $X(s)\notin S_{h_4}^{l_4}$ for some τ, for all s

$(0<s<\tau\leqq t)\,|\,X(0)=E_i^{m+n-j}\}$,

where

$$\lambda_j^i=\sup_{X(t^*)=E_i^{m+n-j}\text{ for all }t^*\in(0,\tau)}\tau \qquad \alpha_{ik}^j=\inf_{X(t)=E_k^{m+n-j},\ X(s)\neq E_i^{m+n-j}\text{ for all }s,\ \lambda_i^j\leqq s<t} t$$

$$S_k^j=\{E_{k+1}^{m+n-(j+1)},\ E_k^{m+n-(j+1)}\}, \qquad S_{h_1}^{l_1}=S_{h_2}^{l_2}\cup S_{h_3}^{l_3},$$

$$S_{h_2}^{l_2}=\{E_{h_2}^{m+n-l_2};\ l_2=j+1, j+2,\cdots,n,\ h_2=0,1,\cdots,i+1\},$$

$$S_{h_3}^{l_3}=\{E_{h_3}^{m+n-l_3};\ l_3=h_3+j-i+1, h_3+j-i+2, \cdots, n, h_3=i+2, i+3, \cdots, n+i-j\},$$

$$S_{h_4}^{l_4}=\{E_{h_4}^{m+n-l_4};\ l_4=h_4+j-i, h_4+j-i+1, \cdots, j-1, h_4=0, 1, \cdots, i-1\}.$$

By the same method as in §3, it is easi lyseen that $F_{ii}^{m+n-j}(t)$, ${}^{(1)}F_{ik}^{*m+n-j}(t)$, ${}^{(2)}G_{ik}^{m+n-j}(t)$, and ${}^{(3)}G_{ik}^{m+n-j}(t)$ have a continuous derivative $f_{ii}^{m+n-j}(t)$, ${}^{(1)}f_{ik}^{*m+n-j}(t)$, ${}^{(2)}g_{ik}^{m+n-j}(t)$, and ${}^{(3)}g_{ik}^{m+n-j}(t)$, respectively, we define the reliability function.

Let us put

(4.5) $\quad {}^{(1)}R_i^{m+n-j}(t)=P\{X(\tau)\in{}^{(1)}S_{ij}$ for all τ, $0\leqq\tau\leqq t\,|\,X(0)=E_i^{m+n-j}\}$

(4.6) $\quad {}^{(2)}R_i^{m+n-j}(t)=P\{X(\tau)\in{}^{(2)}S_{ij}$ for all τ, $0\leqq\tau\leqq t\,|\,X(0)=E_i^{m+n-j}\}$

(4.7) $\quad {}^{(3)}R_i^{m+n-j}(t)=P\{X(\tau)\in{}^{(3)}S_{ij}$ for all τ, $0\leqq\tau\leqq t\,|\,X(0)=E_i^{m+n-j}\}$

where

$${}^{(1)}S_{ij}=\{E_{k_1}^{m+n-l_1};\ l_1=k_1+j-i,\ k_1+j-i+1,\ \cdots,\ n,\ k_1=0, 1,\ \cdots,\ n+i-j,\}.$$

$${}^{(2)}S_{ij}=\{E_{k_2}^{m+n-l_2};\ l_2=k_2+j-i,\ k_2+j-i+1,\ \cdots,\ j,\ k_2=0, 1,\ \cdots,\ i,\}.$$

$${}^{(4)}S_{ij}=\{E_{k_3}^{m+n-l_3};\ l_3=j,\ j+1,\ \cdots,\ n,\ k_3=0, 1,\ \cdots,\ i,\}.$$

$${}^{(5)}S_{ij}=\{E_{k_4}^{m+n-l_4};\ l_4=k_4+j-i,\ k_4+j-i+1,\ \cdots,\ n,\ k_4=i+1,\ i+2,\ \cdots,\ n+i-j.\}$$

$${}^{(3)}S_{ij}={}^{(4)}S_{ij}\cup{}^{(5)}S_{ij}$$

In other word, ${}^{(1)}R_i^{m+n-j}(t)$, ${}^{(2)}R_i^{m+n-j}(t)$, ${}^{(3)}R_i^{m+n-j}(t)$ denote the probability that the system starting from the E_i^{m+n-j} ($j=i, i+1, \cdots, n$; $i=0,1,\cdots,n$) remains within the states $E_{k_1}^{m+n-l_1}$ ($l_1=k_1+j-i, k_1+j-i+1, \cdots, n$; $k_1=0,1,\cdots,n+i-j$), $E_{k_2}^{m+n-l_2}$($l_2=k_2+j-i$, $k_2+j-i+1, \cdots j$; $k_2=0$, $1,\cdots i$), and $E_{k_3}^{m+n-l_3}$($l_3=j$, $j+1,\cdots,n$; $k_3=0,1,\cdots,i$) or $E_{k_4}^{m+n-l_4}$($l_4=k_4+j-i$, $k_4+(j-i)+1,\cdots,n$; $k_4=i+1, i+2, \cdots, n+i-j$) until time t, respectively. The probability ${}^{(1)}R_i^{m+n-j}(t)$ is called the reliability function of the system for the model II. By a standard enumeration of path it follows that

$$(4.8)\quad \left\{\begin{aligned}
{}^{(1)}R_i^{m+n-j}(t)&=e^{-(\lambda_j+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_j+\mu_i)\tau\,(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau\\
&+\int_0^t d\eta\int_0^\eta \mu_i e^{-(\lambda_j+\mu_i)\tau}\sum_{k=1}^{i}{}^{(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)[p_{k-1k}^{j-1j\,(1)}R_k^{m+n-j}(t-\eta)\\
&+p_{k-1k-1}^{j-1j}\,{}^{(1)}R_{k-1}^{m+n-j}(t-\eta)]d\tau+\int_0^t \lambda_j e^{-(\lambda_j+\mu_i)\tau}(p_{ii}^{jj+1\,(3)}R_i^{m+n-(j+1)}(t-\tau)\\
&+p_{ii+1}^{jj+1\,(3)}R_{i+1}^{m+n-(j+1)}(t-\tau))d\tau\\
&+\int_0^t d\eta\int_0^\eta \lambda_j e^{-(\lambda_j+\mu_i)\tau}(p_{ii+1}^{jj+1}\sum_{k=1}^{i+1}{}^{(3)}g_{i+1k}^{m+n-(j+1)}(\eta-\tau)^{(1)}R_{k-1}^{m+n-j}(t-\eta)\\
&+p_{ii}^{jj+1}\sum_{k=1}^{i}{}^{(3)}g_k^{im+n-(j+1)}(\eta-\tau)^{(1)}R_{k-1}^{m+n-j}(t-\eta))d\tau\\
&=e^{-(\lambda_j+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_j+\mu_i)\tau\,(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau\\
&+\int_0^t \lambda_j e^{-(\lambda_j+\mu_i)\tau}(p_{ii}^{jj+1\,(3)}R_i^{m+n-(j+1)}(t-\tau)+p_{ii+1}^{jj+1\,(3)}R_{i+1}^{m+n-(j+1)}(t-\tau)d\tau
\end{aligned}\right.$$

$$
\begin{aligned}
&+\int_0^t d\eta\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\sum_{k=0}^{i}\Big\{\Big(\frac{\mu_i}{\lambda_j+\mu_i}\Big)(p_{k-1k}^{j-1j(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)\\
&+p_{kk}^{j-1j(2)}g_{i-1k}^{m+n-(j-1)}(\eta-\tau)+\Big(\frac{\lambda_j}{\lambda_j+\mu_i}\Big)(p_{ii+1}^{jj+1(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta-\tau)\\
&+p_{ii}^{jj+1(3)}g_{ik+1}^{m+n-(j+1)}(\eta-\tau)\Big\}{}^{(1)}R_k^{m+n-j}(t-\eta)d\tau\\
&={}^{(1)}m_i^j(t)+\sum_{k=0}^{i}\int_0^t{}^{(1)}R_{ik}^{m+n-j}(t-\eta){}^{(1)}f_{ik}^{*m+n-j}(\eta)d\eta,\\
&\qquad\qquad(j=i,\ i+1,\ \cdots,\ n-1;\ i=0,1,\cdots,n-1),\\
&{}^{(1)}R_0^{m+n-j}(t)={}^{(3)}R_0^{m+n-j}(t),\qquad(j=0,1,\cdots,n),\\
&{}^{(1)}R_i^m(t)={}^{(2)}R_i^m(t),\qquad(i=0,1,\cdots,n),
\end{aligned}
$$

$$
(4.9)\quad
\begin{aligned}
&{}^{(2)}R_i^{m+n-j}(t)=e^{-(\lambda_j+\mu_i)t}+\int_0^t\mu_ie^{-(\lambda_j+\mu_i)\tau(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau\\
&+\int_0^t d\eta\int_0^\eta\mu_ie^{-(\lambda_j+\mu_i)\tau}\sum_{k=1}^{i}{}^{(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)(p_{k-1k}^{j-1j(2)}R_k^{m+n-j}(t-\eta)\\
&+p_{k-1k-1}^{j-1j}{}^{(2)}R_{k-1}^{m+n-j}(t-\eta))d\tau\\
&=e^{-(\lambda_j+\mu_i)t}+\int_0^t\mu_ie^{-(\lambda_j+\mu_i)\tau(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau\\
&+\int_0^t d\eta\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\sum_{k=0}^{i}\Big(\frac{\mu_i}{\lambda_j+\mu_i}\Big)(p_{k-1k}^{j-1j(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)\\
&+p_{kk}^{j-1j(2)}g_{i-1k}^{m+n-(j-1)}(\eta-\tau)){}^{(2)}R_k^{m+n-j}(t-\eta)d\tau\\
&={}^{(2)}m_i^j(t)+\sum_{k=0}^{i}\int_0^t{}^{(2)}R_k^{m+n-j}(t-\eta){}^{(2)}f_{ik}^{*m+n-j}(\eta)d\eta,\\
&\qquad\qquad(j=i,\ i+1,\ \cdots,\ n;\ i=1,2,\cdots,n),\\
&{}^{(2)}R_0^{m+n-j}(t)=e^{-\lambda_jt},\qquad(j=0,1,\cdots,n),
\end{aligned}
$$

$$
(4.10)\quad
\begin{aligned}
&{}^{(3)}R_i^{m+n-j}(t)=e^{-(\lambda_j+\mu_i)t}+\int_0^t\lambda_je^{-(\lambda_j+\mu_i)\tau}(p_{ii+1}^{jj+1(3)}R_{i+1}^{m+n-(j+1)}(t-\tau)+p_{ii}^{jj+1}\\
&\times{}^{(3)}R_i^{m+n-(j+1)}(t-\tau))d\tau+\int_0^t d\eta\int_0^\eta\lambda_je^{-(\lambda_j+\mu_i)\tau}(p_{ii+1}^{jj+1}\sum_{k=1}^{i+1}{}^{(3)}g_{i+1k}^{m+n-(j+1)}(\eta-\tau)\\
&\times{}^{(3)}R_{k-1}^{m+n-j}(t-\eta)+p_{ii}^{jj+1}\sum_{k=1}^{i}{}^{(3)}g_{ik}^{m+n-(j+1)}(\eta-\tau){}^{(3)}R_{k-1}^{m+n-j}(t-\eta))d\tau\\
&=e^{-(\lambda_j+\mu_i)t}+\int_0^t\lambda_je^{-(\lambda_j+\mu_i)\tau}(p_{ii+1}^{jj+1(3)}R_{i+1}^{m+n-(j+1)}(t-\tau)+p_{ii}^{jj+1}\\
&\times{}^{(3)}R_i^{m+n-(j+1)}(t-\tau))d\tau+\int_0^t d\eta\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\sum_{k=0}^{i}\Big(\frac{\lambda_j}{\lambda_j+\mu_i}\Big)(p_{ii+1}^{jj+1}\\
&\times{}^{(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta-\tau)+p_{ii}^{jj+1(3)}g_{ik+1}^{m+n-(j+1)}(\eta-\tau)){}^{(3)}R_k^{m+n-j}(t-\eta)d\tau\\
&={}^{(3)}m_i^j(t)+\sum_{k=0}^{i}\int_0^t{}^{(3)}R_k^{m+n-j}(t-\eta){}^{(3)}f_{ik}^{*m+n-j}(\eta)d\eta,\\
&\qquad\qquad(j=i,\ i+1,\ \cdots,\ n-1,\ i=0,1,\cdots,n-1),\\
&{}^{(3)}R_i^m(t)=e^{-(\lambda_n+\mu_i)t},\qquad(i=0,1,\cdots,n).
\end{aligned}
$$

(4.11)
$$\begin{cases} {}^{(2)}g_{ik}^{m+n-j}(t)=\int_0^t d\eta\int_0^\eta \mu_i e^{-(\lambda_j+\mu_i)\tau}\{p_{i-1i}^{j-1j(2)}g_{i-1i-1}^{m+n-(j-1)}(\eta-\tau){}^{(2)}g_{ik}^{m+n-j}(t-\eta) \\ \quad +\sum_{l=k}^{i-1}(p_{ll}^{j-1j(2)}g_{i-1l}^{m+n-(j-1)}(\eta-\tau)+p_{l-1l}^{j-1j(2)}g_{i-1l-1}^{m+n-(j-1)}(\eta-\tau){}^{(2)}g_{lk}^{m+n-j}(t-\eta)\}d\tau \\ \quad =\int_0^t d\eta\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\sum_{l=k}^{i}\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(p_{l-1l}^{j-1j(2)}g_{i-1l-1}^{m+n-(j-1)}(\eta-\tau) \\ \quad +p_{ll}^{j-1j(2)}g_{i-1l}^{m+n-(j-1)}(\eta-\tau)){}^{(2)}g_{lk}^{m+n-j}(t-\eta)d\tau \\ \quad =\sum_{l=k}^{i}\int_0^t {}^{(2)}g_{lk}^{m+n-j}(t-\eta){}^{(2)}f_{il}^{*m+n-j}(\eta)d\eta, \quad (j=i,i+1,\cdots,n;i=0,1,\cdots,n;k<i), \\ {}^{(2)}g_{ii}^{m+n-j}(t)=\lambda_j e^{-(\lambda_j+\mu_i)t}+\int_0^t {}^{(2)}g_{ii}^{m+n-j}(t-\eta){}^{(2)}f_{ii}^{*m+n-j}(\eta)d\eta, \\ \qquad (j=i,\ i+1,\ \cdots,\ n;\ i=1,2,\cdots,\ n). \\ {}^{(2)}g_{00}^{m+n-j}(t)=\lambda_j e^{-\lambda_j t} \qquad (j=0,1,\cdots,\ n), \end{cases}$$

(4.12)
$$\begin{cases} {}^{(3)}g_{ik}^{m+n-j}(t)=\int_0^t d\eta\int_0^\eta \lambda_j e^{-(\lambda_j+\mu_i)\tau}(p_{i+1}^{jj+1}\sum_{l=k+1}^{i+1}{}^{(3)}g_{i+1l}^{m+n-(j+1)}(\eta-\tau){}^{(3)}g_{l-1k}^{m+n-j}(t-\eta) \\ \quad +p_{ii}^{jj+1}\sum_{l=k+1}^{i}{}^{(3)}g_{il}^{m+n-(j+1)}(\eta-\tau){}^{(3)}g_{l-1k}^{m+n-j}(t-\eta))d\tau \\ \quad =\int_0^t d\eta\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\sum_{l=k}^{i}\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(p_{i+1}^{jj+1i(3)}g_{i+1l+1}^{m+n-(j+1)}(\eta-\tau) \\ \quad +p_{ii}^{jj+1(3)}g_{il+1}^{m+n-(j+1)}(\eta-\tau)){}^{(3)}g_{lk}^{m+n-j}(t-\eta)d\tau \\ \quad =\sum_{l=k}^{i}\int_0^t {}^{(3)}g_{lk}^{m+n-j}(t-\eta){}^{(3)}f_{il}^{*m+n-j}(\eta)d\eta, \quad (j=i,i+1,\cdots,n;i=0,1,\cdots,n;k<i), \\ {}^{(3)}g_{ii}^{m+n-j}(t)=\mu_i e^{-(\lambda_i+\mu_i)t}+\int_0^t {}^{(3)}g_{ii}^{m+n-j}(t-\eta){}^{(3)}f_{ii}^{*m+n-j}(\eta)d\eta, \\ \qquad (j=i,\ i+1,\ \cdots,\ n-1;\ i=1,2,\cdots,\ n) \\ {}^{(3)}g_{00}^{m+n-j}(t)=0, \quad {}^{(3)}g_{ii}^{m}(t)=\mu_i e^{-(\lambda_j+\mu_i)t}, \qquad (j=0,1,\cdots,\ n;\ i=1,2,\cdots,\ n). \end{cases}$$

where

$$\mu_0=0, \quad p_{-10}^{j-1j}=0, \quad {}^{(2)}g_{i-1i}^{m+n-(j-1)}(t)=0, \quad {}^{(3)}g_{ii+1}^{m+n-(j+1)}(t)=0,$$

$$\begin{aligned} {}^{(1)}m_i^j(t)&=e^{-(\lambda_j+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_j+\mu_i)\tau(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau \\ &\quad +\int_0^t \lambda_j e^{-(\lambda_j+\mu_i)\tau}(p_{ii}^{jj+1(3)}R_i^{m+n-(j+1)}(t-\tau)+p_{ii}^{jj+1(3)}R_{i+1}^{m+n-(j+1)}(t-\tau))d\tau \\ &\qquad (j=i,\ i+1,\ \cdots,\ n;\ i=0,1,\cdots,\ n), \end{aligned}$$

$${}^{(2)}m_i^j(t)=e^{-(\lambda_j+\mu_i)t}+\int_0^t \mu_i e^{-(\lambda_j+\mu_i)\tau(2)}R_{i-1}^{m+n-(j-1)}(t-\tau)d\tau, \qquad (j=i,\ i+1,\ \cdots,\ n;\ i=0,1,\cdots,\ n)$$

$${}^{(3)}m_i^j(t)=e^{-(\lambda_j+\mu_i)t}+\int_0^t \lambda_j e^{-(\lambda+\mu_i)\tau}(p_{ii}^{jj+1(3)}R_i^{m+n-(j+1)}(t-\tau)+p_{ii}^{jj+1(3)}R_{i+1}^{m+n-(j+1)}(t-\tau))d\tau \qquad (j=i,\ i+1,\ \cdots,\ n;\ i=0,1,\cdots,\ n)$$

(4.13)
$$\begin{cases} {}^{(1)}f_{ik}^{*m+n-j}(\eta)=\int_0^\eta(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\left\{\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(p_{k-1k}^{j-1j(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)\right. \\ \quad +p_{kk}^{j-1j(2)}g_{i-1k}^{m+n-(j-1)}(\eta-\tau)+\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(p_{ii+1}^{jj+1(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta-\tau) \end{cases}$$

$$
\left\{\begin{aligned}
&+p_{ii}^{jj+1(3)}g_{ik+1}^{m+n-(j+1)}(\eta-\tau)\Big\}d\tau\\
&=\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(p_{k-k1}^{jj+1}F_i^{j}*^{(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta)+p_{kk}^{j-1j}F_i^{j}*^{(2)}g_{i-1k}^{m+n-(j-1)}(\eta))\\
&+\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(p_{ii+1}^{jj+1}F_i^{j}*^{(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta)+p_{ii}^{jj+1}F_i^{j}*^{(3)}g_{ik+1}^{m+n-(j+1)}(\eta)),\\
&\qquad\qquad (j=i,\ i+1,\cdots,\ n;i\ =0,1,\cdots,\ n;\ k=0,1,\cdots,\ i)\\
&{}^{(2)}f_{ik}^{*m+n-j}(\eta)=\int_0^{\eta}(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(p_{k-1k}^{j-1j(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta-\tau)\\
&+p_{kk}^{j-1j(2)}g_{i-1k}^{m+n-(j-1)}(\eta-\tau))d\tau\\
&=\left(\frac{\mu_i}{\lambda_j+\mu_i}\right)(p_{k-1k}^{j-1j}F_i^{j}*^{(2)}g_{i-1k-1}^{m+n-(j-1)}(\eta)+p_{kk}^{j-1j}F_i^{j}*^{(2)}g_{i-1k}^{m+n-(j-1)}(\eta))\\
&\qquad\qquad (j=i,\ i+1,\cdots,\ n;\ i=0,1,\cdots,\ n;\ k=0,1,\cdots,\ i),\\
&{}^{(3)}f_{ik}^{*m+n-j}(\eta)=\int_0^{\eta}(\lambda_j+\mu_i)e^{-(\lambda_j+\mu_i)\tau}\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(p_{ii+1}^{jj+1(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta-\tau)\\
&+p_{ii}^{jj+1(3)}g_{ik+1}^{m+n-(j+1)}(\eta-\tau))d\tau\\
&=\left(\frac{\lambda_j}{\lambda_j+\mu_i}\right)(p_{ii+1}^{jj+1}F_i^{j}*^{(3)}g_{i+1k+1}^{m+n-(j+1)}(\eta)+p_{ii}^{jj+1}F_i^{j}*^{(3)}g_{ik+1}^{m+n-(j+1)}(\eta))\\
&\qquad\qquad (j=i,\ i+1,\cdots,\ n;\ i=0,1,\cdots,\ n;\ k=0,1,\cdots,\ i),\\
&F_i^{j}(t)=e^{-(\lambda_j+\mu_i)t},
\end{aligned}\right.
$$

where p_{ik}^{jl} is the conditional transition probability from E_i^{m+n-j} to $E_k^{m+n-l}(k=i,i+1;\ l=j+1;\ j=i,i+1,\cdots,n-1;i=0,1,\cdots,n-1)$ after the failure occured at the state E_i^{m+n-j},and ${}^{(2)}g_{ih}^{m+n-j}(t)dt$ the transition probability that after the system starting from E_i^{m+n-j} remained within the state $E_{k2}^{m+n-l_2}(l_2=k_2+j-i,\ k_2+j-i+1,\ \cdots,\ j;\ k=0,\ 1,\ \cdots,\ i)$, there occure a transition from E_h^{m+n-j} to $E_h^{m+n-(j+1)}$ or $E_{h+1}^{m+n-(j+1)}(h=0,1,\cdots,i)$ during time interval $(t,\ t+dt)$, and ${}^{(3)}g_{ih}^{m+n-j}(t)dt$ the transition probability that after the system starting from E_i^{m+n-j} remained within the state $E_{k3}^{m+n-l_3}(l_3=k_3,\ k_3+1,\ \cdots,n;\ k_3=0,1,\cdots,i)$ or $E_{h4}^{m+n-l_4}(l_4=k_4+j-i,\ k_4+j-i+1,\ \cdots,\ n;\ k_4=i+1,\ i+2,\ \cdots,\ n+i-j)$ and there occure a transition from E_h^{m+n-j} to $E_{h-1}^{m+n-(j-1)}$ $(h=1,2,\cdots,i)$ during time interval $(t,\ t+dt)$.

The equations (4.8)~(4.12) are system renewal equation, the result concerning existence and uniqueness of solution extend in a routin fashion. Moreover it follows by induction on the number of spare that ${}^{(l)}R_i^{m+n-j}(0)=1$, ${}^{(l)}R_i^{m+n-j}(\infty)=0$, ${}^{(l)}R_i^{m+n-j}(t)$ is twice differentiable and ${}^{(l)}R_i^{m+n-j}(t)<0$ $(l=1,2,3)$. This shows that ${}^{(l)}R_i^{m+n-j}(t)$ has the properties assumed in previous §2.

Let us define

$$
(4.14)\quad\left\{\begin{aligned}
&{}^{(l)}\varphi_i^{m+n-j}(s)=\int_0^{\infty}e^{-st(l)}R_i^{m+n-j}(t)dt,\quad {}^{(l)}\varphi_i^{*m+n-j}(s)=\int_0^{\infty}e^{-st(l)}R_i^{\prime m+n-j}(t)dt,\\
&\qquad\qquad (j=i,\ i+1,\cdots,\ n;\ i=0,1,\cdots,\ n;\ l=1,2,3)\\
&{}^{(l)}\psi_{ik}^{m+n-j}(s)=\int_0^{\infty}e^{-st(l)}g_{ik}^{m+n-j}(t)dt,\quad (j=i,i+1,\cdots,n;i=0,1,\cdots,n;k=0,1,\cdots,i;l=2,3)
\end{aligned}\right.
$$

By applying the Laplace transformation to Eqs (4.8)~(4.12) and by using the relations ${}^{(l)}\varphi_i^{*m+n-j}(s)=-1+s\,{}^{(l)}\varphi_i^{m+n-j}(s)$ and ${}^{(2)}\varphi_k^{*m+n-j}(s)=-\sum_{i=0}^{k}{}^{(2)}\psi_{ki}^{m+n-j}(s)$, we obtain

$$
(4.15)\quad \left\{
\begin{aligned}
&{}^{(2)}\varphi_i^{m+n-j}(s)=\{{}^{(2)}\psi_{ii}^{m+n-j}(s)[1+\mu_i{}^{(2)}\varphi_{i-1}^{m+n-(j-1)}(s)+\mu_i\sum_{k=0}^{i-1}(p_{kk}^{j-1j(2)}\psi_{i-1k}^{m+n-(j-1)}(s)\\
&\qquad +p_{k-1k}^{j-1j(2)}\psi_{i-1k-1}^{m+n-(j-1)}(s)){}^{(2)}\varphi_k^{m+n-j}(s)]\}/\lambda_j, \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n),\\
&{}^{(2)}\varphi_i^{*m+n-j}(s)=\{{}^{(2)}\psi_{ii}^{m+n-j}(s)[-\lambda_j+\mu_i\sum_{k=0}^{i-1}(p_{kk}^{j-1j(2)}\psi_{i-1k}^{m+n-(j-1)}(s)\\
&\qquad +p_{k-1k}^{j-1j(2)}\psi_{i-1k-1}^{m+n-(j-1)}(s)){}^{(2)}\varphi_k^{*m+n-j}(s)]\}/\lambda_j, \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n),\\
&{}^{(2)}\varphi_0^{m+n-j}(s)=\frac{1}{s+\lambda_j},\quad {}^{(2)}\varphi_0^{*m+n-j}(s)=-\frac{\lambda_j}{s+\lambda_j}, \qquad (j=0,1,\cdots,n),\\
&{}^{(2)}\psi_{ik}^{m+n-j}(s)=\{\mu_i{}^{(2)}\psi_{ii}^{m+n-j}(s)\sum_{l=k}^{i-1}{}^{(2)}\psi_{lk}^{m+n-j}(s)(p_{ll}^{j-1j(2)}\psi_{i-1l}^{m+n-(j-1)}(s)\\
&\qquad +p_{l-1l}^{j-1j(2)}\psi_{i-1l-1}^{m+n-(j-1)}(s))\}/\lambda_j, \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n),\\
&{}^{(2)}\psi_{ii}^{m+n-j}(s)=\frac{\lambda_j}{s+\lambda_j+\mu_i-\mu_i p_{i-1i}^{j-1j(2)}\psi_{i-1i-1}^{m+n-(j-1)}(s)}, \qquad (j=i,\ i+1,\cdots,n;\ i=0,1,\cdots,n),\\
&{}^{(2)}\psi_{00}^{m+n-j}(s)=\frac{\lambda_j}{s+\lambda_j},\quad {}^{(2)}\psi_{-1-1}^{m+n-j}(s)={}^{(2)}\psi_{-10}^{m+n-j}(s)=0, \qquad (j=0,1,\cdots,n),\\
&\mu_0=0,\quad {}^{(2)}\psi_{ik}^{m+n-j}(s)=0, \qquad (k>i),\\
&{}^{(3)}\psi_{ik}^{m+n-j}(s)=0, \qquad (j>n).
\end{aligned}\right.
$$

$$
(4.16)\quad \left\{
\begin{aligned}
&{}^{(3)}\varphi_i^{m+n-j}(s)=\{{}^{(3)}\psi_{ii}^{m+n-j}(s)[1+\lambda_j(p_{ii+1}^{jj+1(3)}\varphi_i^{m+n-(j+1)}(s)+p_{ii}^{jj+1(3)}\varphi_i^{m+n-(j+1)}(s))\\
&\qquad +\lambda_j\sum_{k=1}^{i}(p_{ii+1}^{jj+1(3)}\psi_{i+1k}^{m+n-(j+1)}(s)+p_{ii}^{jj+1(3)}\psi_{ik}^{m+n-(j+1)}(s)){}^{(3)}\varphi_{k-1}^{m+n-j}(s)]\}/\mu_i,\\
&\qquad\qquad (j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n-1),\\
&{}^{(3)}\varphi_i^{*m+n-j}(s)=\{{}^{(3)}\psi_{ii}^{m+n-j}(s)[-\mu_i+\lambda_j(p_{ii+1}^{jj+1(3)}\varphi_{i+1}^{*m+n-(j+1)}(s)+p_{ii}^{jj+1(3)}\varphi_i^{*m+n-(j+1)}(s)\\
&\qquad +p_{ii+1}^{jj+1(3)}\psi_{i+1i+1}^{m+n-(j+1)}(s))+\lambda_j\sum_{k=1}^{i}(p_{ii+1}^{jj+1(3)}\psi_{i+1k}^{m+n-(j+1)}(s)\\
&\qquad +p_{ii}^{jj+1(3)}\psi_{ik}^{m+n-(j+1)}(s))(1+{}^{(3)}\varphi_{k-1}^{*m+n-j}(s))]\}/\mu_i,\\
&\qquad\qquad (j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n-1),\\
&{}^{(3)}\varphi_0^{m+n-j}(s)=\{1+\lambda_j(p_{01}^{jj+1(3)}\varphi_1^{m+n-(j+1)}(s)\\
&\qquad +p_{00}^{jj+1(3)}\varphi_0^{m+n-(j+1)}(s))\}/s+\lambda_j-\lambda_j p_{01}^{jj+1(3)}\psi_{11}^{m+n-(j+1)}(s), \qquad (j=0,1,\cdots,n),\\
&{}^{(3)}\varphi_0^{*m+n-j}(s)=\{\lambda_j(p_{01}^{jj+1(3)}\varphi_1^{m+n-j}(s)+p_{00}^{jj+1(3)}\varphi_0^{m+n-(j+1)}(s)\\
&\qquad +p_{01}^{jj+1(3)}\psi_{11}^{m+n-(j+1)}(s))\}/s+\lambda_j-\lambda_j p_{01}^{jj+1(3)}\psi_{11}^{m+n-(j+1)}(s), \qquad (j=0,1,\cdots,n),\\
&{}^{(3)}\varphi_i^{m}(s)=\frac{1}{s+\lambda_n+\mu_i},\quad {}^{(3)}\varphi_i^{*m}(s)=\frac{-(\lambda_n+\mu_i)}{s+\lambda_n+\mu_i}, \qquad (i=0,1,\cdots,n),\\
&{}^{(3)}\psi_{ik}^{m+n-j}(s)=\{\lambda_j{}^{(3)}\psi_{ii}^{m+n-j}(s)\sum_{l=k+1}^{i}(p_{ii+1}^{jj+1(3)}\psi_{i+1l}^{m+n-(j+1)}(s)+p_{ii}^{jj+1(3)}\psi_{il}^{m+n-(j+1)}(s))\\
&\qquad \times{}^{(3)}\psi_{l-1k}^{m+n-j}(s)\}/\mu_i, \qquad (j=i,\ i+1,\cdots,n-1;\ k=0,1,\cdots,i-1;\ i=1,2,\cdots,n-1),\\
&{}^{(3)}\psi_{ii}^{m+n-j}(s)=\frac{\mu_i}{s+\lambda_j+\mu_i-\lambda_j p_{ii+1}^{jj+1(3)}\psi_{i+1i+1}^{m+n-(j+1)}(s)}, \qquad (j=i,i+1,\cdots,n-1;i=1,2,\cdots,n-1),\\
&{}^{(3)}\psi_i^{m+n-j}(s)=0,\quad {}^{(3)}\psi_{ii}^{m}(s)=\frac{\mu_i}{s+\lambda_n+\mu_i}, \qquad (j=0,1,\cdots,n;\ i=1,2,\cdots,n),\\
&\mu_0=0,\quad {}^{(3)}\psi_{ik}^{m+n-j}(s)=0, \qquad (k>i),\\
&{}^{(3)}\psi_{ik}^{m+n-j}(s)=0, \qquad (j>n).
\end{aligned}\right.
$$

$$(4.17)\begin{cases} {}^{(1)}\varphi_i^{m+n-j}(s)=\{1+\mu_i{}^{(2)}\varphi_{i-1}^{m+n-(j-1)}(s)+\lambda_j(p_{ii}^{jj+1(3)}\varphi_i^{m+n-(j+1)}(s)+p_{ii+1}^{jj+1(3)}\varphi_{i+1}^{m+n-(j+1)}(s)) \\ \quad +\sum_{k=0}^{i-1}(s+\lambda_j+\mu_i){}^{(1)}\psi_{ik}^{*m+n-j}(s){}^{(1)}\varphi_k^{m+n-j}(s)\}/(s+\lambda_j+\mu_i)(1-{}^{(1)}\psi_{ii}^{*m+n-j}(s)), \\ \qquad (j=i,\ i+1,\ \cdots,\ n-1;\ i=1,2,\cdots,n-1), \\ {}^{(1)}\varphi_i^{*m+n-j}(s)=\{\mu_i{}^{(2)}\varphi_{i-1}^{*m+n-j}(s)+\lambda_j(p_{ii}^{jj+1(3)}\varphi_i^{*m+n-(j+1)}(s)+p_{i+1}^{jj+1i(3)}\varphi_{i+1}^{*m+n-(j+1)}(s)) \\ \quad +(s+\lambda_j+\mu_i)[{}^{(1)}\psi_{ii}^{*m+n-j}(s)+\sum_{k=0}^{i-1}{}^{(1)}\psi_{ik}^{*m+n-j}(s)(1{}^{(1)}\varphi_k^{*m+n-j}(s))]\}/(s+\lambda_j+\mu_i) \\ \quad (1-{}^{(1)}\psi_{ii}^{*m+n-j}(s)), \qquad (j=i,i+1,\cdots,n-1;\ i=1,2,\cdots,n-1), \\ {}^{(1)}\varphi_0^{m+n-j}(s)={}^{(3)}\varphi_0^{m+n-j}(s),\ {}^{(1)}\varphi_i^{m}(s)={}^{(2)}\varphi_i^{m}(s), \qquad (j=0,1,\cdots,n;\ i=0,1,\cdots,n), \end{cases}$$

where

$$(s+\lambda_j+\mu_i){}^{(1)}\psi_{ik}^{*m+n-j}(s)=\mu_i(p_{k-1k}^{j-1j(2)}\psi_{i-1k-1}^{m+n-(j-1)}(s)+p_{kk}^{j-1j\,(2)}\psi_{i-1k}^{m+n-(j-1)}(s))$$
$$+\lambda_j(p_{i+1}^{jj+1(3)}\psi_{i+1k+1}^{m+n-(j+1)}(s)+p_{ii}^{jj+1(3)}\psi_{ik+1}^{m+n-(j+1)}(s)),$$
$$(j=i,i+1,\cdots,n;\ k=1,2,\cdots,i;\ i=0,1,\cdots,n),$$

$$\mu_0=0,\ {}^{(2)}\psi_{ik}^{m+n-j}(s)={}^{(3)}\psi_{ik}^{m+n-j}(s)=0, \qquad (j>n),$$

$${}^{(2)}\psi_{-1-1}^{m+n-j}(s)=0,\ {}^{(2)}\psi_{ik}^{m+n-j}(s)={}^{(3)}\psi_{ik}^{m+n-j}(s)=0, \qquad (k>i).$$

The case when the following holds is importants.

$$(4.18)\begin{cases} \lambda_j=(m+n-j)\lambda,\ \mu_l=l\mu, & (j=0,1,\cdots,n;\ l=1,2,\cdots,n), \\ p_{k-1k-1}^{j-1j}=\dfrac{(n-j+1)\lambda}{(m+n-j+1)\lambda}=\dfrac{n-j+1}{m+n-j+1}, & (j=k,\ k+1,\cdots,n;\ k=1,2,\cdots,n), \\ p_{k-1k}^{j-1j}=\dfrac{m\lambda}{(m+n-j+1)\lambda}=\dfrac{m}{(m+n-j+1)}, & (j=k,\ k+1,\cdots,n;\ k=1,2,\cdots,n), \end{cases}$$

This is the case that the system consists of equipments identical and independent to each other and the reliabilities in stand-by operating and main operating are all equal. Consequently we have proved the following theorem.

THEOREM 4.1 *The Laplace transform* ${}^{(1)}\varphi_i^{m+n-j}(s)$ *of the reliability function in the model II is given by* (4.17). *The reliability function can be obtain by the inverse Laplace transform of* (4.17).

Next we calculate the MTBF T_{i+1}^{j+1} of the system. After some simple calculation we may derive the following results

$$\lim_{s\to 0}{}^{(l)}\psi_i^{*m+n-j}(s)=-1, \qquad (j=i,\ i+1,\cdots,n;\ i=0,1,\cdots,n;\ l=1,2,3),$$

$${}^{(l)}T_{i+1}^{j+1}=\lim_{s\to 0}{}^{(l)}\varphi_i^{m+n-j}(s), \qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n;\ l=2,3)$$

Hence from (4.15) and (4.16), we obtain

$$
\left\{
\begin{aligned}
&{}^{(2)}T^{j+1}_{i+1}=\{{}^{(2)}L^{j}_{ii}[1+\mu_i{}^{(2)}T^{j}_{i}+\mu_i\sum_{k=0}^{i-1}(p^{j-1j}_{kk}{}^{(2)}L^{j-1}_{i-1k}+p^{j-1j}_{k-1k}{}^{(2)}L^{i-1}_{j-1k-1})^{(2)}T^{j+1}_{k+1}]\}/\lambda_j,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=i,\ i+1,\cdots,\ n;\ i=1,2,\cdots,\ n),\\
&{}^{(3)}T^{j+1}_{i+1}=\{{}^{(3)}L_{ii}[1+\lambda_j(p^{jj+1(3)}_{ii+1}T^{j+2}_{i+2}+p^{jj+1(3)}_{ii}T^{j+2}_{i+1})+\lambda_j\sum_{k=1}^{i}(p^{jj+1(3)}_{ii+1}L^{j+1}_{i+1k}+p^{jj+1(3)}_{ii}L^{j+1}_{k})\\
&\qquad\times{}^{(3)}T^{j+1}_{k}]\}/\mu_i,\qquad\qquad\qquad\qquad\qquad (j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,\ n),\\
&{}^{(3)}T^{j+1}_{1}=\{1+\lambda_j(p^{jj+1(3)}_{01}T^{j+2}_{2}+p^{jj+1(3)}_{00}T^{j+2}_{1})\}/\lambda_j(1-p^{jj+1(3)}_{01}L^{j+1}_{11}),\ {}^{2}T^{j+1}_{1}=\frac{1}{\lambda_j},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=0,\ 1,\cdots,\ n),
\end{aligned}
\right.
\tag{4.19}
$$

where

$$
{}^{(2)}L^{j-1}_{i-1-1}=0,\quad {}^{(2)}L^{j}_{ii}=\lim_{s\to 0}{}^{(2)}\psi^{m+n-j}_{ii}(s)=\frac{\lambda_j}{\lambda_j+\mu_i-\mu_i p^{j-1j}_{i-1i}{}^{(2)}L^{j-1}_{i-1i-1}},
$$
$$
(j=i,\ i+1,\cdots,\ n;\ i=1,2,\cdots,\ n),
$$

$$
{}^{(2)}L^{j}_{ik}=\lim_{s\to 0}{}^{(2)}\psi^{m+n-j}_{ik}(s)=\{\mu_i{}^{(2)}L^{j}_{ii}\sum_{l=k}^{i-1}(p^{j-1j(2)}_{ll}L^{j-1}_{i-1l}+p^{j-1j(2)}_{l-1l}L^{j-1}_{i-1l-1})\}/\lambda_j,
$$
$$
(j=i,\ +1,\cdots,\ n;\ k=1,2,\cdots,\ i-1;\ i=1,2,\cdots,\ n),
$$

$$
{}^{(3)}T^{n+1}_{i+1}=\frac{1}{\lambda_n+\mu_i},\quad {}^{(3)}L^{n}_{ii}=\frac{\mu_i}{\lambda_n+\mu_i},\quad {}^{(3)}L^{j}_{00}={}^{(2)}L^{j}_{00}=0,\qquad (j=0,1,\cdots,\ n;\ i=0,1,\cdots,\ n),
$$

$$
{}^{(3)}L^{j}_{ii}=\lim_{s\to 0}{}^{(3)}\psi^{m+n-j}_{ii}(s)=\frac{\mu_i}{\lambda_j+\mu_i-\lambda_j p^{jj+1(3)}_{ii+1}L^{j+1}_{i+1i+1}},\quad (j=i,i+1,\cdots,n-1;i=1,2,\cdots,n-1),
$$

$$
{}^{(3)}L^{j}_{ik}=\lim_{s\to 0}{}^{(3)}\psi^{m+n-j}_{ik}(s)=\{\lambda_j{}^{(3)}L^{j}_{ii}\sum_{l=k+1}^{i}(p^{jj+1(3)}_{ii+1}L^{j+1}_{i+1l}+p^{jj+1(3)}_{ii}L^{j+1}_{il})^{(3)}L^{j}_{l-1k}\}/\mu_i,
$$
$$
(j=i,i+1,\cdots,n;\ k=1,2,\cdots,i-1;\ i=1,2,\cdots,n),
$$

$$
\mu_0=0,\quad {}^{(3)}T^{l}_{h}=0,\quad {}^{(2)}L^{j}_{ik}={}^{(3)}L^{j}_{ik}=0,\qquad (h>n+1;\ j>n),
$$

$$
{}^{(2)}L^{j}_{ik}={}^{(3)}L^{j}_{ik}=0,\qquad (k>i).
$$

Specially when (4.18) holds, we have

$$
\left\{
\begin{aligned}
&{}^{(2)}T^{j+1}_{i+1}=\{{}^{(2)}L^{j}_{ii}[(m+n-j+1)(1+i\mu{}^{(2)}T^{j}_{i})+i\mu\sum_{k=0}^{i-1}((n-j+1){}^{(2)}L^{j-1}_{i-1k}\\
&\qquad+m{}^{(2)}L^{j-1}_{i-1k-1})^{(2)}T^{j+1}_{k+1}]\}/(m+n-j+1)(m+n-j)\lambda,\ (j=i,i+1,\cdots,n;i=1,2,\cdots,n),\\
&{}^{(3)}T^{j+1}_{i+1}=\{{}^{(3)}L^{j}_{ii}[1+\lambda(m{}^{(3)}T^{j+2}_{i+2}+(n-j){}^{(3)}T^{j+2}_{i+1}+\lambda\sum_{k=1}^{i}(m{}^{(3)}L^{j+1}_{i+1k}+(n-j){}^{(3)}L^{j+1}_{ik})^{(3)}T^{j+1}_{k+1}]\}/i\mu,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=i,i+1,\cdots,n;\ i=1,2,\cdots,n),\\
&{}^{(3)}T^{j+1}_{1}=\{1+\lambda(m{}^{(3)}T^{j+2}_{2}+(n-j){}^{(3)}T^{j+2}_{1})^{(3)}T^{j+2}_{1}\}/(m+n-j)\lambda-m{}^{(3)}L^{j+1}_{11},\ {}^{(2)}T^{j+1}_{1}\\
&\qquad=\frac{1}{(m+n-j)\lambda},\qquad\qquad\qquad\qquad\qquad (j=0,1,\cdots,\ n),
\end{aligned}
\right.
\tag{4.20}
$$

where

$$
{}^{(2)}L^{j-1}_{i-1-1}=0,{}^{(2)}L^{j}_{ii}=\frac{(m+n-j+1)(m+n-j)\lambda}{(m+n-j+1)[(m+n-j)\lambda+i\mu]-im\mu{}^{(2)}L^{j-1}_{i-1i-1}},
$$
$$
(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,\ n),
$$

$$ {}^{(2)}L_{ik}^{j}=\{i\mu^{(2)}L_{ii}^{j}\sum_{l=k}^{i-1}{}^{(2)}L_{lk}^{j}[(n-j+1)^{(2)}L_{i-1l-1}^{j-1}+m^{(2)}L_{i-1l-1}^{j-1}]\}/(m+n-j+1)(m+n-j)\lambda, $$

$$ (j=i,i+1,\cdots,n;\ k=0,1,\cdots,i-1;\ i=1,2,\cdots,n), $$

$$ {}^{(3)}T_{i+1}^{n+1}=\frac{1}{m\lambda+i\mu},\quad {}^{(3)}L_{ii}^{n}=\frac{i\mu}{m\lambda+i\mu},\quad {}^{(3)}L_{00}^{j}={}^{(2)}L_{00}^{j}=0,\qquad (j=0,1,\cdots,n;\ i=1,\cdots,n), $$

$$ {}^{(3)}L_{ii}^{j}=\frac{i\mu}{(m+n-j)\lambda+i\mu-m\lambda^{(3)}L_{i+1i+1}^{j+1}},\qquad (j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n-1), $$

$$ {}^{(3)}L_{ik}^{j}=\{\lambda^{(3)}L_{ii}^{j}\sum_{l=k+1}^{i}[m^{(3)}L_{i+1l}^{j+1}+(n-j)^{(3)}L_{il}^{j+1}]^{(3)}L_{l+k}^{j}/i\mu, $$

$$ (j=i,\ i+1,\cdots,n;\ k=0,1,\cdots,i-1;\ i=1,2,\cdots,n), $$

$$ \mu_0=0,\quad {}^{(3)}T_{h}^{l}=0,\quad {}^{(2)}L_{ik}^{j}={}^{(3)}L_{ik}^{j}=0,\qquad (h>n+1,\ j>n), $$

$$ {}^{(2)}L_{ik}^{j}={}^{(3)}L_{ik}^{j}=0,\qquad (k>i), $$

moreover from (4.17) and (4.19), we get

$$ (4.21)\quad \begin{cases} {}^{(1)}T_{i+1}^{j+1}=\lim_{s\to 0} m^{(1)}\varphi_{i}^{m+n-j}(s) \\ \quad =\dfrac{1+\mu_i{}^{(2)}T_i^j+\lambda_j(p_{ii}^{jj+1(3)}T_{i+1}^{j+2}+p_{ii+1}^{jj+1(3)}T_{i+2}^{j+2})+(\lambda_j+\mu_i)\sum_{k=0}^{i-1}{}^{(1)}L_{ik}^{j}{}^{(1)}T_{k+1}^{j+1}}{(\lambda_j+\mu_i)(1-{}^{1}L_{ii}^{j})}, \\ \qquad (j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n), \\ {}^{(1)}T_{1}^{j+1}={}^{(3)}T_{1}^{j+1},\quad {}^{(1)}T_{i+1}^{n+1}={}^{(2)}T_{i+1}^{n+1},\qquad (j=0,1,\cdots,n;\ i=1,\cdots,n), \end{cases} $$

where

$$ {}^{(1)}L_{ik}^{j}=\lim_{s\to 0} m^{(1)}\psi_{ik}^{*m+n-j}(s)=\{\mu_i(p_{k-1k}^{j-1j(2)}L_{i-1k-1}^{j-1}+p_{kk}^{j-1j(2)}L_{i-1k}^{j-1}) $$

$$ +\lambda_j(p_{ii+1}^{jj+1(3)}L_{i+1k+1}^{j+1}+p_{ii}^{jj+1(3)}L_{ik+1}^{j+1})\}/\lambda_j+\mu_i, $$

$$ (j=i,\ i+1,\cdots,n;\ k=0,1,\cdots,i;\ i=0,1,\cdots,n), $$

$$ {}^{(1)}L_{ik}^{j}=0,\qquad (k>i). $$

and where ${}^{(2)}T_i^j$ and ${}^{(3)}T_i^j$ are given as (4.19), and equal to

$$ (4.22)\quad \begin{cases} {}^{(1)}T_{i+1}^{j+1}=\dfrac{1+i\mu^{(2)}T_i^j+\lambda[(n-j)^{(3)}T_{i+1}^{j+2}+m^{(3)}T_{i+2}^{j+2}]+[(m+n-j)\lambda+i\mu]\sum_{k=0}^{i-1}{}^{(1)}L_{ik}^{j}{}^{(1)}T_{k+1}^{j+1}}{[(m+n-j)\lambda+i\mu](1-{}^{(1)}L_{ii}^{j})}, \\ \qquad (j=i,\ i+1,\cdots,n-1,\ i=1,2,\cdots,n), \\ {}^{(1)}T_{1}^{j+1}={}^{(3)}T_{1}^{j+1},\quad {}^{(1)}T_{i+1}^{j+1}={}^{(2)}T_{i+1}^{n+1},\qquad (j=0,1,\cdots,n;\ i=0,1,\cdots,n), \end{cases} $$

where

$$ {}^{(1)}L_{ik}^{j}=\{i\mu[m^{(2)}L_{i-1k-1}^{j-1}+(n-j+1)^{(2)}L_{i-1k}^{j-1}]+(m+n-j+1)\lambda[m^{(3)}L_{i+1k+1}^{j+1} $$

$$ +(n-j)^{(3)}L_{ik+1}^{j+1}]\}/(m+n-j+1)[(m+n-j)\lambda+i\mu], $$

$$ (j=i,\ i+1,\cdots,n;\ k=0,1,\cdots,i-1;\ i=0,1,\cdots,n), $$

$$ {}^{(1)}L_{ik}^{j}=0,\qquad (k>i). $$

and where ${}^{(2)}T_i^j$ and ${}^{(3)}T_i^j$ are given as (4.20), when (4.18) hold true.

Our result may be summarized following theorem.

THEOREM 4.2 *The MTBF* ${}^{(1)}T_{i+1}^{j+1}$ *of the model II is expressed by* (4.21) *and given by* (4.22), *when* (4.18) *is true. Moreober the relative improvement* $\triangle T_{h+1r+1}$ *in MTBF of the system when the spare increase from h to r is given by* $[T_{i+1}^{j+1}]_{n=r}/[T_{i+1}^{j+1}]_{n=h}$.

Now we calculate the variance

Let us put

$$ {}^{(l)}E_{i+1}^{j+1}(T^2)=\int_0^\infty t^{2(l)}f_i^{m+n-j}(t)dt,\quad ({}^{(l)}f_i^{m+n-j}(t)={}^{(l)}F_i'^{m+n-j}(t)=(1-{}^{(l)}R_i^{m+n-j}(t))'), $$

then it follows from (4.15) and (4.16) that

$$ (4.23)\quad \begin{cases} {}^{(2)}E_{i+1}^{j+1}(T^2)=-2\lim\limits_{s\to 0}{}^{(2)}\varphi_i^{m+n-j\,\prime}(s) \\ \quad =\{\mu_i({}^{(2)}L_{ii}^j)^2[{}^{(2)}E_i^j(T^2)+\sum\limits_{k=0}^{i-1}[(p_{kk}^{j-1j(2)}L_{i-1k}^{j-1}+p_{k-1k}^{j-1j(2)}L_{i-1k-1}^{j-1})^{(2)}E_{k+1}^{j+1}(T^2) \\ \quad -2(p_{kk}^{j-1j(2)}D_{i-1k}^{j-1}+p_{k-1k}^{j-1j(2)}D_{i-1k-1}^{j-1})^{(2)}T_{k+1}^{j+1}]]-2\lambda_j{}^{(2)}D_{ii}^{j\,(2)}T_{i+1}^{j+1}\}/\lambda_j{}^{(2)}L_{ii}^j, \\ \qquad\qquad (j=i,\ i+1,\cdots,\ n;\ i=1,2,\cdots,\ n), \\ {}^{(2)}E_1^{j+1}(T^2)=\dfrac{2}{(\lambda_j)^2}, \qquad\qquad (j=0,1,\cdots,\ n), \\ {}^{(3)}E_{i+1}^{j+1}(T^2)=-2\lim\limits_{s\to 0}{}^{(3)}\varphi_i^{m+n-j\,\prime}(s) \\ \quad =\{\lambda_j({}^{(3)}L_{ii}^j)^2[p_{ii+1}^{jj+1(3)}E_{i+2}^{j+2}(T^2)+p_{ii}^{jj+1(3)}E_{i+1}^{j+2}(T^2)+\sum\limits_{k=1}^{i}[(p_{ii+1}^{jj+1(3)}L_{i+1k}^{j+1} \\ \quad +p_{ii}^{jj+1(3)}L_{ik}^{j+1})^{(3)}E_k^{j+1}(T^2)-2(p_{ii+1}^{jj+1(3)}D_{i+1k}^{j+1}+p_{ii}^{jj+1(3)}D_{ik}^{j+1})^{(3)}T_k^{j+1}]] \\ \quad -2\mu^{i(3)}D_{ii}^{j\,(3)}T_{i+1}^{j+1}\}/\mu_i{}^{(3)}L_{ii}^j, \qquad (j=i,\ i+1,\cdots,\ n-1;\ i=1,2,\cdots,\ n-1), \\ {}^{(3)}E_1^{j+1}(T^2)=\{p_{01}^{jj+1(3)}E_2^{j+2}(T^2)+p_{00}^{jj+1(3)}E_1^{j+2}(T^2)+2(1-\lambda_j p_{01}^{jj+1(3)}D_{11}^{j+1}) \\ \quad \times(1-p_{01}^{jj+1(3)}L_{11}^{j+1})^{(3)}T_1^{j+1}\}/\lambda_j(1-p_{01}^{jj+1(3)}L_{11}^{j+1})^2, \qquad (j=0,1,\cdots,\ n), \\ {}^{(3)}E_{i+1}^{n+1}(T^2)=\dfrac{2}{(\lambda_n+\mu_i)^2}, \qquad\qquad (i=0,1,\cdots,\ n). \end{cases} $$

where

$$ {}^{(2)}D_{ik}^j=\lim_{s\to 0}{}^{(2)}\varphi_{ik}^{m+n-j\,\prime}(s)=\{\lambda_j{}^{(2)}D_{ii}^{j\,(2)}L_{ik}^j+\mu_i({}^{(2)}L_{ii}^j)^2\sum_{l=k}^{i-1}[(p_{ll}^{j-1j(2)}L_{i-1l}^{j-1}+p_{-1l}^{j-1j(2)}L_{i-1l-1}^{j-1})^{(2)}D_{lk}^j $$
$$ +(p_{ll}^{j-1j(2)}D_{i-1l}^{j-1}+p_{l-1l}^{j-1j(2)}D_{i-1l-1}^{j-1})^{(2)}L_{lk}^j]\}/\lambda_j{}^{(2)}L_{ii}^j, $$
$$ (j=i,\ i+1,\cdots,\ n;\ k=0,1,\cdots,\ i-1;\ i=1,2,\ n), $$
$$ {}^{(2)}D_{ii}^j=\lim_{s\to 0}{}^{(2)}\varphi_{ik}^{m+n-j\,\prime}(s)=-\{(1-\mu_i p_{i-1i}^{j-1j(2)}D_{i-1i-1}^j)({}^{(2)}L_{ii}^j)^2\}/\lambda_j, $$
$$ (j=i,\ i+1,\cdots,\ n;i=1,2,\cdots,n), $$

$$^{(2)}D_{00}^{j}=\lim_{s\to 0}{}^{(2)}\psi_{00}^{m+n-j'}(s)=-\frac{1}{\lambda_j}, \qquad (j=0,1,\cdots,n),$$

$$^{(3)}D_{ik}^{j}=\lim_{s\to 0}{}^{(3)}\psi_{ik}^{m+n-j'}(s)=\{\mu_i{}^{(3)}D_{ii}^{j}{}^{(3)}L_{ik}^{j}+\lambda_j({}^{(3)}L_{ii}^{j})^2\sum_{l=k+1}^{i}[(p_{ii+1}^{jj+1(3)}L_{i+1l}^{j+1}+p_{ii}^{jj+1(3)}L_{il}^{j+1})^{(3)}D_{l-1k}^{j}$$
$$+(p_{ii+1}^{jj+1(3)}D_{i+1l}^{j+1}+p_{ii}^{jj+1(3)}D_{il}^{j+1})^{(3)}L_{l-1k}^{j}]\}/\mu_i{}^{(3)}L_{ii}^{j},$$
$$(j=i,\ i+1,\cdots,n;\ k=0,1,\cdots,i-1;\ i=1,2,\cdots,n),$$

$$^{(3)}D_{ii}^{j}=\lim_{s\to 0}{}^{(3)}\psi_{ii}^{m+n-j'}(s)=-\{(1-\lambda_j p_{ii+1}^{jj+1(3)}D_{i+1i+1}^{j+1})({}^{(3)}L_{ii}^{j})^2\}/\mu_i,$$
$$(j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n-1),$$

$$^{(3)}D_{ii}^{n}=\lim_{s\to 0}{}^{(3)}\psi_{00}^{m'}(s)=-\frac{\mu_i}{(\lambda_n+\mu_i)^2}, \qquad (i=0,1,\cdots,n),$$

$$^{(3)}D_{00}^{j}=0, \qquad (j=0,1,\cdots,n).$$

and where $^{(2)}L_{ik}^{j}$, $^{(2)}T_i^j$, $^{(3)}L_{ik}^{j}$ and $^{(3)}T_i^j$ are given as (4.19), and given by

(4.24)

$$^{(2)}E_{i+1}^{j+1}(T^2)=\{(m+n-j+1)i\mu({}^{2}L_{ii}^{j})^2[{}^{(2)}E_i^j(T^2)+\sum_{k=0}^{i-1}[((n-j+1)^{(2)}L_{i-1k}^{j-1}+m^{(2)}L_{i-1k-1}^{j-1})$$
$$\times{}^{(2)}E_{j+1}^{k+1}(T^2)-2((n-j+1)^{(2)}D_{i-1k}^{j-1}+m^2D_{i-1k-1}^{j-1})^{(2)}T_{k+1}^{j+1}]]-2(m+n-j+1)$$
$$\times(m+n-j)\lambda^{(2)}D_{ii}^{j}{}^{(2)}T_{i+1}^{j+1}\}/(m+n-j+1)(m+n-j)\lambda^{(2)}L_{ii}^{j},$$
$$(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n),$$

$$^{(2)}E_1^{j+1}(T^2)=\frac{2}{(m+n-j)^2\lambda^2}, \qquad (j=0,1,\cdots,n),$$

$$^{(3)}E_{i+1}^{j+1}(T^2)=\{\lambda({}^{(3)}L_{ii}^{j})^2m^{(3)}E_{i+2}^{j+2}(T^2)+(n-j)^{(3)}E_{i+1}^{j+2}(T^2)+\sum_{k=1}^{i}[(m^{(3)}L_{i+1k}^{j+1}+(n-j)^{(3)}L_{ik}^{j+1})$$
$$\times{}^{(3)}E_k^{j+1}(T^2)-2(m^{(3)}D_{i+1k}^{j+1}+(n-j)^{(3)}D_{ik}^{j+1})^{(3)}T_k^{j+1}]]-2i\mu^{(3)}D_{ii}^{j}{}^{(3)}T_{i+1}^{j+1}\}/i\mu^{(3)}L_{ii}^{j},$$
$$(j=i,\ i+1,\cdots,n-1;\ i=1,2,\cdots,n-1),$$

$$^{(3)}E_1^{j+1}(T^2)=\{m^{(3)}E_2^{j+2}(T^2)+(n-j)^{(3)}E_1^{j+1}(T^2)+2(1-m\lambda^{(3)}D_{11}^{j+1})[(m+n-j)$$
$$-m^{(3)}L_{11}^{j+1}]^{(3)}T_1^{j+1}\}/\{(m+n-j)-m^{(3)}L_{11}^{j+1}\}^2\lambda, \qquad (j=0,1,\cdots,n),$$

$$^{(3)}E_{i+1}^{n}(T^2)=\frac{2}{(m\lambda+i\mu)^2}, \qquad (i=0,1,\cdots,n).$$

where

$$^{(2)}D_{ik}^{j}=\{(m+n-j+1)(m+n-j)\lambda^{(2)}D_{ii}^{j}{}^{(2)}L_{ik}^{j}+i\mu({}^{(2)}L_{ii}^{j})^2\sum_{l=k}^{i-1}[((n-j+1)^{(2)}L_{i-1l}^{j-1}$$
$$+m^{(2)}L_{i-1l-1}^{j-1})^{(2)}D_{lk}^{j}+((n-j+1)^{(2)}D_{i-1l}^{j-1}+m^{(2)}D_{i-1l-1}^{j-1})$$
$$\times{}^{(2)}L_{lk}^{j}]\}/(m+n-j+1)(m+n-j)\lambda^2L_i^j,$$
$$(j=i,i+1,\cdots,n;\ k=0,1,\cdots,i-1;\ i=1,2,\cdots,n)$$

$$^{(2)}D_{ii}^{j}=-\{[(m+n-j+1)-im\mu^{(2)}D_{i-1i-1}^{j-1}]({}^{(2)}L_{ii}^{j})^2\}/(m+n-j+1)(m+n-j)\lambda,$$
$$(j=i,\ i+1,\cdots,n;\ i=1,2,\cdots,n),$$

$$^{(2)}D_{00}^{j}=-\frac{1}{(m+n-j)\lambda}, \qquad (j=0,1,\cdots,n)$$

$$^{(3)}D^j_{ik}=\{i\mu^{(3)}D^j_{ii}{}^{(3)}L^j_{ik}+\lambda({}^{(3)}L^l_{ii})^2\sum_{l=k+1}^{i}[(m^{(3)}L^{j+1}_{i+1l}+(n-j)^{(3)}L^{j+1}_{il})^{(3)}D^j_{l-1k}+(m^{(3)}D^{j+1}_{i+1l}$$
$$+(n-j)^{(3)}D^{j+1}_{il})^{(3)}L^j_{l-1k}]\}/i\mu^{(3)}L^j_{ii}, \qquad (j=i,i+1,\cdots,n;k=0,1,\cdots,i-1;i=1,2,\cdots,n),$$
$$^{(3)}D^j_{ii}=-\{(1-m\lambda^{(3)}D^{j+1}_{i+1i+1})({}^{(3)}L^j_{ii})^2\}/i\mu, \qquad (j=i,i+1,\cdots,n-1;i=1,2,\cdots,n-1)$$
$$^{(3)}D^n_{ii}=-\frac{i\mu}{(m\lambda+i\mu)^2}, \qquad (i=1,2,\cdots,n),$$
$$^{(3)}D^j_{00}=0, \qquad (j=0,1,\cdots,n).$$

and where $^{(2)}L^j_{ik}$, $^{(2)}T^j_i$, $^{(3)}L^j_{ik}$ and $^{(3)}T^j_i$ are given as (4.20), when (4.18) is true. Hence it follows from (4.17) and (4.23) that

$$(4.25)\quad \begin{cases} {}^{(1)}E^{j+1}_{i+1}(T^2)=-2\lim_{s\to0}{}^{(1)}\varphi'^{m+n-j}_i(s) \\ \quad =\mu_i{}^{(2)}E^j_i(T^2)+\lambda_j(p^{jj+1(3)}_{ii}E^{j+1}_{i+1}(T^2)+p^{jj+1(3)}_{ii+1}E^{j+2}_{i+2}(T^2))+\sum_{k=0}^{j-1}[(\lambda_j+\mu_i) \\ \quad \times({}^{(1)}L^j_{ik}E^{j+1}_{k+1}(T^2)-2^{(1)}D^j_{ik}{}^{(1)}T^{j+1}_{k+1})-2^{(1)}L^j_{ik}{}^{(1)}T^{j+1}_{k+1}]+2[1-{}^{(1)}L^j_{ii}-(\lambda_j+\mu_i)^{(1)}D^j_{ii}] \\ \quad \times{}^{(1)}T^{j+1}_{i+1}/(\lambda_j+\mu_i)(1-{}^{(1)}L^j_{ii}), \qquad (j=i,\ i+1,\cdots,\ n-1;\ i=1,2,\cdots,\ n-1) \\ {}^{(1)}E^{j+1}_1(T^2)={}^{(3)}E^{j+1}_1(T^2), \qquad (j=0,1,\cdots,\ n), \\ {}^{(1)}E^{m+1}_{i+1}(T^2)={}^{(2)}E^{m+1}_{i+1}(T^2), \qquad (i=0,1,\cdots,\ n). \end{cases}$$

where

$$^{(1)}D^j_{ik}=\lim_{s\to0}{}^{(1)}\psi'^{*m+n-j}_{ik}(s)=\{\mu_i(p^{j-1j(2)}_{k-1k}D^{j-1}_{i-1k-1}+p^{j-1j(2)}_{kk}D^{j-1}_{i-1k})+\lambda_j(p^{jj+1(3)}_{ii+1}D^{j+1}_{i+1k+1}$$
$$+p^{jj+1(3)}_{ii}D^{j+1}_{ik+1})\}/(\lambda_j+\mu_i), \qquad (j=i,i+1,\cdots,n;k=0,1,\cdots,i;i=0,1,\cdots,n),$$
$$\mu_0=0,\ {}^{(2)}D^j_{ik}=0,\ {}^{(3)}D^j_{ik}=0, \qquad (k>i),$$

and where $^{(1)}L^j_{ik}$, $^{(1)}T^j_i$, $^{(2)}E^j_i(T^2)$ and $^{(3)}E^j_i(T^2)$ are given as (4.21) and (4.23), specially when (4.18) holds true, we have

$$(4.26)\quad \begin{cases} {}^{(1)}E^{j+1}_{i+1}(T^2)=\{i\mu^{(2)}E^j_i(T^2)+\lambda[(n-j)^{(3)}E^{j+2}_{i+1}(T^2)+m^{(3)}E^{j+2}_{i+2}(T^2)]+\sum_{k=0}^{i-1}[((m+n-j)\lambda \\ \quad +i\mu)({}^{(1)}L^j_{ik}{}^{(1)}E^{j+1}_{k+1}(T^2)-2^{(1)}D^j_{ik}{}^{(1)}T^{j+1}_{k+1})-2^{(1)}L^j_{ik}{}^{(1)}T^{j+1}_{k+1}] \\ \quad +2[1-{}^{(1)}L^j_{ii}-((m+n-j)\lambda+i\mu)^{(1)}D^j_{ii}]^{(1)}T^{j+1}_{i+1}\}/[(m+n-j)\lambda+i\mu](1-{}^{(1)}L^j_{ii}), \\ \qquad (j=i,\ i+1,\cdots,\ n-1;\ i=1,2,\cdots,\ n-1), \\ {}^{(1)}E^{j+1}_1(T^2)={}^{(3)}E^{j+1}_1(T^2), \qquad (j=0,1,\cdots,\ n), \\ {}^{(1)}E^{m+1}_{i+1}(T^2)={}^{(2)}E^{m+1}_{i+1}(T^2), \qquad (j=0,1,\cdots,\ n), \end{cases}$$

where

$$^{(1)}D^{ji}_k=\{i\mu[m^{(2)}D^{j-1}_{i-1k-1}+(n-j+1)^{(2)}D^{j-1}_{i-1k}]+(m+n-j+1)\lambda[m^{(3)}D^{j+1}_{i+1k+1}$$
$$+(n-j)^{(3)}D^{j+1}_{ik+1}]\}/(m+n-j+1)[(m+n-j)\lambda+i\mu],$$
$$(j=i,i+1,\cdots,n;k=0,1,\cdots,i;i=0,1,\cdots,n),$$

$\mu_0=0, \ {}^{(2)}D^j_{ik}=0, \ {}^{(3)}D^j_{ik}=0, \qquad (k>i),$

and where ${}^{(1)}L^j_{ik}$, ${}^{(1)}T^j_i$, ${}^{(2)}E^j_i(T^2)$, and ${}^{(3)}E^j_i(T^2)$ are given as (4.22) and (4.24). We have obtain the following theorem

THEOREM 4.3 *The variance* ${}^{(1)}V^{j+1}_{i+1}$ *of the model II is given by the following formula.*

$$(4.27)\quad \begin{cases} {}^{(1)}V^{j+1}_{i+1}=\{{}^{(1)}T^{j+1}_{i+1}[2(1-{}^{(1)}L^j_{ii}-(\lambda_j+\mu_i){}^{(1)}D^j_{ii})-(\lambda_j+\mu_i)(1-{}^{(1)}L^j_{ii}){}^{(1)}T^{j+1}_{i+1}]+[\mu_i{}^{(2)}E^j_i(T^2) \\ \quad +\lambda_j(p^{jj+1}_{ii}{}^{(3)}E^{j+2}_{i+1}(T^2)+p^{jj+1}_{ii+1}{}^{(3)}E^{j+2}_{i+2}(T^2)]+\sum_{k=0}^{i-1}[(\lambda_j+\mu_i)({}^{(1)}L^j_{ik}{}^{(1)}E^{j+1}_{k+1}(T^2) \\ \quad -2{}^{(1)}D^j_{ik}{}^{(1)}T^{j+1}_{k+1})-2{}^{(1)}L^j_{ik}{}^{(1)}T^{j+1}_{k+1}]\}/(\lambda_j+\mu_i)(1-{}^{(1)}L^j_{ii}), \\ \qquad\qquad (j=i,i+1,\cdots,n-1;i=1,2,\cdots,n-1), \\ {}^{(1)}V^{j+1}_1={}^{(3)}V^{j+1}_1={}^{(3)}E^{j+1}_{i+1}(T^2)-({}^{(3)}T^{j+1}_{i+1})^2, \qquad (j=0,1,\cdots,n), \\ {}^{(1)}V^{m+1}_{i+1}={}^{(2)}V^{m+1}_{i+1}={}^{(2)}E^{j+1}_{i+1}(T^2)-({}^{(2)}T^{j+1}_{i+1})^2, \qquad (i=0,1,\cdots,n), \end{cases}$$

where ${}^{(1)}L^j_{ik}$, ${}^{(2)}E^j_i$, ${}^{(3)}E^j_i$, *and* ${}^{(1)}D^j_{ik}$ *are given as* (4.21), (4.23), *and* (4.25), *and equal to*

$$(4.28)\quad \begin{cases} {}^{(1)}V^{j+1}_{i+1}=\{{}^{(1)}T^{j+1}_{i+1}[2(1-{}^{(1)}L^j_{ii}-((m+n-j)\lambda+i\mu){}^{(1)}D^j_{ii}-((m+n-j)\lambda+i\mu) \\ \quad \times(1-{}^{(1)}L^j_{ii}){}^{(1)}T^{j+1}_{i+1}]+[i\mu{}^{(2)}E^j_i(T^2)+\lambda((n-j){}^{(3)}E^{j+2}_{i+1}(T^2)+m{}^{(3)}E^{j+2}_{i+2}(T^2))] \\ \quad +\sum_{k=0}^{i-1}[((m+n-j)\lambda+i\mu)({}^{(1)}L^j_{ik}{}^{(1)}E^{j+1}_{k+1}(T^2)-2{}^{(1)}D^j_{ik}{}^{(1)}T^{j+1}_{k+1})-2{}^{(1)}L^j_{ik} \\ \quad \times{}^{(1)}T^{j+1}_{k+1}]\}/[(m+n-j)\lambda+i\mu](1-{}^{(1)}L^j_{ii}), \ (j=i,i+1,\cdots,n-1;i=1,2,\cdots,n-1), \\ {}^{(1)}V^{j+1}_1={}^{(3)}V^{j+1}_1={}^{(3)}E^{j+1}_{i+1}(T^2)-({}^{(3)}T^{j+1}_{i+1})^2, \qquad (j=0,1,\cdots,n), \\ {}^{(1)}V^{m+1}_1={}^{(2)}V^{m+1}_{i+1}={}^{(2)}E^{j+1}_{i+1}(T^2)-({}^{(2)}T^{j+1}_{i+1})^2, \qquad (i=0,1,\cdots,n), \end{cases}$$

where ${}^{(1)}L^j_{ik}$, ${}^{(1)}T^j_i$, ${}^{(2)}E^j_i$, ${}^{(3)}E^j_i$, *and* ${}^{(1)}D^j_{ik}$ *are given as* (4.22) (4.24), *and* (4.26), *when* (4.18) *holds true.*

EXAMPLE 3. *For the case* $m=n=1$, $i=1$, *since* $\lambda_0=2\lambda$, $\lambda^1=\lambda$, $\mu^1=\mu$, *and* $p^{01}_{00}=p^{01}_{01}=\frac{1}{2}$, *we have*

$${}^{(1)}\varphi^1_1(s)={}^{(2)}\varphi^1_1(s)=\frac{(s+2\lambda)(s+\lambda+\mu)}{(s+\lambda)^2(s+2\lambda+\mu)}$$

which leads to that

$${}^{(1)}R^1_1(t)={}^{(2)}R^1_1(t)=e^{-\lambda t}+\frac{\lambda\mu}{\lambda+\mu}te^{-\lambda t}+\frac{\lambda\mu}{(\lambda+\mu)^2}e^{-\lambda t}\{e^{-(\lambda+\mu)t}-1\}$$

$${}^{(1)}T^2_2={}^{(2)}T^2_2=\frac{2(\lambda+\mu)}{\lambda(2\lambda+\mu)}, \quad \triangle T_{12}=\frac{2(\lambda+\mu)}{(2\lambda+\mu)}$$

$${}^{(1)}V^2_2={}^{(2)}V^2_2=\frac{4}{(2\lambda+\mu)^2}+\frac{10\mu}{\lambda(2\lambda+\mu)^2}+\frac{2\mu^2}{\lambda^2(2\lambda+\mu)^2}$$

EXAMPLE 4. *For hte case* $m=n=1$, $i=0$, *since* $\lambda_0=2\lambda$, $\lambda_1=\lambda$, $\mu_1=\mu$, *and* $p_{00}^{01}=p_{01}^{01}=\frac{1}{2}$, *we get*

$$^{(1)}\varphi_0^2(s)={}^{(3)}\varphi_0^2(s)=\frac{s^2+(4\lambda+\mu)s+\lambda(3\lambda+2\mu)}{(s+\lambda)(s+2\lambda+\mu)},$$

then

$$^{(1)}R_0^2(t)={}^{(3)}R_0^2(t)=e^{-\lambda t}+\frac{\lambda\mu}{\lambda+\mu}te^{-\lambda t}+\frac{\lambda^2}{(\lambda+\mu)^2}e^{-\lambda t}\{1-e^{-(\lambda+\mu)t}\},$$

$$^{(1)}T_1^2={}^{(3)}T_1^1=\frac{3\lambda+2\mu}{\lambda(2\lambda+\mu)},\quad \triangle T_{12}=\frac{3\lambda+\mu}{2\lambda+\mu},$$

$$^{(1)}V_1^2={}^{(3)}V_1^1=\frac{5}{(2\lambda+\mu)^2}+\frac{8\mu}{\lambda(2\lambda+\mu)^2}+\frac{2\mu^2}{\lambda^2(2\lambda+\mu)^2}.$$

§ 5. Multidimentional allocation process.

We first calculate the reliability function that will be needed in the analysis in this section. In the system stated in § 1, we assume that the failure is not detected when an equipment failed and therefore it will never be repaired if it fails. The reliability function $R_i(t)$ can be obtain by the same method in § 3. After some simple calculation, we get

$$(5.1)\qquad R_i(t)=\sum_{k=i}^{n}A_k e^{-\lambda_k t},$$

where

$$A_k=\frac{\prod_{l=i,\neq k}^{n}\lambda_l}{\prod_{l=i}^{n}(\lambda_l-\lambda_k)},\qquad (i=0,1,\cdots,n).$$

Specially when $\lambda_k=(m+n-k)\lambda$ holds, then

$$(5.2)\qquad R_i(t)=\sum_{k=i}^{n}(-1)^{n-k}\frac{(m+n)!}{(n-k)!\,k!\,(m+n-k)(m-1)!}e^{-(m+n-k)\lambda t}$$

Consider a multi-stage electronic system in which the reliability may be taken to be product of the reliabilities of the each stage and jth stage consist of identical m_j equipments. To improve the reliability of a particular stage, we can put a number of units in parallel or stand-by. Suppose that we have a choice of types of equipments to be used at each stage. We could, if we wished so, allow combinations of types of items at each stage, without affecting the validity of the following treatment. Let $R_j(m_j, n_j, l, t)$ be the reliability of the jth stage when n_j units of l type are put in parallel or stand-by at the jth stage and let the quantity $R_j(m_j, n_j, 1, t)$, $R_j(m_j, n_j, 2, t)$, and $R_j(m_j, n_j, 3, t)$ are given by (5.1), (3.30), and theorem 4.1, respectively, where m_j and t are fixed paramer and $m_j=0,1,\cdots$; $n_j=0,1,\cdots$; $j=1,2,\cdots,n$; $l=1,2,3$. We shall further suppose that we have two types of constraints, cost and weight. Let $c_j(l)$, $w_j(l)(l=1,2,3)$

denote the unit cost and weight for l type equipment at jth stage.

Given the overall restriction on weight of w and cost of c, we wish, as before, to determine which type of and what quantity of equipment we should use at each stage, so as to maximize the reliability of the devices. We can now introduce a Laglange multiplier and proceed as to obtain a one-dimentional version. Let us $f_N(c)$ denotes the maximum reliability of an N-stage device where c is the upperbound of cost admited to expence over the system. Then we have the recurrence relation from the principle of optimality.

(5.3) $$f_N(c)=\max_{l}\left\{\max_{0\leq n_N\leq [c/c_N(l)]}[R_N(m_N, n_N, l, t)e^{-\lambda n_N w_N(l)}f_{N-1}(c-n_N c_N(l))]\right\},$$

$N=2, 3, \cdots$ with

$$f_1(c)=\max_{l}\left\{\max_{0\leq n_1<[c/c_1(l)]}[R_1(m_1, n_1, l, t)e^{-\lambda n_1 w_1(l)}]\right\}.$$

where λ is a Laglange multiplier, to be determined so that the constrains of the weight is met. Once a again, let us note that each n_j and l are constrained to assume only the values $0, 1, 2, \cdots$ and $1, 2, 3$, respectively.

§ 6. **Acknowledgement.** The auther wishes to express his thanks to Prof. T. Kitagawa, Dr. A. Kudô, and Mr. N. Furukawa for their encourgement and suggestions.

Reference

[1] CHUNG, K. L., Probabilistic Methods in Markov Chains. Proc. Fourth Berkeley Symposium on Math. Statist. and Probability, Vol. 11, p35–56. University of California Press. 1961.

[2] BELLMAN, R. Dynamic Programming, Princeton, N. J. Princeton Univ. Press, 1957.

[3] BELLMAN, R. and DREYFUS, S. Applied Dynamic Programming, Princeton, N. J. Princeton Univ. Press, 1962.

[4] KARLIN, S., and J. MCGREGOR. The Differential Equation of Birth-and-Death Processes and the Stieltjes Moment Problem, Trans. Amer. Math. Soc., (1957), 85, 489–546.

[5] KARLIN, S., and J. MCGREGOR. The Classification of Birth-and-Death Processes, Trans. Amer. Math. Soc., (1957), 86, 366–400.

[6] BAZORSKY, I. Reliability Theory and Practice. Printive Hall. (1961).

[7] KODAMA, M. A Problem on a Reliability of the System with Redundancy. Kumamoto J. Sci. A. Vol. 6, No. 2, 1964.