Kumamoto J. Sci., Ser. A, Vol. 6, Nos. 3/4, pp. 130–134 (1694)

ON SOME KIND OF SET-VALUED TRANSFORMATION

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(Received October 5, 1964)

1. In this note we shall be concerned with the correspondences between a complex manifold and the set of linear submanifolds of a complex projective space, analytically defined, more general than usual holomorphic mappings. A primitive form of such correspondences will be found in Hopf's *σ-process*, though it is covered by the notion of meromorphic mapping already introduced. Hopf's *σ-process* of the complex number space C^n at the origin is constructed by the aid of the equations: $z_i p_j - z_j p_i = 0$, $1 \leqq i, j \leqq n$, where $z_1, z_2, \cdots, z_n$ are the coordinates of C^n, and $p_1, p_2, \cdots, p_n$ are the homogeneous coordinates of $(n-1)$-dimensional complex projective space P^{n-1}. These equations are regarded to define a transformation of C^n to P^{n-1}, the values of which are the linear submanifolds of P^{n-1}. By generalizing the coefficients of the equations above to general holomorphic functions we are lead to a transformation of C^n to P^{n-1} the values of which are the linear submanifolds of not neccesarily equal dimension, of P^{n-1}. The purpose of this note is to characterize such a transformation by usual holomorphic or meromorphic mapping with values in some complex Grassmann manifold.

By *meromorphic mapping* of a complex manifold X to another complex manifold Y, we mean a "meromorphic mapping" in the sense of REMMERT. Then the graph G of a meromorphic mapping is a proper modification of X. Through this note we assume the knowledge of the construction and elementary properties of complex Grassmann manifold, see [5].

2. Let X be a complex manifold and τ be a transformation of X to the set of linear submanifolds of n-dimensional complex projective space P^n with homogeneous coordinates $(p_0, p_1, \cdots, p_n)$. Then τ will be given by a system of equations:

$$(1) \qquad \sum_{j=0}^{n} a_{ij}(x) p_j = 0, \ (1 \leq i \leq l),$$

where $a_{ij}(x)$'s are the functions defined in X. The system of functions $\{a_{ij}(x)\}$ is not neccesarily unique with respect to τ. But the rank of the matrix $(a_{ij}(x))$ is uniquely determined at every point of X. The quantity $n - rank(a_{ij}(x))$ is denoted by $d_\tau(x)$ and called the *dimension of* τ *at* x. $d_\tau = inf_{x \in X} d_\tau(x)$ is called the minimal dimension of τ, or in short the *dimension of* τ. By definition $d_\tau(x)$ is the dimension of the submanifold $\tau(x)$; $d(x) \leqq n$.

The transformation τ is said to be *meromorphic at* $x \in X$, if there exists some choice of the matrix of (1) such that all the coefficients $a_{ij}(x)$'s are holomorphic in some neighborhood of x. τ is said to be *holomorphic at* x especially when τ is meromorphic and the rank of the matrix $(a_{ij}(x))$ is constant in some neighborhood of x. τ is said to be holomorphic (meromorphic) *in* X, if τ is holomorphic (meromorphic) at every point of X.

First we consider the holomorphic case. The general (meromorphic) case will be treated in **3**.

The matrix function $(a_{ij}(x))$ needs not to be defined in X. Since the meromorphy of τ is locally defined, it is sufficient to give locally defined $(a_{ij}(x))$. Once a $(a_{ij}(x))$ is given, there are various $(a_{ij}(x))$'s that are equivalent to the original $(a_{ij}(x))$. Then each $(a_{ij}(x))$ from those equivalent ones is called a *local representation* (of the matrix of the equations defining τ). Through this note a transformation denoted by *k-dimensional meromorphic transformation* is a set-valued transformation defined above. For simplicity the space appearing is assumed to be connected, and therefore $d_\tau(x)$ is constant.

Assume: $d_\tau = inf_{x\in X} d_\tau(x) = k$. Then, by definition the *rank* $(a_{ij}(x)) = n-k$ and is independent of the local representation $(a_{ij}(x))$ of τ. Now we shall proceed with a fixed local representation $(a_{ij}(x))$. Since $d_\tau(x) = d_\tau = k$, there exist a $(n-k)$-minor $(a_{i_\alpha j_\beta}(x))$ of $(a_{ij}(x))$. For simplicity we change indeces i_α, j_β to α, β, and assume both α and β varie from 0 to $n-k-1$: $0 \leq \alpha, \beta \leq n-k-1$. Then the equation (1) is solved with respect to $p_0, p_1, \cdots, p_{n-k-1}$:

$$(2) \qquad p_j = \sum_{i=n-k}^{n} b_{ji}(x)\, p_i, \qquad (0 \leq j \leq n-k-1).$$

Thus the linear submanifold $\tau(x)$ is regarded to be the orbit of the point $(p_0, p_1, \cdots, p_n)$ given by

$$(3) \qquad \begin{cases} p_j = \sum_{i=0}^{k} b_{j\, n-k+i}(x)\, u_i, & (0 \leq j \leq n-k-1), \\ p_j = \sum_{i=0}^{k} \delta_{j\, n-k+i}(x)\, u_i, & (n-k \leq j \leq n), \end{cases}$$

where u_i's $(0 \leq i \leq k)$ varie independently in C^{k+1} and $\delta_{j\, n-k+i}$ is the KRONECKER's symbol.

3. In this section we shall construct a holomorphic mapping φ, that characterizes τ, of X to the complex Grassmann manifold $H(k, n)$ which represents the set *of* k-dimensional linear submanifolds of n-dimensional complex projective space p^n.

Since $u_0, u_1, \cdots, u_k$ in (3) are independent, we can choose $k+1$ independent vectors $(u_0^\alpha, u_1^\alpha, \cdots, u_k^\alpha)$, $0 \leq \alpha \leq k$. Through (3) these vectors give rise to another system of $k+1$ independent vectors $(p_0^\alpha, p_1^\alpha, \cdots, p_n^\alpha)$, $0 \leq \alpha \leq k$. Then the system of quantities:

$$(4) \qquad P_{\lambda_0\lambda_1\cdots\lambda_k} = \begin{vmatrix} p_{\lambda_0}^0 & p_{\lambda_1}^0 & & p_{\lambda_k}^0 \\ p_{\lambda_0}^1 & p_{\lambda_1}^1 & & p_{\lambda_k}^1 \\ \vdots & & & \\ p_{\lambda_0}^k & p_{\lambda_1}^k & & p_{\lambda_k}^k \end{vmatrix}, \quad 0 \leq \lambda_0, \lambda_1, \cdots, \lambda_k \leq n,$$

gives the Plücker coordinates of the k-dimensional linear submanifold $\tau(x)$ of P^n. As is easily verified, the Plücker coordinates (4) are independent of special choice of the system of independent vectors $(u_0^\alpha, u_1^\alpha, \cdots, u_k^\alpha)$, and are therefore denoted by $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$.

By the definition of τ, the matrix function $(a_{ij}(x))$ of the local representation (1) is holomorphic in an open set U. From now on we proceed with fixed U. As the first step we show the

Proposition 1. *The Plücker coordinates $P_{\lambda_0\lambda_1\cdots\lambda_k}$ of the image $\tau(x)$ are independent of the special choice of the local representation* (1) *of τ.*

For the proof let $(a'_{ij}(x))$ and $(a''_{ij}(x))$ be different two matrices providing the local representation of τ. Since these matrices represent the same transformation τ, $\sum_{j=0}^{n} a_{ij}(x) p_j$, $0 \leq i \leq l'$ and $\sum_{j=0}^{n} a_{ij}(x) p_j$, $0 \leq i \leq l''$ generate the same ideal associated to the linear manifold $\tau(x)$, and therefore the matrices $A'=(a'_{ij}(x))$ and $A''=(a''_{ij}(x))$ are related with each other by two holomorphic matrices $M=(m_{\mu\nu}(x))$ and $N=(n_{\zeta\xi}(x))$: $A'=MA''$ and $A''=NA'$. It is easily verified that the matrices M and N induce a nonsingular linear transformation carrying the system of independent vectors $(p_0^\alpha, p_1^\alpha, \cdots, p_n^\alpha)$, $0 \leq \alpha \leq k$, constructed by the aid of the local representation A' to another system of independent vectors constructed by the aid of another A''. On the other hand the Plücker coordinates $P_{\lambda_0\lambda_1\cdots\lambda_k}$ are independent of the special choice of the system of mutually independent $k+1$ vectors $(p_0^\alpha, p_1^\alpha, \cdots, p_k^\alpha)$ as is mentioned above. Thus the proof is completed.

Since $a_{ij}(x)$'s are all holomorphic, $P_{\lambda_0\lambda_1\cdots\lambda_k}$'s, which may be denoted $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ by **Prop. 1**, are all holomorphic in U, and by definition $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ do not vanish simultaneously. Hence the Plücker coordinates $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$, $0 \leq \lambda_0, \lambda_1, \cdots, \lambda_k \leq n$, define a holomorphic mapping of U to the Grassmann manifold $H(k,n)$. Denote it φ_U. To every point X there exists a neighborhood U furnished with φ_U constructed above. From the collection of these φ_U we can construct a mapping φ of X to $H\ (k,n)$. Thus we have the

Proposition 2. *Let X be a complex manifold and τ be a k-dimensional holomorphic transformation of X to the n-dimensional complex projective space P^n. Then a holomorphic mapping φ of X to the Grassmann manifold $H(k,n)$ is canonically constructed from τ.*

The converse of **Prop. 2** is stated as follows:

Proposition 3. *Let X be a complex manifold and φ be a holomolphic mapping of X to the Grassmann manifold $H(k,n)$. Then, a k-dimensional holomorphic transformation τ of X to n-dimensional complex projective space P^n is canonically constructed from φ.*

Let P^N be the ambiant space of $H(k,n)$, $N=\binom{n+1}{k+1}-1$. As the homogeneous coordinates of P^N we can take the system of numbers $P_{\lambda_0\lambda_1\cdots\lambda_k}$, $\{\lambda_0, \lambda_1, \cdots, \lambda_k\} \subset \{1,2,\cdots,n\}$ which is alternate in it indices. Then $H(k,n)$ is defined by

$$\sum_{i=0}^{k+1} (-1)^i P_{\mu_0\mu_1\cdots\mu_{k-1}\lambda_i} P_{\lambda_0\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_{k+1}} = 0$$

To every point of X there exists an open neighborhood U in which φ is represented by $\binom{n+1}{k+1}$ holomorphic functions $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$. Then a fundamental property of $H(k,n)$ asserts that the relations:

$$\sum_{i=0}^{k+1} (-1)^i p_{\lambda_i} P_{\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_{k+1}}(x) = 0, \tag{5}$$

$(\lambda_0, \lambda_1, \cdots, \lambda^n)$ varies in the set of combinations of $(1,2,3,\cdots,n)$, induces a k-dimensional transformation τ_U of U. It is verified that τ_U is holomorphic. From the collection of τ_U we construct a k-dimensional holomorphic transformation τ of X to P^n. In other words, a local representation of τ is given by (5).

Combining **Prop. 1** with **Prop. 2** we obtain the

Theorem I. *A k-dimensional holomorphic transformation of a complex manifold X to the n-dimensional complex projective space P^n is represented by a holomorphic mapping of X to the Grassmann manifold $H(k,n)$.*

4. In the following we consider the case of k-dimensional meromorphic transformation with $d_\tau(x)$ not neccesarily constant. First we shall show the

Proposition 4. *Let τ be a k-dimensional meromorphic transformation of X to P^n. Then a meromorphic mapping φ of X to $H(k,n)$ is canonically constructed from τ.*

Put $N=\{x\in X: d_\tau(x)>n\}$. Then by definition N is a proper analytic subset of X. $d^\tau(x)$ is equal to the constant k in $X-N$, and therefore τ is holomorphic in $X-N$. By **Th. I** a holomorphic mapping φ of $X-N$ to $H(k,n)$ is canonically constructed from $\tau|X-N$.

Now let U be any open set such that $U\cap N\neq\phi$, in which a local representation $(a_{ij}(x))$ of τ exists. Let us denote by $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ the functions defining φ in $U-N$, constructed from $(a_{ij}(x))$ as in **2**. Then, as is seen in **2**, $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$'s are all rationally related with the functions $a_{ij}(x)$. Since $a_{ij}(x)$'s are holomorphic in U, $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ are all meromorphically extended to U. It is almost obvious that thus $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$'s define a meromorphic mapping of U to $H(k,n)$. Thus φ is meromorphically extended from $X-N$ to X.

We are able to prove a proposition analogous to **Prop. 3** as follows:

Proposition 5. *Let φ be a meromorphic mapping of a complex manifold X to the Grassmann manifold $H(k,n)$. Then a k-dimensional meromorphic transformation τ of X to the n-dimensional complex projective space P^n is canonically constructed from φ.*

Since φ is a meromorphic mapping of X to $H(k,n)$, φ is regarded to be a meromorphic mapping of X to the N-dimensional projective space P^N which is the ambiant space of $H(k,n)$, $N=\binom{n+1}{k+1}-1$. Let $\widetilde{X}$ be the graph of φ. Then the tri-tuple $(\widetilde{X},\pi,X)$ is a proper modification of X, where π is the natural projection of $\widetilde{X}$ in $X\times P^N$ to X. If U is an arbitrary relatively compact holomorphically complete open subset of X such that $\widetilde{U}=\tau^{-1}(U)$ is an analytic subset of $U\times P^N$, by a theorem stated in [3], the structure of U is given as follows: there exists a finite number of homogeneous polynomials $P_1(x,p_0,p_1,\cdots,p_N)$, $\cdots,P_m(x,p_0,p_1,\cdots,p_N)$ with the coe fficients holomorphic in U such that $\widetilde{U}$ is given by $\widetilde{U}=\{(x,p): P_1(x,p)=\cdots=P_N(x,p)=0\}$. There exists an analytic set $\widetilde{N}$ in $\widetilde{U}$ such that π is biholomorphic in $\widetilde{U}-\widetilde{N}$, $\pi(\widetilde{U}-\widetilde{N})$ is dense in U, and, by meromorphiy of φ, $\pi(\widetilde{N})$ is analytic in U; the tri-tuple $(\widetilde{U},\pi,U)$ is a proper modification of U. Since π is biholomorphic in the open dense subset $\widetilde{U}-\widetilde{N}$, all polynomials $P_1(x,p),\cdots,P_N(x,p)$ must be linear. Hence the singularity of φ, if it appears, is induced only by simultaneous vanishing of the holomorphic coefficients of $P_i(x,p)$'s. Since φ is a mapping of X, hence of U, to $H(k,n)$, we can adopt the notation $P_{\lambda_0\lambda_1\cdots\lambda_k}$ instead of $p_0,p_1,\cdots,p_N$ as before. At last as the solutions of the equations: $P_1(x,p)=P_2(x,p)=\cdots=P_m(x,p)=0$, we obtain the functions $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ holomorphic in $U-N$, and meromorphic in $U, N=\pi(\widetilde{N})$.

Now, we construct the transformation τ in the present **Prop.**. $\varphi\circ\pi$ is holomorphic in $\widetilde{U}$. By **Th. I** we can construct a k-dimensional holomorphic transformation $\tilde{\tau}_{\dot{U}}$ of U to P^n.

A local representation of $\tau_U = (\tilde{\tau}_U | \widetilde{U} - \widetilde{N}) \circ \pi_{-1}$ in some neighborhood of a point of $U-N$ is constructed from the meromorphic functions $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$ through (5):

$$(6) \qquad \sum_{i=0}^{k+1} (-1)^i p_{\lambda_i} P_{\lambda_0\cdots\hat{\lambda}_i\cdots\lambda_{k+1}}(x) = 0.$$

Let us denote by $\Delta(x)$ the least common multiple of the denominators of $P_{\lambda_0\lambda_1\cdots\lambda_k}(x)$. Then

$$(7) \qquad \sum_{i=0}^{k+1} (-1)_i p_{\lambda_i} \Delta(x) P_{\lambda_0\cdots\lambda_i\cdots\lambda_{k+1}}(x) = 0$$

is again a local representation of τ_U in $U-N$, and defines a k-dimensional meromorphic transformation of U to P^n. In other words τ_U is extended from $U-N$ to U. Since U was arbitrarily chosen, we can construct a k-dimensional meromorphic transformation τ of X to P^n from the collection of τ^U. Thus **Prop. 5** is proved.

Summarizing the **Prop. 4** and **5** we obtain

Theorem II. *A k-dimensional meromorphic transformation of a complex manifold X to the complex projective space P^n is canonically represented by a meromorphic mapping of X to the complex Grassmann manifold $H(k, n)$.*

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