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ON WEAK COMPACTNESS AND REFLEXIVITY OF SEPARABLE BANACH SPACES

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1. Preliminary. In his paper [1][(1)], V. Klee has presented a conjecture on weak compactness of subsets of separable Banach spaces: a closed convex subset C of a separable Banach space E is weakly compact if and only if every continuous linear functional on E attains a maximum on C. Hereafter, if a set C shares the latter property, we say that the set C has the property (M) in E. The purpose of the present paper is to give a partial answer of the conjecture under an additional condition, that is, to show that the conjecture holds in case of C being a closed convex body.

In this paper, [3] should be consulted with as far as terminologies are concerned.

2. We begin with the next lemmas.

LEMMA 1. *A weakly closed and weakly bounded subset C of a reflexive Banach space B is weakly compact.*

PROOF. Let (x_α) be a universal net in C. Weak boundedness of C implies its boundedness and hence (x_α) is bounded. Consequently, a family (x_α) is equicontinuous as a subset of the bidual B^{**} of B. Furthermore, there exists $\lim_\alpha f(x_\alpha)$ for each $f \in B^*$ because of weak boundedness of C. We consider a functional $\bar{x}$ on B^* such as $\bar{x}(f) = \lim_\alpha f(x_\alpha)$ for each $f \in B^*$. Clearly, $\bar{x}$ is a continuous linear functional on B^* as a limit functional of a net (x_α) of equicontinuous linear functionals on B^*. Therefore, x belongs to B^{**}, consequently to B because of the reflexivity of B. Thus the proof is completed.

LEMMA 2. *Suppose D is a bounded, closed, convex, symmetric and absorbing subset of a Banach space. Then, the Minkowski functional p of D is a norm with respect to which the set D is the closed unit ball: $D = \{x : p(x) \leqq 1\}$.*

PROOF. The Minkowski functional p of D is clearly a pseudo-norm because D being convex symmetric and absorbing. Now, Let $p(x)$ be equal to 0 and k be an arbitrary positive number. Then, it is straightforward from the definition of the Minkowski functional that there exists a positive number r such that $k^{-1} > r > 0$ and x/r belongs to D. Therefore, there exists a constant M such that $M \geqq \|x\|/r \geqq k \cdot \|x\|$, since the set D is bounded, where $\|x\|$ is the norm of a point x in the Banach space B. The number k may be arbitrarily large. Consequently, an equality $x = 0$ holds. Thus p is a norm. It follows immediately from the definition of p that D is the closed unit ball with respect to the norm p. ([4], p.

(1) Numbers in brackets refer to the references at the end of the paper.

112).

The main result in the present paper is the following.

THEOREM. *If a closed convex body C in a separable Banach space B has the property* (*M*) *in B, then the space B is reflexive and C is weakly compact. In this case, every closed convex subset which has the property* (*M*) *in B, is weakly compact.*

PROOF. Suppose a subset C of B is a closed convex body which has the property (M) in B. Then, it follows that every continuous linear functional f on B attains a minimum on C, since $-f$ attains a maximum on C, and consequently, attains both maximum and minimum on the closed symmetric convex envelope D of C. This fact implies weak boundedness and hence boundedness of D. Since C is a convex body, D is also a convex body and furthermore it has the origin 0 as an interior point by virtue of symmetricity and convexity, and hence is absorbing. By lemma 2, it follows that the Minkowski functional p of the set D is a norm and D is the closed unit ball with respect to the norm p.

Now, the set D is a bounded neighbourhood of 0. It follows that there exists closed balls S_1 and S_2 with center 0 such that $S_1 \subset D \subset S_2$. This fact implies that the original topology of B is identical with the topology generated by the norm p, consequently the normed space with the norm p, denoted by B_0, is homeomorphic with the original Banach space B and B_0 becomes a separable Banach space. Therefore, by virtue of the James's theorem ([2], p. 167), the space B_0 is reflexive and hence the original space B is also reflexive. Furthermore, the set D is weakly compact in B_0 and hence weakly compact in B, since B_0 is homeomorphic with B. The set C is a closed convex subset of B and hence weakly closed subset of D. It follows that C is a weakly compact subset of B. The latter half of the theorem is an immediate consequence of lemma 1.

COROLLARY 1. *If a weakly closed subset C with non-empty interior of a separable Banach space B has the property* (*M*) *in B, then it is weakly compact.*

PROOF. Clearly, the closed convex envelope of C has the property (M) in B and hence is weakly compact by the theorem. Consequently, the set C is weakly compact because it is a weakly closed subset of D.

COROLLARY 2. *Let C be a weakly closed subset of a separable Banach space B and let the interior of C with respect to the closed linear extension of C be non-empty. If the set C has the property* (*M*) *in B, then it is weakly compact.*

PROOF. Let E denotes the closed linear extension of C. The separability and completeness of B is inherited by the subspace E ([4], p. 6) and hence E is a separable Banach space. Every continuous linear functional $\bar{f}$ on E has a linear continuous extension f on B which attains a maximum on C by the assumption. Therefore, $\bar{f}$ attains a maximum on C. Thus C has the property (M) in E. Clearly, the set C is weakly closed in E, since it is weakly closed in B. It follows from corollary 1 that C is weakly compact in E and hence weakly compact in B.

COROLLARY 3. *Suppose C is a weakly closed subset of a separable Banach space B and suppose a set D is the closed symmetric convex envelope of C and* $\inf\{\|x\| : x \in ex_{\{0\}}D - \{0\}\} > 0$, *where* $ex_{\{0\}}D$ *is the set of all extreme point in D relative to* $\{0\}$. *If the set C has the property* (*M*) *in B, then it is weakly compact.*

PROOF. Let E denotes the linear extension of C. We shall show that E is a closed linear subspace of B. Now, let x be a point of E which is different from 0. Then, it has the linear form $\sum_1^n \alpha_i y_i$ of points y_i of C and it may be assumed that α_i's are positive numbers and y_i's belong to D without loss of generality. Put $r_1 = \sum_1^n \beta_i$ and $z = r_1^{-1} \sum_1^n \beta_i y_i$. Then, the point z belongs to D and x is represented by a multiple $r_1 z$ of z. Since D is bounded, $\sup\{r' > 0 : r'z \in D\}$, denoted by $r_0 \neq 0$, is finite. The point $r_0 z$, denoted by y, belongs to D, since D is a closed set. Furthermore, it belongs to $ex_{\{0\}}D - \{0\}$ because D is convex. Thus, any point of E is a multiple ry of a point y of $ex_{\{0\}}D - \{0\}$. Now, let (x_α) be a net in E converging to a point x of B different from 0, and let x_α be $r_\alpha y_\alpha$, where y_α's belong to $ex_{\{0\}}D - \{0\}$. By boundedness of D and the assumption, there exists a pair of positive numbers ε and M such that $0 < \varepsilon < \|y_\alpha\| < M$ for all α. Since $\{x_\alpha\}$ converges to $x \neq 0$, we assume without loss of generality that there exists a pair of positive numbers η and K such that, for every α, $0 < \eta < \|x_\alpha\| = r_\alpha \cdot \|y_\alpha\| < K$. This implies that $\eta \cdot M^{-1} < r_\alpha < K \cdot \varepsilon^{-1}$, for every α. Let (r_β) be a universal subnet of (r_α). The universal net (r_β) of bounded scalars has a limit $r \neq 0$ and hence the net $(r_\beta^{-1} x_\beta)$ converges to $r^{-1}x$. In other words, the net (y_β) converges to y. Since D is closed and $y_\beta \in D$, the limit point y belongs to D and hence x belongs to E. Thus E is a closed linear subspace of B. Furthermore, it has been verified that D is absorbing and hence a barrel in E. Therefore, the set D is a neighbourhood of 0 in E and has a non-empty interior because E is a Banach space. It follows from corollary 2 that D is weakly compact and so is the set C.

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References

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