Kumamoto J. Sci., Ser. A, Vol. 7, No. 2, pp. 21-27 (1965)

ON INFINITESIMAL TRANSFORMATIONS OF CERTAIN ALMOST-TACHIBANA SPACE

Masanori KŌZAKI and Takeo OHKUBO

Kurume Technical College (M. K) and Department of Mathematics, Faculty of Science, Kumamoto University (T. O)
(Received September 30, 1965)

§ 0. Introduction.

S. Tachibana[1] proved that in an n-dimensional compact almost-Tachibana space with vanishing $\hat{K}$ an infinitesimal conformal (or projective) transformation is necessarily an isometry. On the other hand, S. Tachibana[2] and H. Nakagawa[3] discussed automorphisms of almost-Tachibana spaces under certain conditions.

In this paper, we shall consider an infinitesimal conformal (or projective or holomorphically projective) transformation of an n-dimensional compact almost-Tachibana space satisfying the equation $R^{*}_{ji}=aR_{ji}+b\,g_{ji}$ (a, b ; scalars or constants) and deduce Tachibana's results and that in an n-dimensional compact almost-Tachibana space of non-positive comstant holomorphic curvature an infinitesimal conformal (or projective or holomorphically projective) transformation is necessarily an isometry. Next, we shall discuss an automorphism of our space.

§ 1. Preliminaries.

Let M be an n-dimensional almost-Hermitian space with almost-complex structure φ_i^h and positive definite Riemannian metric g_{ji} satisfying the relations :

(1. 1) $$\varphi_i^r\varphi_r^h=-\delta_i^h,$$

(1. 2) $$g_{rs}\varphi_j^r\varphi_i^s=g_{ji}.$$

From (1.2) it follows that tensors $\varphi_{ji}=\varphi_j^r g_{ri}$ and $\varphi^{ji}=\varphi_r^i g^{rj}$ are skew symmetric. Let $\left\{ {h \atop ji} \right\}$, R_{kji}^h and $R_{ji}=R_{ij}$ be Christoffel symbols, Riemannian curvature tentor and Ricci tensor respectively. We denote be ∇_j the operator of covariant derivative with respect to $\left\{ {h \atop ij} \right\}$. It is well known that with respect to the canonical connection[4)5)] defined by

$$\Gamma_{ji}^h=\left\{ {h \atop ji} \right\}-\frac{1}{2}\varphi_r^h\nabla_j\varphi_i^r,$$

1) S. Tachibana (6), 2) S. Tachibana (7), 3) H. Nakagawa (1) 4) S. Tachibana (5), 5) S. Tachibana (6),

tensors g_{ji} and φ_i^h are covariantly constant. If we denote by $K_{lji}{}^h$ the curvature tensor of Γ_{ji}^h and put

$$\hat{K}_{ji}=2K_{jir}{}^t\varphi_t^r, \quad R^*_{ji}=(1/2)\varphi^{tr}R_{trsi}\varphi_j^s,$$

then we have

$$\hat{K}_{ji}=4\varphi_j^r R^*_{ri}-\varphi_s^t\nabla_j\varphi_r^s\nabla_i\varphi_t^r. \tag{1.3}$$

If an almost-Hermitian space satisfies

$$\nabla_j\varphi_{ih}+\nabla_i\varphi_{jh}=0, \tag{1.4}$$

then it is called an almost-Tachibana space. In an almost-Tachibana space, it is well known that the following equations are valid[6)7)8)],

$$R^*_{rs}\varphi_j^r\varphi_i^s=R^*_{ji}, \quad R^*_{ji}=R^*_{ij}, \tag{1.5}$$

$$(R_{ji}-R^*_{ji})v^jv^i\geqq 0 \text{ for any vector field } v^h, \tag{1.6}$$

$$R-R^*=\text{constant}\geqq 0, \text{ where } R=g^{rs}R_{rs}, \ R^*=g^{rs}R^*_{rs}, \tag{1.7}$$

$$\nabla^h N(v)_h=0 \text{ for any vector field } v^h, \text{ where } N(v)_h=\varphi_h^t(\nabla_t\varphi_{rs})\nabla^r v^s. \tag{1.8}$$

§ 2. Certain almost-Tachibana spaces.

It is known that the followings are valid ;

Lemma 2.1.[9)] *In order that the form $\hat{K}$ vanishes it is necessary and sufficient that* $5R^*_{ji}=R_{ji}$.

Lemma 2.2.[10)] *In an n-dimensional ($n\geqq 4$) almost-Tachibana space of constant holomorphic curvature k, we have*

$$3R^*_{ji}=-R_{ji}+(n+2)kg_{ji} \qquad (k: \text{constant}).$$

It is well known that in an almost-Hermitian space the differential form $\hat{K}=\hat{K}_{ji}dx^j\wedge dx^i$ is closed[11)]. Now, we consider an almost-Tachibana space (every eigen value of $R_{ji}\neq R/n$) satisfying the equation

$$R^*_{ji}=aR_{ji}+bg_{ji} \quad (a, b: \text{scalars}). \tag{2.1}$$

In an almost-Tachibana space, from (1.3) we have

$$\hat{K}_{ji}=\varphi_j^r(4R^*_{ri}-R_{ri}). \tag{2.2}$$

Transvecting (2.2) with φ_k^j, we get

$$\varphi_k^j\hat{K}_{ji}=-5R^*_{ki}+R_{ki}=-5(R^*_{ki}-R_{ki})-4R_{ki},$$

---

6) S. Sawaki (2), 7) S. Sawaki (3), 8) S. Tachibana (4), 9) S. Tachibana (6). 10) H. Nakagawa (1). 11) S. Tachibana (5), this form $\hat{K}$ is a generalization of Chern 2-form of Hermitian spaces.

from which we have

$$\nabla^k(\varphi_k^i \hat{K}_{ji}) = -(5/2)\nabla_i(R^* - R) - 2\nabla_i R.$$

If $\nabla_k \hat{K}_{ji}=0$, taking account of (1.7) and the last equation we get $\nabla_i R=0$. Hence R and R^* are constants.

If we contract (2.1) with g_{ji} and operate ∇^j to it, we have

$$(\nabla^j a)R + (\nabla^j b)n = 0. \tag{2.3}$$

On the other hand, if we operate ∇^j to (2.1), then we have

$$(\nabla^j a)R_{ji} + (\nabla^j b)g_{ji} = 0. \tag{2.4}$$

From (2.3) and (2.4) we get

$$(\nabla^j a)(R_{ji} - (R/n)g_{ji}) = 0,$$

from which by our assumption, we obtain $\nabla^j a=0$ and $\nabla^j b=0$. Hence a and b are constants. Therefore we have the following

Theorem 2.1. *In an almost-Tachibana space (every eigen value of $R_{ji} \neq R/n$) satisfying the relation $R^*_{ji} = aR_{ji} + bg_{ji}$ (a, b : scalars), if $\Delta_k \hat{K}_{ji}=0$, then a and b are constants.*

§ 3. Infinitesimal conformal transformation.

Consider an almost-Tachibana space satisfying the equation $R^*_{ji}=aR_{ji}+bg_{ji}$ (a, b : constants). Then for a vector field v^h we have

$$\nabla^j(R^*_{ji}v^i) = a\nabla^j(R_{ji}v^i) + b\nabla^j v_j. \tag{3.1}$$

Now let v^h be an infinitesimal conformal transformation, then by definition there exists a scalar function ρ satisfying $\underset{v}{\pounds} g_{ji} = 2\rho g_{ji}$ and as is well known the equation

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j \delta_i^h + \rho_i \delta_j^h - \rho^h g_{ji}, \quad \rho_i \equiv \nabla_i \rho \tag{3.2}$$

holds good, where we denote by $\underset{v}{\pounds}$ the operator of Lie derivative with respect to a vector field v^h. From (3.2) we have

$$\rho_i = (1/n)\nabla_i f, \quad f \equiv \nabla_i v^i,$$

and

$$\nabla^r \Delta_r v^h + R_r^h v^r = (2-n)\rho^h. \tag{3.3}$$

On the other hand, transvecting (3.2) with $\varphi_l^j \varphi_h^i$ we get

$$\varphi_l^j \nabla_j(\varphi_h^i \nabla_i v^h) + N(v)_l + 2R^*_{lr}v^r = -2\rho_l. \tag{3.4}$$

If we operate ∇^l to this equation and take account of (2.3), then we obtain

(3. 5) $$\nabla^l(R^{*}_{lr}v^r)=-\nabla^l\rho_l.$$

Hence from (3.1) and (3.5) we have

(3. 6) $$a\nabla^j(R_{ji}v^i)=-\nabla^l\rho_l-bf.$$

Next operating ∇_h to (3.3) we have $\nabla^r\nabla_r f+2\nabla^r(R_{ri}v^i)=(2-n)\nabla^h\rho_h$, from which we get $\Delta f=\lambda f$ $(\lambda=nb/(na-a-1))$ by virtue of (3.6). From the last equation, we have

Theorem 3.1. *In a compact almost-Tachibana space satisfying the relation* $R^{*}_{ji}=aR_{ji}+bg_{ji}$, *if* λ *is non-negative, an infinitesimal conformal transformation is necessarily an isometry.*

Especially, in a compact almost-Tachibana space with vanishing $\hat{K}$ or of constant holomorphic curvature we have

$$\lambda=0 \quad \text{or} \quad \lambda=-nk.$$

Hence we have the following

Corollary 1.[12] *In an* n $(\neq 6)$*-dimensional compact almost-Tachibana space with vanishing* $\hat{K}$*, an infinitesimal conformal transformation is necessarily an isometry.*

Corollary 2. *In an* n*-dimensional compact almost-Tachibana space of non-positive constant holomorphic curvature, an infinitesimal conformal transformation is necessarily an isometry.*

§ 4. Infinitesimal projective transformation.

In the same way we can deal with an infinitesimal projective transformation. Let v^h be such a transformation, then by definition there exists a vector field ρ_i such that

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\}=\rho_j\delta^h_i+\rho_i\delta^h_j.$$

From the above we have

$$\nabla^r\nabla_r v^h+R^h_r v^r=2\rho^h, \quad \rho_i=1/(n+1)\nabla_i f, \quad f\equiv\nabla_r v^r.$$

In an almost-Tachibana space satisfying the relation $R^{*}_{ji}=aR_{ji}+bg_{ji}$ (a, b : constants), we have

$$\Delta f=\lambda f, \quad \lambda=2b(n+1)/(na-a-2).$$

---

12) S. Tachibana (6),

From the last equation, we have the following

Theorem 4.1. *In a compact almost-Tachibana space satisfying the relation* $R^*_{ji} = aR_{ji} + bg_{ji}$, *if* λ *is non-negative, an infinitesimal projective transformation is necessarily an isometry.*

Corollary 1.[13] *In an* $n(\neq 11)$*-dimensional compact almost-Tachibana space with vanishing* $\hat{K}$, *an infinitesimal projective transformation is necessarily an isometry.*

Corollary 2. *In an* n*-dimensional compact almost-Tachibana spaec of non-positive constant holomorphic curvature, an infinitesimal projective transformation is necessarily an isometry.*

§ 5. Infinitesimal holomorphically projeective transformation.

In the same way we can deal with an infinitesimal holomorphically projective transformation (we call such a transformation HP-transformation). Let v^h be HP-transformation, then by definition there exists a vector field ρ_i such that

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j \delta^h_i + \rho_i \delta^h_j - \tilde{\rho}_j \varphi^h_i - \tilde{\rho}_i \varphi^h_j, \quad \tilde{\rho}_j = \varphi^h_j \rho_h.$$

From this equation we have

$$\nabla^r \nabla_r v^h + R^h_r v^r = 0, \quad \rho_i = 1/(n+2) \nabla_i f, \quad f \equiv \nabla^i v_i.$$

In an n-dimensional almost-Tachibana space satisfying the relation $R^*_{ji} = aR_{ji} + bg_{ji}$ (a, b : constants), we have

$$\Delta f = \lambda f, \quad \lambda = 2(n+2)b/(na+2a-2),$$

from which we obtain the following

Theorem 5.1. *In an* n*-dimensional compact almost-Tachibana space satisfying the relation* $R^*_{ji} = aR_{ji} + bg_{ji}$, *if* λ *is non-negative an HP-transformation is necessarily an isometry.*

Corollary 1. *In an* $n(\neq 8)$*-dimensional compact almost-Tachibana space with vanishing* $\hat{K}$, *an HP-transformation is necessarily an isometry.*

Corollary 2. *In an* n*-dimensional compact almost-Tachibana space of non-positive constant holomorphic curvature, an HP-transformation is necessarily an isometry.*

§ 6. Automorphism.

13) S. Tachibana (6).

If an isometry is almost-analytic, it is called an automorphism. In an almost-Tachibana space, if a vector field v^h is almost-analytic then the followings are valid :

$$\nabla^r \nabla_r v_i + R_{ir} v^r = 0, \tag{6.1}$$

$$2N(v)_i = (R^*_{ir} - R_{ir}) v^r. \tag{6.2}$$

In an n-dimensional almost-Tachibana space satisfying the relation $R^*_{ji} = aR_{ji} + bg_{ij}$ (a, b : constants), from (6. 2) we have

$$2N(v)_i = (a-1) R_{ji} v^j + b g_{ji} v^j.$$

If we operate ∇^i to the above equation we have

$$(a-1) R_{ji} \nabla^j v^i + bf = 0, \quad f \equiv \nabla^i v_i. \tag{6.3}$$

On the other hand, from (6.1) we have

$$\nabla^r \nabla_r f + 2R_{ir} \nabla^i v^r = 0. \tag{6.4}$$

From (6.3) and (6.4) we obtain

$$\Delta f = \lambda f, \quad \lambda = 2b/(a-1),$$

and hence we have the following

Theorem 6.1. *In an n-dimensional compact almost-Tachibana space satisfying the relation* $R^*_{ji} = aR_{ji} + bg_{ji}$, *if* λ *is non-negative, an almost-analytic vector is an automorphism.*

From this theorem, we have the following

Corollary 1.[14] *In an n-dimensional compact almost-Tachibana space with vanishing* $\hat{K}$, *an almost-analytic vector is an automorphism.*

Corollary 2.[15] *In an n-dimensional compact almost-Tachibana space of non-positive constant holomorphic curvature, an almost-analytic vector is an automorphism.*

If an $n(>2)$-dimensional almost-Tachibana space is conformally flat, then we have

$$(n-2) R^*_{ji} = 2R_{ji} - R/(n-1) g_{ji}, \quad R = \text{constant} \geqq 0,$$

and hence we have the following

Corollary 3.[16] *In an* $n(>4)$*-dimensional compact conformally flat almost-Tachibana space, an almost-analytic vector is an automorphism.*

---

14) S. Tachibana (5), 15) H. Nakagawa (1), 16) S. Tachibana (7).

Bibliography

(1) H. Nakagawa: On automorpisms of an almost Tachibana space of constant holomorphic curvature. Sci. Rep. Tōkyo Kyoiku Daigaku, Vol. 7, No. 185 (1963) 269–274.

(2) S. Sawaki: On infinitesimal transformations of almost-Kählerian space and K-space. Kōdai Math. Sem. Rep. Vol. 16 (1964) 105–115.

(3) S. Sawaki: On the Matsushima's theorem in a compact Einstein K-space. Tōhoku Math. Jour. Vol. 13 (1961) 455–465.

(4) S. Tachibana: On almost analytic vectors in certain almost Hermitian manifolds. Tōhoku Math. Jour. Vol. 11 (1959) 351–363.

(5) S. Tachibana: On automorphisms of certain compact almost-Hermitian spaces. Tōhoku Math. Jour. Vol. 13 (1961) 179–185.

(6) S. Tachibana: On infinitesimal conformal and projective transformations of compact K-space. Tōhoku Math. Jour. Vol. 13 (1961) 386–392.

(7) S. Tachibana: On automorphisms of conformally flat K-spaces. Jour. Math. Soc. Japan, Vol. 13 (1961) 183–188.