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# A DELIVERY-LAG INVENTORY CONTROL PROCESS WITH EMERGENCY AND NON-STATIONARY STOCHASTIC DEMANDS II

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## §1. Introduction.

In this paper we consider an $n$-period, one-commodity dynamic inventory model with non-stationary stochastic demands, and with $k$ (constant) period-lag delivery of regular orders, and with $k$ "emergency" orders, characterized by delivery lags $\mu_1=0$, $\mu_2=1$, $\cdots$, $\mu_k=k-1$. Let $m^i_{r,n-j+1}(i=1, 2, \cdots, m;\ j=1, 2, \cdots, n;\ r=0, 1, \cdots, k-1$ for $n\geqq k$, $r=0, 1, \cdots, n-1$ for $n<k)$ denote the emergency quantity of the time lag $r$ for the period $j$, at the begining of which the demand density is given by $\varphi_i$ and let $m^i_{k,n-j+1}(n\geqq k)$ denote the regular quantity. The cumlative demand in each period is non-negative random variable whose distribution may change from period to period by a Markov transition law with matrix $P=\|p_{ij}\|$ $(i, j=1, 2, \cdots, m.)$ where $p_{ij}\geqq 0$ and $\sum_{j=1}^{m} p_{ij}=1$ for each $i$. It is assumed that the demand density does not change during one period. In other words, when the demand deneity is $\varphi_i$ during a period, one of the following period changes to $\varphi_j$ with probability $p_{ij}$ $(i, j=1, 2, \cdots, m)$. We shall show simple and more explicit properties for optimal policy, with assumptions a little bit changed, than those discussed previously the general case in [8]

The inventory periods $I_1$, $I_2$, $\cdots$, $I_n$, are numbered from left to right. At the begining of the $j$th period $(j=1, 2, \cdots, n)$ two action have to be taken (i) placing $k$ "emergency" order, (ii) issuing a regular order to be delivered at the end of the $j+k-1$ period. The delivery lag $\lambda=k$ is constant throughout the rest of the paper. We impose the following conditions on the model.

(1.1) The interval in ordering is $k$-period.

(1.2) There is backlogging of excess demand.

(1.3) The known distribution function of demand is absolutely continuous with respect to the Lebesgue measure. The density will be denote by $\varphi_i(\xi)(i=1, 2, \cdots, m)$.

(1.4) The holding cost function $h(\eta)$ and the penalty cost function $p(\eta)$ are twice differentiable, positive convex function for positive arguments.

(1.5) There is credit function $v(\eta)$ defined by

$$v(\eta)=\begin{cases} v\eta & \eta\geqq 0, \\ 0 & \eta<0. \end{cases}$$

The reduced penalty cost, the net penalty cost, is defined in the following way. If at the begining of some period the order of size $z$ to be delivered at the end of the period, has been known and a demand $D=\xi$ occures, then the net penalty cost for this period is

$$p(\xi-y)-v[\min(z,\ \xi-y)]$$

where $y$ is the starting stock level of that period.

(1.6) There is a concave, twice differentiable salvage gain function $w(\eta)$ that is increasing for $\eta>0$, and is zero for $\eta\leqq 0$

(1.7) The ordering cost function $c_k(\eta_k)$ for regular orders to be delivered $k$ period later is given by

$$c_k(\eta_k)=\begin{cases} c_k\eta_k & \eta_k\geqq 0, \\ 0 & \eta_k<0. \end{cases}$$

The ordering cost function $c_j(\eta_j)$ for emergency orders to be delivered $j$ period later is given by

$$c_j(\eta_j)=\begin{cases} c_j\eta_j & \eta_j>0, \\ 0 & \eta_j\leqq 0. \end{cases} \qquad (j=0,\ 1,\ \cdots,\ k-1).$$

with $c_0>c_1>\cdots>c_k>0$.

(1.8) (a) $\alpha^j\lim\limits_{\eta\to\infty} w'(\eta)=\alpha^j\lim\limits_{\eta\to\infty}\int_0^\eta w'(\eta-\xi)\varphi_i(\xi)\,d\xi<c_j<\alpha^{j-1}v.$

(b) $w'(0)<v.$ (c) $0<\alpha\leqq 1,$ $(j=1,\ 2,\ \cdots,\ k;\ i=1,\ 2,\ \cdots,\ m)$

(1.9) $L(\eta,\ \varphi_i)-v\int_0^\eta(\eta-\xi)\varphi_i(\xi)\,d\xi$ is convex.

where $L(\eta,\ \varphi_i)$, the expected one-period loss arising from penalty and holding costs, is given by

$$L(\eta,\ \varphi_i)=\begin{cases} \int_0^\eta h(\eta-\xi)\varphi_i(\xi)\,d\xi+\int_\eta^\infty p(\xi-\eta)\varphi_i(\xi)\,d\xi & \eta>0 \\ \int_0^\infty p(\xi-\eta)\varphi_i(\xi)\,d\xi & \eta\leqq 0, \end{cases} \tag{1.10}$$

$i=1,\ 2,\ \cdots,\ m.$

we shall assume that all integrals occuring in this paper exist and are finite, and all integration and differentiation where needed can be interchanged. This impose certain restrictions on the class of demand densities.

Let $f_n(x, \varphi_i)$ denote the total discounted expected loss an $n$-period inventory model, where the demand density in first period is $\varphi_i$, $x$ is the inventory on hand after the receipt of prior orders at the begining of the first period, and an optimal ordering policy is followed in period 1, 2, …, $n$. We obtain for $n \geqq k$ from the principle of optimalty

(1.11)
$$f_n(x, \varphi_i) = \min_{m_0 \geqq 0, m_1 \geqq 0, \ldots, m_n \geqq 0} \{\sum_{j=0}^{k} c_j m_j + L(x+m_0, \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty L(x+m_0$$
$$+ m_1 - t, \varphi_{i_1}) \varphi_i(t)\, dt + \cdots + \alpha^{k-1} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} p_{ii_1} \cdots p_{i_{k-2}i_{k-1}} \int_0^\infty \cdots \int_0^\infty L(x+m_0$$
$$+ m_1 + \cdots + m_{k-1} - t - t_1 \cdots - t_{k-2}, \varphi_{i_{k-1}}) \varphi_i(t) \cdots \varphi_{i_{k-2}}(t_{k-2})\, dt \cdots dt_{k-2}$$
$$+ [V(x + m_0 + m_1, \varphi_i) - V(x + m_0, \varphi_i)] + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty [V(x + m_0 + m_1 + m_2$$
$$- t, \varphi_{i_1}) - V(x + m_0 + m_1 - t, \varphi_{i_1})] \varphi_i(t)\, dt + \cdots + \alpha^{k-1} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} p_{ii_1}$$
$$\cdots p_{i_{k-2}i_{k-1}} \int_0^\infty \cdots \int_0^\infty [V(x + m_0 + m_1 + \cdots + m_k - t - t_1 - \cdots - t_{k-2}, \varphi_{i_{k-1}})$$
$$- V(x + m_0 + m_1 + \cdots + m_{k-1} - t - t_1 - \cdots - t_{k-2}, \varphi_{i_{k-1}})] \varphi_i(t) \cdots \varphi_{i_{k-2}}(t_{k-2})\, dt$$
$$\cdots dt_{k-2} + \alpha^k \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_k=1}^{m} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{k-1}i_k} \int_0^\infty \cdots \int_0^\infty f_{n-k}(x + m_0 + m_1 + \cdots$$
$$+ m_k - t - t_1 - \cdots - t_{k-1}, \varphi_{i_k}) \varphi_i(t) \varphi_{i_1}(t_1) \cdots \varphi_{i_{k-1}}(t_{k-1})\, dt\, dt_1 \cdots dt_{k-1}\}$$
$$= \min_{u_k \geqq u_{k-1} \geqq \ldots \geqq u_1 \geqq u_0 \geqq x} \{\sum_{j=1}^{k} [(c_{j-1} - c_j) u_{j-1} + L_{j-1}(u_{j-1}, \varphi_i) + V_{j-2}(u_{j-1}, \varphi_i)$$
$$- V_{j-1}(u_{j-1}, \varphi)] + c_k u_k + V_{k-1}(u_k, \varphi_i) + f_{n-k, k}(u_k, \varphi)\} - c_0 x$$
$$i = 1, 2, \ldots, m.$$

where

$$i_0 = i,\ x + m_0 = u_0,\ x + m_0 + m_1 = u_1, \cdots,\ x + m_0 + m_1 + \cdots + m_k = u_k$$

$$L_l(x, \varphi_i) = \alpha \sum_{j=1}^{m} p_{ij} \int_0^\infty L_{l-1}(x - t, \varphi_j), \varphi_i(t)\, dt \qquad i = 1, 2, \ldots, m,$$

$$V_l(x, \varphi_i) = \alpha \sum_{j=1}^{m} p_{ij} \int_0^\infty V_{l-1}(x - t, \varphi_j), \varphi_i(t)\, dt \qquad i = 1, 2, \ldots, m,$$

$$W_l(x, \varphi_i) = \alpha \sum_{j=1}^{m} p_{ij} \int_0^x W_{l-1}(x - t, \varphi_j) \varphi_i(t)\, dt \qquad i = 1, 2, \ldots, m,$$

$$f_{n-k,l}(x, \varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty f_{n-k,l-1}(x-t, \varphi_j)\varphi_i(t)\,dt \qquad i=1, 2, \cdots, m,$$

$$f_{n-k,0}(x, \varphi_i)=f_{n-k}(x, \varphi_i),\ L_{-1}(x, \varphi_i)=V_{-1}(x, \varphi_i)=0 \qquad i=1, 2, \cdots, m,$$

$$L_0(x, \varphi_i)=L(x, \varphi_i),\ V_0(x, \varphi_i)=V(x, \varphi_i) \qquad i=1, 2, \cdots, m,$$

$$f_0(x, \varphi_i)=-W_0(x, \varphi_i)=-w(x) \qquad i=1, 2 \cdots, m,$$

$$V(x, \varphi_i)=\begin{cases}-vx+v\int_0^x (x-\xi)\varphi_i(\xi)\,d\xi & x\geqq 0\\ -vx & x<0\end{cases}$$

$$i=1, 2, \cdots, m.$$

It is noticed that $f_{0k}(x, \varphi_i)=-W_k(x)$ for $x>0$. From the same method as in the $n\geqq k$, we have for $n<k$

$$f_n(x, \varphi_i)=\min_{u_n\geqq u_{n-1}\geqq\cdots\geqq u_1\geqq 0}\{\sum_{j=1}^{n}[(c_{j-1}-c_j)u_{j-1}+L_{j-1}(u_{j-1}, \varphi_i)+V_{j-2}(u_{j-1}, \varphi_i)$$
$$-V_{j-1}(u_{j-1}, \varphi_i)]+c_nu_n+V_{n-1}(u_n, \varphi_i)-W_n(u_n, \varphi_i)\}-c_0x$$
$$i=1, 2, \cdots, m.$$

## §2. Optimal policy under two modes of delivery ($k=1$)

In this section we will analyze the optimal policies of a dynamic inventory model with a one-period-lag delivery of regulars, and with a no-time-lag delivery of emergency orders. We impose the following conditions that guarantee uniqueness and finiteness of critical numbers.

(2.1) (a) $\lim_{x\to-\infty} L'(x, \varphi_i)+c_0-c_1+v<0$ (b) $\lim_{x\to+\infty} L'(x, \varphi_i)-\alpha\lim_{x\to+\infty} w'(x)>0$

(c) $v<p'(0), h(0)=p(0)=0$ (d) $\varphi_i(\xi)>0$ for $\xi>0$ $\quad i=1, 2, \cdots, m.$

From (1.11), we get

$$(2.2)\quad f_n(x, \varphi_i)=\min_{u_1\geqq u_0\geqq x}\{(c_0-c_1)(u_0-x)+L(u_0, \varphi_i)-V(u_0, \varphi_i)+c_1(u_1-x)$$
$$+V(u_1, \varphi_i)+f_{n-1,1}(x, \varphi_i)\} \qquad i=1, 2, \cdots, m.$$

Let us define

$$(2.3)\quad \bar{L}^{(1)}(x, \varphi_i)=\min_{u_0\geqq x}\{(c_0-c_1)(u_0-x)+L(u_0, \varphi_i)-V(u_0, \varphi_i)\}$$
$$i=1, 2, \cdots, m.$$

$$(2.4)\quad M^{(1)}(x, \varphi_i)=c_0-c_1+L'(x, \varphi_i)-V'(x, \varphi_i) \qquad i=1, 2, \cdots, m.$$

It is clear from the conditions of (2.1) that each $M^{(1)}(x, \varphi_i)$ has a unique zero, $\hat{x}_1(\varphi_i)$. If $p(\eta)$ is linear, then we get from (2.1a) $M^{(1)}(0, \varphi_i)=c_0-c_1-p(0)+v<0$; hence $\hat{x}_1(\varphi_i)$ is even positive. It follows that $u_0=\hat{x}_1(\varphi_i)$ for $x<\hat{x}_1(\varphi_i)$ and $u_0=x$ for $x\geqq \hat{x}_1(\varphi_i)$; consequently $\bar{L}^{(1)}(x, \varphi_i)$ is given by

$$(2.5)\qquad \bar{L}^{(1)}(x, \varphi_i)=\begin{cases}(c_0-c_1)(\hat{x}_1(\varphi_i)-x)+L(\hat{x}_1(\varphi_i), \varphi_i)-V(\hat{x}_1(\varphi_i), \varphi_i) & x<\hat{x}_1(\varphi_i)\\ L(x, \varphi_i)-V(x, \varphi_i) & x\geqq \hat{x}_1(\varphi_i)\end{cases}$$

$$i=1, 2, \cdots, m$$

Returning to (2.2) we consider the following cases to accomplish the desired minization.

Case (a) $x\geqq \hat{x}_1(\varphi_i)$. In this case we have $u_0=x$, then it follows from (2.5) that

$$(2.6)\qquad f_n(x, \varphi_i)=L(x, \varphi_i)-V(x, \varphi_i)+\min_{u_1\geqq x}\{c_1(u_1-x)+V(u_1, \varphi_i)+ f_{n-1,1}(u_1, \varphi_i)\}\qquad i=1, 2, \cdots, m.$$

Case (b) $x<\hat{x}_1(\varphi_i)$ Considering the additional restriction, $u_1\geqq u_0$, we have $u_0=\hat{x}_1(\varphi_i)$, if $u_1\geqq \hat{x}_1(\varphi_i)$, if $u_1<\hat{x}_1(\varphi_i)$, $u_0=\hat{x}_1(\varphi_i)$ is contradictory to the restriction $u_1>u_0$; hence from the convexity of $L(x, \varphi_i)$ we must have $u_1=u_0<\hat{x}_1(\varphi_i)$ Therefore, we get

I. If $u_1\geqq \hat{x}(\varphi_i)$.

$$(2.7)\qquad f_n(x, \varphi_i)=(c_0-c_1)(\hat{x}_1(\varphi_i)-x)+L(\hat{x}_1(\varphi_i), \varphi_i)-V(\hat{x}_1(\varphi_i), \varphi_i) + \min_{u_1\geqq x}\{c_1(u_1-x)+V(u_1, \varphi_i)+f_{n-1,1}(u_1, \varphi_i)\}\qquad i=1, 2, \cdots, m$$

II. If $u_1<\hat{x}_1(\varphi_i)$

$$(2.8)\qquad f_n(x, \varphi_i)=\min_{\hat{x}_1(\varphi_i)>u_1\geqq x}\{(c_0-c_1)(u_1-x)+L(u_1, \varphi_i)-V(u_1, \varphi_i)+c_1(u_1-x) + V(u_1, \varphi_i)+f_{n-1,1}(u_1, \varphi_i)\}$$

$$=(c_0-c_1)(\hat{x}_1(\varphi_i)-x)+L(\hat{x}_1(\varphi_i), \varphi_i)-V(\hat{x}_1(\varphi_i), \varphi_i) + \min_{x_1(\varphi_i)>u_1\geqq x}\{c_1(u_1-x)+(c_0-c_1)(u_1-\hat{x}_1(\varphi_i))+L(u_1, \varphi_i)-V(u_1, \varphi_i) -(L(\hat{x}_1(\varphi_i), \varphi_i)-V(\hat{x}_1(\varphi_i), \varphi_i)+V(u_1, \varphi_i) +f_{n-1,1}(u_1, \varphi_i)\}$$

Let us define

$$\Lambda^{(1)}(x, \varphi_i) = \begin{cases} (c_0 - c_1)(x - \hat{x}_1(\varphi_i)) + (L(x, \varphi_i) - V(x, \varphi_i)) \\ \qquad - (L(\hat{x}_1(\varphi_i)\varphi_i) - V(\hat{x}_1(\varphi_i), \varphi_i)) & x < \hat{x}_1(\varphi_i) \\ 0 & x \geqq \hat{x}_1(\varphi_i) \end{cases} \tag{2.9}$$

$i = 1, 2, \cdots, m.$

Using (2.5) and (2.9), we may represent expressions (2.6), (2.7), and (2.8) by a single expression

$$f_n(x, \varphi_i) = \bar{L}^{(1)}(x, \varphi_i) + \min_{u_1 \geqq x} \{c_1(u_1 - x) + \Lambda^{(1)}(u_1, \varphi_i) + V(u_1, \varphi_i) + f_{n-1,1}(u_1, \varphi_i)\} \qquad i = 1, 2, \cdots, m. \tag{2.10}$$

Let us define

$$G_{1n}(u, \varphi_i) = c_1 u + \Lambda^{(1)}(u, \varphi_i) + V(u, \varphi_i) + f_{n-1,1}(u, \varphi_i) \qquad i = 1, 2, \cdots, m. \tag{2.11}$$

At $u = \hat{x}_1(\varphi_i)$, the derivative $G'_{1n}(u, \varphi_i)$ is given by

$$G'_{1n}(\hat{x}_1, (\varphi_i), \varphi_i) = c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) + f'_{n-1,1}(\hat{x}_1(\varphi_i), \varphi_i) \tag{2.12}$$

THEOREM 2.1 *Let conditions* (1.1)~(1.9) *hold, If* (2.1) *are satisfied, then for each* $i$ (i) *the optimal ordering policy is of the following form.*

*Case* 1 *If* $\bar{x}_{1n}(\varphi_i) > \hat{x}_1(\varphi_i)$, *it is optimal to order*:

1. *for* $x < \hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i) - x$ *at* $c_0$ and amount $\bar{x}_{1n}(\varphi_i) - \hat{x}_1(\varphi_i)$ *at* $c_1$;
2. *for* $\hat{x}_1(\varphi_i) \leqq x < \bar{x}_{1n}(\varphi_i)$, *amount* $\bar{x}_{1n}(\varphi_i) - x$ *at* $c_1$;
3. *for* $x \geqq \bar{x}_{1n}(\varphi_i)$, *none.* $\qquad i = 1, 2, \cdots, m.$

*Case* 11 *If* $\bar{x}_{1n}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$, *it is optimal to order*:

1. *for* $x < \bar{x}_{1n}(\varphi_i)$, *amount* $\bar{x}_{1n}(\varphi_i) - x$ *at* $c_0$;
2. *for* $x \geqq \bar{x}_{1n}(\varphi_i)$, *none.* $\qquad i = 1, 2, \cdots, m,$

*where* $\bar{x}_{1n}(\varphi_i)$ *is a unique root of each equations* $G'_{1n}(x, \varphi_i) = 0$. (ii) $f_n(x, \varphi_i)$ *is a convex function of* $x$, *decreasing for* $x$ *small enough, increasing for* $x$ *large enough and* $f'_n(x, \varphi_i) \geqq -c_0$.

Proof (by induction). We first deal with the case of $n = 1$. Then we have

$$\lim_{u_1 \to \infty} G'_{11}(u, \varphi_i) = c_1 - \alpha \lim_{u \to \infty} w'(u_1) > 0 \qquad \text{by (1.8a)} \tag{2.13}$$

$$\lim_{u_1 \to -\infty} G'_{11}(u_1, \varphi_i) = c_0 + \lim_{u_1 \to -\infty} L'(u_1, \varphi_i) < c_1 - v < 0 \qquad \text{by (1.8a) and (2.1a)} \tag{2.14}$$

Two case may be considered, according to the sign of $G'_{11}(\hat{x}_1(\varphi_i), \varphi_i)$:

Case (a) $G'_{11}(\hat{x}(\varphi_i), \varphi_i) < 0$. Since $G_{11}(u, \varphi_i)$ is strictly convex for $u > 0$, $G'_{11}(0, \varphi_i) < 0$, and (2.13) is valid, there exists a unique positive $\bar{x}_{11}(\varphi_i)$ such that $\bar{x}_{11}(\varphi_i) > \hat{x}_1(\varphi_i)$ and

$$(2.15)\qquad C'_{11}(\bar{x}_{11}(\varphi_i), \varphi_i)=0=c_1+V'(\bar{x}_{11}(\varphi_i), \varphi_i)-\alpha\int_0^{\bar{x}_{11}(\varphi_i)} w'(\bar{x}_{11}(\varphi_i)-t)\varphi_i(t)\,dt \qquad i=1, 2, \cdots, m.$$

It follows that we have in (2.10) for the $n=1$ $u_1=\bar{x}_{11}(\varphi_i)$ if $x<\bar{x}_{11}(\varphi_i)$ and $u_1=x$ if $x\geqq\bar{x}_{11}(\varphi_i)$. Therefore, we obtain from (2.5), (2.9) and (2.10)

$$f_1(x, \varphi_i)=c_0(\hat{x}_1(\varphi_i)-x)+c_1(\bar{x}_{11}(\varphi_i)-\hat{x}_1(\varphi_i))+L(\hat{x}_1(\varphi_i), \varphi_i)-V(\hat{x}_1(\varphi_i), \varphi_i)$$

$$+V(\bar{x}_{11}(\varphi_i), \varphi_i)-\alpha\int_0^{\bar{x}_{11}(\varphi_i)} w(\bar{x}_{11}(\varphi_i)-t)\varphi_i(t)\,dt \qquad x<\hat{x}_1(\varphi_i)$$

$$(2.16)\qquad =c_1(\bar{x}_{11}(\varphi_i)-x)+L(x, \varphi_i)-V(x, \varphi_i)+V(\bar{x}_{11}(\varphi_i), \varphi_i)$$

$$-\alpha\int_0^{\bar{x}_{11}(\varphi_i)} w(\bar{x}_{11}(\varphi_i)-t)\varphi_i(t)\,dt \qquad \hat{x}_1(\varphi_i)\leqq x<\bar{x}_{11}(\varphi_i)$$

$$=L(x, \varphi_i)-\alpha\int_0^x w(x-t)\varphi_i(t)\,dt \qquad x\geqq\bar{x}_{11}(\varphi_i)$$

$$f'_1(x, \varphi_i)=-c_0 \qquad x<\hat{x}_1(\varphi_i)$$

$$(2.17)\qquad =-c_1+L'(x, \varphi_j)-V(x, \varphi_i) \qquad \hat{x}_1(\varphi_i)<x<x_{11}(\varphi_i)$$

$$=L'(x, \varphi_i)-\alpha\int_0^x w'(x-t)\varphi_i(t)\,dt \qquad x>\bar{x}_{11}(\varphi_i)$$

It follows from (2.16) that Case 1 for $n=1$ holds. From (2.17) and (2.1b), we get (ii) for the $n=1$.

Case (b) $C'_{11}(\hat{x}_1(\varphi_i), \varphi_i)\geqq 0$. Then (2.11) is given by

$$(2.18)\qquad G'_{11}(x, \varphi_i)=c_0+L'(x, \varphi_i)-\alpha\int_0^x w'(x-t)\varphi_i(t)\,dt \qquad \text{for } x<\hat{x}_1(\varphi_i)$$

Since (2.1c.d), (1.8b) and (2.14) are valid, there exists a unique $\bar{x}_{11}(\varphi_i)$ such that $\hat{x}_1(\varphi_i)\geqq\bar{x}_{11}(\varphi_i)$, and

$$(2.19)\qquad G'_{11}(\bar{x}_{11}(\varphi_i), \varphi_i)=0=c_0+L'(\bar{x}_{11}(\varphi_i), \varphi_i)-\alpha\int_0^{\bar{x}_{11}(\varphi_i)} w'(\bar{x}_{11}(\varphi_i)-t)\varphi_i(t)\,dt$$

If $p(\eta)$ is linear, $\bar{x}_{11}(\varphi_i)$ is even positive since $G'_{11}(0, \varphi_i)=c_0+L'(0, \varphi)<c_1-v$ by (1.8a) and (2.1a). By use of $\bar{x}_{11}(\varphi_i)$, we obtain from (2.10)

$$(2.20)\qquad f_1(x, \varphi_i)=c_0(\bar{x}_{11}(\varphi_i)-x)+L(\bar{x}_{11}(\varphi_i), \varphi_i)-\alpha\int_0^{\bar{x}_{11}(\varphi_i)} w(\bar{x}_{11}(\varphi_i)-t)\varphi_i(t)\,dt \qquad x<\bar{x}_{11}(\varphi_i)$$

$$=L(x, \varphi_i)-\alpha\int_0^x w(x-t)\varphi_i(t)\,dt \qquad x\geqq\bar{x}_{11}(\varphi_i)$$

$$(2.21)\qquad f'_1(x, \varphi_i)=-c_0 \qquad x<\bar{x}_{11}(\varphi_i)$$

$$= L'(x, \varphi_i) - \alpha\int_0^x w'(x-t)\varphi_i(t)\,dt \qquad x > \bar{x}_{11}(\varphi_i)$$

It follows from (2.20) that the Case II for $n=1$ holds. The expression (2.17) and (2.21) used with (2.1b), verifies part (ii). Assuming that the theorem 2.1 hold for the integer $n-1$, and suppose that $\bar{x}_{1,n-1}(\varphi_i)$ and $\hat{x}_1(\varphi_i)$ have the properties that

(i) $G'_{1,n-1}(\hat{x}_1, (\varphi_i), \varphi_i) < 0$

$$(2.22) \qquad f'_{n-1}(x, \varphi_i) = \begin{cases} -c_0 & x < \hat{x}_1(\varphi_i) \\ -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & \hat{x}_1(\varphi_i) < x < \bar{x}_{1,n-1}(\varphi_i) \\ L'(x, \varphi_i) + f'_{n-1,1}(x, \varphi_i) & x > \bar{x}_{1,n-1}(\varphi_i) \end{cases}$$

and $\bar{x}_{1,n-1}(\varphi_i) > 0$

(ii) $G'_{1,n-1}(x_1(\varphi_i), \varphi_i) \geqq 0.$

$$(2.23) \qquad f'_{n-1}(x, \varphi_i) = \begin{cases} -c_0 & x < \bar{x}_{1,n-1}(\varphi_i) \\ L'(x, \varphi_i) + f'_{n-1,1}(x, \varphi_i) & x > \bar{x}_{1,n-1}(\varphi_i) \end{cases}$$

Then we have

$$(2.24) \qquad \lim_{x\to\infty} G'_{1n}(x, \varphi_i) = c_1 + \alpha \lim_{x\to\infty} \sum_{i_1=1}^m p_{ii_1}\int_0^\infty f'_{n-1}(x-t, \varphi_{i_1})\varphi_i(t)\,dt > 0$$

Case (a) $G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Then $G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq c_1 - \alpha c_0 + V'(\hat{x}_1(\varphi_i), \hat{x}_i)$, due to the fact that $f'_{n-2}(x, \varphi_i) > -c_0$ for all $x$. Hence, by the assumption of this case, we must have $c_1 - \alpha c_0 + V'(\hat{x}_1(\varphi_i), \varphi_i) < 0$. We have from the inductive assumption, (2.9) and (2.11)

$$(2.25) \qquad G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) + \alpha\sum_{i_1=1}^m p_{ii_1}\int_0^\infty f'_{n-1}(\hat{x}_1(\varphi_i) - t, \varphi_{i_1})\varphi_i(t)\,dt$$

$$= c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$$

If $\hat{x}_1(\varphi_i) > 0$, it is clear that $G'_{1n}(0, \varphi_i) < 0$. The hypothesis $\bar{x}_{1,n-1}(\varphi_i) > 0$ implies

$$-c_1 + L'(0, \varphi_i) - V'(0, \varphi_i) = v - c_1 - \int_0^\infty p'(t)\varphi_i(t)\,dt < v - p'(0) - c_1 < 0$$

hence

$$G'_{1n}(0, \varphi_i) = c_1 - v + \alpha\sum_{i_1=1}^m p_{ii_1}\int_0^\infty f'_{n-1}(-t, \varphi_{i_1})\varphi_i(t)\,dt < c_1 - v$$

$$+ \alpha\sum_{i_1=1}^m p_{ii_1} f'_{n-1}(0, \varphi_{i_1}) < 0$$

Since $G_{1n}(u, \varphi_i)$ is strictly convex for $u > 0$ by (1.8b) and (2.1c.d), and (2.24) is valid, there exists a unique positive $\bar{x}_{1n}(\varphi_i)$ such that and $\bar{x}_{1n}(\varphi_i) > \hat{x}_1(\varphi_i)$. Hence, from the same analysis as in the case $n=1$, $\bar{x}_{1n}(\varphi_i)$ and $\hat{x}_1(\varphi_i)$ uniquely define the optimal policy which is identical to the Case 1, and also $f'_n(x, \varphi_i)$ is given by

$\bar{x}_{1n}(\varphi_i) > \hat{x}_1(\varphi_i)$;

$$(2.26) \qquad f'_n(x, \varphi_i) = \begin{cases} -c_0 & x < \hat{x}_1(\varphi_i) \\ -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & \hat{x}_1(\varphi_i) < x < \bar{x}_{1n}(\varphi_i) \\ L'(x, \varphi_i) + f'_{n-1,1}(x, \varphi_i) & x > \bar{x}_{1n}(\varphi_i) \end{cases}$$

Case (b) $G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. If we had $c_1 - \alpha c_0 + V'(\hat{x}_1(\varphi_i), \varphi_i) < 0$, then we may have $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$, or we may still have $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, since $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq c_1 - \alpha c_0 + V'(\hat{x}_1(\varphi_i), \varphi_i) < 0$. The first case has already been analyzed. If we have $c_1 - \alpha c_0 + V'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, we must have $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$ by due to the fact that $f'_{n-1}(x, \varphi_i) \geqq -c_0$ for all $x$. In the case $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, we have from the inductive assumption

$$(2.27) \qquad \lim_{x\to-\infty} G'_{1n}(x, \varphi_i) = c_0 + \lim_{x\to-\infty} L'(x, \varphi_i) + \alpha \lim_{x\to-\infty} \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty f'_{n-1}(x-t, \varphi_{i_1}) \varphi_i(t)\, dt$$
$$= c_0 + \lim_{x\to-\infty} L'(x, \varphi_i) - \alpha c_0 < c_1 - v - \alpha c_0 < 0$$

On the other hand, (2.24) hold, Hence there is an $\bar{x}_{1n}(\varphi_i)$ such that $\bar{x}_{1n}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$ and

$$(2.28) \qquad G'_{1n}(\bar{x}_{1n}(\varphi_i), \varphi_i) = 0 = c_0 + L'(\bar{x}_{1n}(\varphi_i), \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty f'_{n-1}(\bar{x}_{1n}(\varphi_i) - t, \varphi_{i_1}) \varphi_i(t)\, dt \qquad i = 1, 2, \cdots, m.$$

Uniqueness follows from (2.1c, d) and (1.8b). The last assertion is immediate. Therefore, $\hat{x}_1(\varphi_i)$ and $\bar{x}_{1n}(\varphi_i)$ uniquely determine the optimal policy which is identical to the Case II. $f'_n(x, \varphi_i)$ is given by

$\bar{x}_{1n}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$;

$$(2.29) \qquad f'_n(x, \varphi_i) = \begin{cases} -c_0 & x < \bar{x}_{1n}(\varphi_i), \\ L'(x, \varphi_i) + f'_{n-1,1}(x, \varphi_i) & x > \bar{x}_{1n}(x_i). \end{cases}$$

It is noticed that if $p(\eta)$ is linear, then $\hat{x}_1(\varphi_i) > 0$ and $\bar{x}_{1n}(\varphi_i) > 0$.
In order to advance the further discussion, let us designate by $\tilde{x}_1^{(0)}(\varphi_i)$ and $\tilde{x}_1^{(1)}(\varphi_i)$ the unique roots of the equations

$$(2.30) \qquad F_1^{(0)}(x, \varphi_i) = c_0(1-\alpha) + L'(x, \varphi_i) = 0 \qquad i = 1, 2, \cdots, m.$$

$$(2.31)\qquad F_1^{(1)}(x, \varphi_i) = c_1 + V'(x, \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} g^{(1)}(x-t, \varphi_{i_1})\varphi_i(t)\,dt = 0 \qquad i=1, 2, \cdots, m,$$

where $g^{(1)}(x, \varphi_i)$ is define by

$$(2.32)\qquad \begin{aligned} g^{(1)}(x, \varphi_i) &= -c_0 && x < \hat{x}_1(\varphi_i) \\ &= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) && x > \hat{x}_1(\varphi_i) \end{aligned} \qquad i=1, 2, \cdots, m.$$

From the previous analysis, it is clear that

$$(2.33)\qquad f_n'(x, \varphi_i) \geqq g^{(1)}(x, \varphi_i) \geqq -c_0 \text{ for all } x \qquad i=1, 2, \cdots, m.$$

THEOREM 2.2. If conditions of Theorem 2.1 are satisfied, then for $n \geqq 2$

(I) $f_{n-1}'(x, \varphi_i) \geqq f_n'(x, \varphi_i)$ for $x < \bar{x}_{1n}(\varphi_i)$ $\qquad i=1, 2, \cdots, m.$

(II) Case I $c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$;

$\bar{x}_{1n}(\varphi_i) \leqq \bar{x}_{1,n+1}(\varphi_i) \leqq \tilde{x}_1^{(1)}(\varphi_i)$ $\qquad i=1, 2, \cdots, m,$;

there also exists a uniques $j \geqq 2$ such that $\bar{x}_{1j}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{1,j+1}(\varphi_i)$.

Case II $c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 \geqq 0$;

$\bar{x}_{1n}(\varphi_i) \leqq \bar{x}_{1,n+1}(\varphi_i) \leqq \tilde{x}_1^{(0)}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$ $\qquad i=1, 2, \cdots, m.$

Proof (by induction). Suppose first that $n=2$. Then it follows from (2.26) and (2.29) that we have in both cases, $\bar{x}_{11}(\varphi_i) \leqq \bar{x}_{12}(\varphi_i)$ and $\bar{x}_{11}(\varphi_i) > \bar{x}_{12}(\varphi_i)$

$$(2.34)\qquad f_1'(x, \varphi_i) \geqq f_2'(x, \varphi_i) \text{ for } x < \bar{x}_{12}(\varphi_i) \qquad i=1, 2, \cdots, m.$$

Case I$F_1^{(1)}(\hat{x}_1(\varphi_i)\varphi_i) = c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$. Then we have $\hat{x}_1(\varphi_i) < \tilde{x}_1^{(1)}(\varphi_i)$. Since

$$(2.35)\qquad \begin{aligned} G_{12}'(\hat{x}_1(\varphi_i), \varphi_i) &= c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} f_1'(\hat{x}_1(\varphi_i) - t, \varphi_{i_1})\varphi_i(t)\,dt \\ &\geqq c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0 \qquad \text{by (2.23)} \end{aligned}$$

Two subcases are possible; Case I. A $G_{12}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. Then again two subscases are possible, since $G_{13}'(\hat{x}_1(\varphi_i)\varphi_i) \geqq c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$

Case I. A $G_{12}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G_{13}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, in this case, we have $\hat{x}_1(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$ and $\hat{x}_1(\varphi_i) \geqq \bar{x}_{13}(\varphi_i)$. Since

$$(2.36)\qquad \begin{aligned} G_{13}'(\bar{x}_{12}(\varphi_i), \varphi_i) &= G_{13}'(\bar{x}_{12}(\varphi_i), \varphi_i) - G_{12}'(\bar{x}_{12}(\varphi_i), \varphi_i) \\ &= \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^{\infty} [f_2'(\bar{x}_{12}(\varphi_i) - t, \varphi_{i_1}) - f_1'(\bar{x}_{12}(\varphi_i) - t, \varphi_{i_1})]\varphi_i(t)\,dt \leqq 0 \end{aligned}$$

by (2.34), it follows that $\tilde{x}_1^{(1)}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$

Case I. A $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Then we get $\bar{x}_{13}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$. Since

$$G'_{13}(\tilde{x}_1^{(1)}(\varphi_i), \varphi_i) = G'_{13}(\tilde{x}_1^{(1)}(\varphi_i), \varphi_i) - F_1^{(1)}(\tilde{x}_1^{(1)}(\varphi_i), \varphi_i) \tag{2.37}$$

$$= \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty [f'_2(\tilde{x}_1^{(1)}(\varphi_i) - t, \varphi_{i_1}) - g(\tilde{x}_1^{(1)}(\varphi_i) - t, \varphi_{i_1})] \varphi_i(t)\, dt > 0$$

by (2.33), we have $\tilde{x}_1^{(1)}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$

Case I. B $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Then we get $\hat{x}_1(\varphi_i) < \bar{x}_{12}(\varphi_i)$. The relation

$$G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) + \alpha \sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty f'_2(\hat{x}_1)\varphi_i) - t, \varphi_{i_1}) \varphi_i(t)\, dt \tag{2.38}$$

$$\leqq G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) < 0$$

implies $\hat{x}_1(\varphi_i) < \bar{x}_{12}(\varphi_i)$. From the same calculation as (2.36) and (2.37), we obtain $\tilde{x}_1^{(1)}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i) > \hat{x}_1(\varphi_i)$

Case II $c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 \geqq 0$. Then we have $\hat{x}_1(\varphi_i) \geqq \tilde{x}_1^{(1)}(\varphi_i)$. Since we have $G'_{12}(\hat{x}_{12}(\varphi_i), \varphi_i) \geqq 0$ and $G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$ by (2.35), by the analysis as in the case I.A, we get $\hat{x}_1(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$. Since

$$F_1^{(0)}(\hat{x}_1(\varphi_i), \varphi_i) = F_1^{(0)}(\hat{x}_1(\varphi_i), \varphi_i) - M^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) \tag{2.39}$$

$$= c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 \geqq 0;$$

hence we have $\hat{x}_1(\varphi_i) \geqq \tilde{x}_1^{(0)}(\varphi_i)$. Therefore, we have

$$G'_{13}(\tilde{x}_1^{(0)}(\varphi_i), \varphi_i) = G'_{13}(\tilde{x}_1^{(0)}(\varphi_i), \varphi_i) - F_1^{(0)}(\tilde{x}_1^{(0)}(\varphi_i), \varphi_i) \tag{2.40}$$

$$= \alpha \{\sum_{i_1=1}^{m} p_{ii_1} \int_0^\infty [f'_2(\hat{x}_1(\varphi_i) - t, \varphi_{i_1}) + c_0] \varphi_i(t)\, dt\} \geqq 0$$

by (2.18) and (2.33); we get $\hat{x}_1(\varphi_i) \geqq \tilde{x}_1^{(0)}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$ In each cases, it is easily seen that

$$f'_2(x, \varphi_i) \geqq f'_3(x, \varphi_i) \text{ for } x < \bar{x}_{13}(\varphi_i) \tag{2.41}$$

In order to verify the latter case of Case I, we shall examine the relationship amoung $\bar{x}_{13}(\varphi_i)$, $\bar{x}_{14}(\varphi_i)$, $\hat{x}_1(\varphi_i)$ and $\tilde{x}_1^{(1)}(\varphi_i)$. The following cases may be considered, according to the sign $G'_{14}(\hat{x}_1(\varphi_i), \varphi_i)$. From the same analysis in the case $n = 2$, we obtain the next results.

Case I. $A_1$ $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{14}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$

$$\tilde{x}_1^{(1)}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{14}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i) \tag{2.42}$$

Case I. $A_2$ $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{14}(\hat{x}_1(\varphi_i), \varphi_i) < 0$.

(2.43) $$\tilde{x}_1^{(1)}(\varphi_i) \geqq \bar{x}_{14}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$$

Case I. $A_3$ $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0, \quad G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) < 0, \quad G'_{14}(\hat{x}_1(\varphi_i), \varphi_i) < 0$

(2.44) $$\tilde{x}_1^{(1)}(\varphi_i) \geqq \bar{x}_{14}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{12}(\varphi_i)$$

Case I. $B_1$ $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) < 0, G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) < 0. \quad G'_{14}(\hat{x}_1(\varphi_i), \varphi_i) < 0$

(2.45) $$\tilde{x}_1^{(1)}(\varphi_i) \geqq \bar{x}_{14}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i) > \hat{x}_1(\varphi_i)$$

Case II. $A_1$ $G'_{12}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0, \quad G'_{13}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0, \quad G'_{14}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$

(2.46) $$\hat{x}_1(\varphi_i) \geqq \tilde{x}_1^{(0)}(\varphi_i) \geqq \bar{x}_{14}(\varphi_i) \geqq \bar{x}_{13}(\varphi_i) \geqq \bar{x}_{12}(\varphi_i), \quad \hat{x}_1(\varphi_i) \geqq \tilde{x}_1^{(1)}(\varphi_i).$$

If we assume the validity of the theorem for the integer $n-1 (n-1 \geqq 2)$, the following possible situations may occur (Fig. 1).

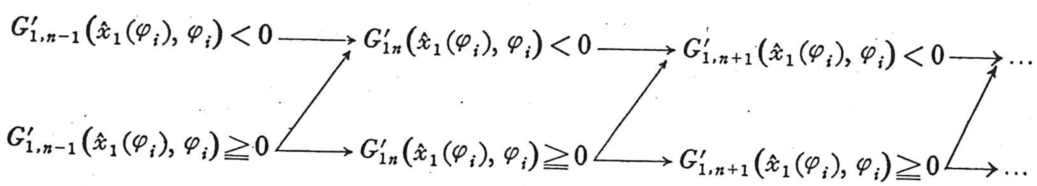


Fig. 1

Case I. $A_1$ $c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$, $G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. Then from (2.23) and (2.29), we obtain $f'_{n-1}(x, \varphi_i) \geqq f'_n(x, \varphi_i)$ for $x < \bar{x}_{1n}(\varphi_i)$. Since.

$$G'_{1,n+1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0$$

Two subcases are possible (Fig. 1);

Case I. $A_{11}$ $c_1 + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0, \quad G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0,$

$$G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0, \quad G'_{1,n+1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$$

Then we have $\tilde{x}_1^{(1)}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{1n}(\varphi_i) \geqq \bar{x}_{1,n-1}(\varphi_i)$ and $\hat{x}_1(\varphi_i) \geqq \bar{x}_{1,n+1}(\varphi_i)$. Since

$$G'_{1,n+1}(\bar{x}_{1n}(\varphi_i), \varphi_i) = G'_{1,n+1}(\bar{x}_{1n}(\varphi_i), \varphi_i) - G'_{1n}(\bar{x}_{1n}(\varphi_i), \varphi_i)$$

$$= \alpha \sum_{i_1=1}^{m} p_{i i_1} \int_0^\infty [f'_n(\bar{x}_{1n}(\varphi_i) - t, \varphi_{i_1}) - f'_{n-1}(\bar{x}_{1n}(\varphi_i) - t, \varphi_{i_1})] \varphi_i(t)\, dt \leqq 0$$

it follows that $\tilde{x}_1^{(1)}(\varphi_i) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{1,n+1}(\varphi_i) \geqq \bar{x}_{1n}(\varphi_i) \geqq \bar{x}_{1,n-1}(\varphi_i)$.

Case I. $A_{12}$ $c_1 - V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha c_0 < 0, \quad G'_{1,n-1}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0,$

$$G'_{1n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0, \quad G'_{1,n+1}(\hat{x}_1(\varphi_i), \varphi_i) < 0.$$

Then we get $\tilde{x}_1^{(1)}(\varphi_1) > \hat{x}_1(\varphi_i) \geqq \bar{x}_{1n}(\varphi_i) \geqq \bar{x}_{1,n-1}(\varphi_i)$ and $\hat{x}_1(\varphi_i) < \bar{x}_{1,n+1}(\varphi_i)$. Since

$$G'_{1,n+1}(\tilde{x}_1^{(1)}(\varphi_i),\varphi_i)=G'_{1,n+1}(\tilde{x}_1^{(1)}(\varphi_i),\varphi_i)-F_1^{(1)}(\tilde{x}_1^{(1)}(\varphi_i),\varphi_i)$$

$$=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty[f'_n(\tilde{x}_1^{(1)}(\varphi_i)-t,\varphi_{i_1})-g(\tilde{x}_1^{(1)}(\varphi_i)-t,\varphi_{i_1})]\varphi_i(t)\,dt\geqq 0,$$

it follows that $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1,n+1}(\varphi_i)>\hat{x}_1(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)$.

Case I. $A_2$ $c_1+V'(\hat{x}_1(\varphi_i),\varphi_i)-\alpha c_0<0$, $G'_{1,n-1}(\hat{x}_1(\varphi_i),\varphi_i)\geqq 0$, $G'_{1n}(\hat{x}_1(\varphi_i),\varphi_i)<0$. Then we have $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)>\hat{x}_1(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)$. It follows from (2.23) and (2.26) that $f'_{n-1}(x,\varphi_i)\geqq f'_n(x,\varphi_i)$ for $x<\bar{x}_{1n}(\varphi_i)$. Since

$$G'_{1,n+1}(\bar{x}_{1n}(\varphi_i),\varphi_i)\leqq G'_{1n}(\bar{x}_{1n}(\varphi_i),\varphi_i)=0$$

and

$$G'_{1,n+1}(\tilde{x}_1^{(1)}(\varphi_i),\varphi_i)\geqq 0$$

it follows that $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1,n+1}(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)>\hat{x}_1(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)$.

Case I. $B_1$ $c_1+V'(\hat{x}_1(\varphi_i),\varphi_i)-\alpha c_0<0$. $G'_{1,n-1}(\hat{x}(\varphi_i),\varphi_i)<0$, $G'_{1n}(\hat{x}_1(\varphi_i),\varphi_i)<0$. Then we have $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)>\hat{x}_1(\varphi_i)$. It follows from (2.22) and (2.26) $f'_{n-1}(x,\varphi_i)\geqq f'_n(x,\varphi_i)$ for $x<\bar{x}_{1n}(\varphi_i)$. Since we have $G'_{1,n+1}(\hat{x}_1(\varphi_i),\varphi_i)\leqq G'_{1n}(\hat{x}_1(\varphi_i),\varphi_i)<0$ and $G'_{1,n+1}(\tilde{x}^{(1)}(\varphi_i),\varphi_i)\geqq 0$, it follows that $\hat{x}_1(\varphi_i)\leqq\bar{x}_{1,n+1}(\varphi_i)$ and $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1,n+1}(\varphi_i)$. Hence, we get $\tilde{x}_1^{(1)}(\varphi_i)\geqq\bar{x}_{1,n+1}(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)>\hat{x}_1(\varphi_i)$

Case II $c_1+V'(\hat{x}_1(\varphi_i),\varphi_i)-\alpha c_0\geqq 0$, $G'_{1,n-1}(\hat{x}_1(\varphi_i),\varphi_i)\geqq 0$, $G'_{1n}(\hat{x}_1(\varphi_i),\varphi_i)\geqq 0$. Then we obtain $\bar{x}_{1,n-1}(\varphi_i)\leqq\bar{x}_{1n}(\varphi_i)\leqq\tilde{x}_1^{(0)}(\varphi_i)\leqq\hat{x}_1(\varphi_i)$. It follows from (2.23) and (2.29) $f'_{n-1}(x,\varphi_i)\geqq f'_n(x,\varphi_i)$ for $x\leqq\bar{x}_{1n}(\varphi_i)$. Since

$$G'_{1,n+1}(\tilde{x}_1^{(0)}(\varphi_i),\varphi_i)=G'_{1,n+1}(\tilde{x}_1^{(0)}(\varphi_i),\varphi_i)-F_1^{(0)}(\tilde{x}_1^{(0)}(\varphi_i),\varphi_i)$$

$$=\alpha\sum_{i_1=1}^{m}p_{ii_1}\int_0^\infty[f'_n(\tilde{x}_1^{(0)}(\varphi_i)-t,\varphi_{i_1})+c_0]\varphi_i(t)\,dt\geqq 0$$

and

$$G'_{1,n+1}(\bar{x}_{1n}(\varphi_i),\varphi_i)=G'_{1,n+1}(\bar{x}_{1n}(\varphi_i),\varphi_i)-G'_{1n}(\bar{x}_{1n}(\varphi_i),\varphi_i)\leqq 0$$

it follows that $\hat{x}_1(\varphi_i)\geqq\tilde{x}_1^{(0)}(\varphi_i)\geqq\bar{x}_{1,n+1}(\varphi_i)\geqq\bar{x}_{1n}(\varphi_i)\geqq\bar{x}_{1,n-1}(\varphi_i)$. The proof is complete.

## §3. Optimal policy under three modes of delivery ($k=2$)

In this section we analyze the dynamic model for the case $k=2$. We obtain from (1.11)

(3.1) $$f_n(x,\varphi_i)=\min_{u_2\geqq u_1\geqq u_0\geqq x}\{(c_0-c_1)(u_0-x)+L(u_0,\varphi_i)-V(u_0,\varphi_i)$$

$$+(c_1-c_2)(u_1-x)+L_1(u_1, \varphi_i)-V_1(u_1, \varphi_i)+V(u_1, \varphi_i)$$
$$+c_2(u_2-x)+V_1(u_2, \varphi_i)+f_{n-2,2}(u_2, \varphi_i)\} \qquad n \geqq 2$$
$$i=1, 2, \cdots, m,$$

$$f_1(x, \varphi_i)=\min_{u_1 \geqq u_0 \geqq x}\{(c_0-c_1)(u_0-x)+L(u_0, \varphi_i)-V(u_0, \varphi_i)+c_1(u-x)$$
$$+V(u_1, \varphi_i)-W_1(u_1, \varphi_i)\} \qquad i=1, 2, \cdots, m$$

If we real, at this point, the technique which was applied to reduced (2.2) to (2.10) in §2, then it will be readily seen that (3.1) may be reduced by the same technique to (3.2)

$$f_n(x, \varphi_i)=\tilde{L}^{(1)}(x, \varphi_i)+\min_{u_2 \geqq u_1 \geqq x}\{(c_1-c_2)(u_1-x)+\Lambda^{(1)}(u_1, \varphi_i)+L_1(u_1, \varphi_i)$$
$$-V_1(u_1, \varphi_i)+V(u_1, \varphi_i)+c_2(u_2-x)+V_1(u_2, \varphi_i)+f_{n-2,2}(u_2, \varphi_i)\} \qquad n \geqq 2 \tag{3.2}$$
$$i=1, 2, \cdots, m.$$

where $\tilde{L}^{(1)}(x, \varphi_i)$ and $\Lambda^{(1)}(u_1, \varphi_i)$ are given by (2.3) and (2.9), respectively. Let us define

$$\tilde{L}^{(2)}(x, \varphi_i)=\min_{u_1 \geqq x}\{(c_1-c_2)(u_1-x)+\Lambda^{(1)}(u_1, \varphi_i)+L_1(u_1, \varphi_i)-V_1(u_1, \varphi_i)$$
$$+V(u_1 \varphi_i)\} \qquad i=1, 3, \cdots, m, \tag{3.3}$$

$$M^{(2)}(x, \varphi_i)=c_1-c_2+\Lambda^{(1)\prime}(x, \varphi_i)+L_1'(x, \varphi_j)-V_1'(x, \varphi_i)+V'(x, \varphi_i) \tag{3.4}$$
$$i=1, 2, \cdots, m.$$

It is easily seen that exists a unique root $\hat{x}_2(\varphi_i)$ for each equation $M^{(2)}(x, \varphi_i)=0$ since the conditions (2.1c.d) are satisfied, $\lim_{x\to\infty} M^{(2)}(x, \varphi_i)>0$ by (1.9), (1.7), (1.11) and (2.1b), and $\lim_{x\to-\infty} M^{(2)}(x, \varphi_i)<0$ by (1.7), (1.8) and (2.1a). If $p(\eta)$ is linear, $\hat{x}_2(\varphi_i)$ is even positive since $M^{(2)}(t, \varphi_i)<(c_0-c_1-p'(0)+v)+c_1-v-c_2+\alpha(v-p'(0)) <0$ for all $t \leqq 0$ Hence, we obtain

$$\tilde{L}^{(2)}(x, \varphi_i)=(c_1-c_2)(\hat{x}_2(\varphi_i)-x)+\Lambda^{(1)}(\hat{x}_2(\varphi_i), \varphi_i)+L_1(\hat{x}_2(\varphi_i), \varphi_i)$$
$$-V_1(\hat{x}_2(\varphi_i), \varphi_i)+V(\hat{x}_2(\varphi_i), \varphi_i) \qquad x<\hat{x}_2(\varphi_i)$$
$$=\Lambda^{(1)}(x, \varphi_2)+L_1(x, \varphi_i)-V_1(x, \varphi_i)+V(x, \varphi_i) \qquad x \geqq \hat{x}_2(\varphi_i) \tag{3.5}$$
$$i=1, 2, \cdots, m.$$

$\tilde{L}^{(2)}(x, \varphi_i)$ is clearly convex in $x$. By the same technique which was applied to reduce (2.2) to (2.10), we get

$$f_n(x, \varphi_i)=\tilde{L}^{(1)}(x, \varphi_i)+\tilde{L}^{(2)}(x, \varphi_i)+\min_{u_2 \geqq x}\{c_2(u_2-x)+\Lambda^{(2)}(u_2, \varphi_i) \tag{3.6}$$

$$+V_1(u_2,\varphi_i)+f_{n-2,2}(u_2,\varphi_i)\} \qquad i=1,2,\cdots,m$$

where $\Lambda^{(2)}(x,\varphi_i)$ is given by

$$(3.7)\quad \Lambda^{(2)}(x,\varphi_i)=(c_1-c_2)(x-\hat{x}_2(\varphi_i))+(\Lambda^{(1)}(x,\varphi_i)+L_1(x,\varphi_i)-V_1(x,\varphi_i)$$
$$+V(x,\varphi_i))-(\Lambda^{(1)}(\hat{x}_2(\varphi_i),\varphi_i)+L_1(\hat{x}_2(\varphi_i),\varphi_i)-V_1(\hat{x}_2(\varphi_i),\varphi_i)$$
$$+V(\hat{x}_2(\varphi_i),\varphi_i)) \qquad x<\hat{x}_2(\varphi_i)$$
$$=0 \qquad x\geqq\hat{x}_2(\varphi_i)$$
$$i=1,2,\cdots,m.$$

We again notice that $\Lambda^{(2)}(x,\varphi_i)$ is convex in $x$. Let us define

$$(3.8)\quad G_{2n}(u,\varphi_i)=c_2u+\Lambda^{(2)}(u,\varphi_i)+V_1(u,\varphi_i)+f_{n-2,2}(u,\varphi_i) \qquad i=1,2,\cdots,m.$$

At $x=\hat{x}_j$ $(j=1,2)$, the derivative $G'_{2n}(u,\varphi_i)$ are given by

$$(3.9)\quad G'_{2n}(\hat{x}_2(\varphi_i),\varphi_i)=c_2+V'_1(\hat{x}_2\varphi_i)+f'_{n-2,2}(\hat{x}_2(\varphi_i),\varphi_i)$$
$$G'_{2n}(\hat{x}_1(\varphi_i),\varphi_i)=c_1+L'_1(\hat{x}_1\varphi_i)+V'(\hat{x}_1\varphi_i)+f'_{n-2,2}(\hat{x}_1(\varphi_i),\varphi_i)$$

THEOREM 3.1. *If conditions of Theorem 2.1 are satisfied and if* $\hat{x}_2(\varphi_i)>\hat{x}_1(\varphi_i)$, *then* (i) *the optimal ordering policy is of the following form for* $n\geqq 2$.

*Case I* *If* $\bar{x}_{2n}(\varphi_i)>\hat{x}_2(\varphi_i)$, *it is optimal to order*:

1. *for* $x<\hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i)-x$ *at* $c_0$, *amount* $\hat{x}_2(\varphi_i)-\hat{x}_1(\varphi_i)$ *at* $c_1$, *and amount* $\bar{x}_{2n}(\varphi_i)-\hat{x}_2(\varphi_i)$ *at* $c_2$;
2. *for* $\hat{x}_1(\varphi_i)\leqq x<\hat{x}_2(\varphi_i)$, *amount* $\hat{x}_2(\varphi_i)-x$ *at* $c_1$, *and amount* $\bar{x}_{2n}(\varphi_i)-\hat{x}_2(\varphi_i)$ *at* $c_2$;
3. *for* $\hat{x}_2(\varphi_i)\leqq x<\bar{x}_{2n}(\varphi_i)$, *amount* $\bar{x}_{2n}(\varphi_i)-x$ *at* $c_2$;
4. *for* $x>\bar{x}_{2n}(\varphi_i)$, *none.*

Case II *If* $\hat{x}_2(\varphi_i)\geqq\bar{x}_{2n}(\varphi_i)>\hat{x}_1(\varphi_i)$, *it is optimal to order*:

1. *for* $x<\hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i)-x$ *at* $c_0$, *and amount* $\bar{x}_{2n}(\varphi_i)-\hat{x}_1(\varphi_i)$ *at* $c_1$;
2. *for* $\hat{x}_1(\varphi_i)\leqq x<\bar{x}_{2n}(\varphi_i)$, *amount* $\bar{x}_{2n}(\varphi_i)-x$ *at* $c_1$;
3. *for* $x\geqq\bar{x}_{2n}(\varphi_i)$, *none.*

Case III *If* $\bar{x}_{2n}(\varphi_i)\leqq\hat{x}_1(\varphi_i)$, *it is optimal to order*:

1. *for* $x<\bar{x}_{2n}(\varphi_i)$, *amount* $\bar{x}_{2n}(\varphi_i)-x$ *at* $c_0$;
2. *for* $x\geqq\bar{x}_{2n}(\varphi_i)$, *none.*

*where* $\bar{x}_{2n}(\varphi_i)$ *is a unique root of each equation,* $G'_{2n}(x,\varphi_i)=0$. (ii) $f_n(x,\varphi_i)$ *is a convex function of* $x$, *decreasing for* $x$ *small enough, increasing for* $x$ *large enough and* $f'_n(x,\varphi_i)\geqq -c_0$ *for all* $x$. (iii) *For* $n=1$, *we obtain the same result as in the case* $n=1$ *of Theorem* 2.1

Proof (by induction). It is clear from (3.1) that (iii) is valid. We begin with the case $n=2$. From (1.8a), (2.1a), (3.7), and (3.8) we have

$$\text{(3.10)}\quad \lim_{u_2\to\infty} G'_{22}(u_2, \varphi_i) = c_2 + \lim_{u_2\to\infty} V'_1(u_2, \varphi_i) + \lim_{u_2\to\infty} f'_{02}(u_2, \varphi_i)$$
$$= c_2 - \alpha^2 \lim_{u_2\to\infty} w'(u_2) > 0$$

$$\text{(3.11)}\quad \lim_{u_2\to-\infty} G'_{22}(u_2, \varphi_i) = c_2 + \lim_{u_2\to-\infty} \Lambda'^{(2)}(u_2, \varphi_i) - \alpha v + \lim_{u_2\to-\infty} f'_{02}(u_2, \varphi_i)$$
$$= c_0 + \lim_{u_2\to-\infty} L'(u_2, \varphi_i) + \lim_{u_2\to-\infty} L'_1(u_2, \varphi_i) < c_1 - v + \alpha(c_1 - c_0 - v) < 0$$

It follows from (2.1c.d) that each equation $G'_{22}(x, \varphi_i) = 0$ posses a unique root. If $p(\eta)$ is linear, then $G'_{22}(0, \varphi_i) = c_1 - v + (c_0 - c_1 - p'(0) + v) - \alpha p'(0) < 0$; hence $\bar{x}_{2n}(\varphi_i)$ is even positive. If we assume that

$$\text{(3.12)}\quad \hat{x}_2(\varphi_i) > \hat{x}_1(\varphi_i),$$

then there are three possibilities requiring separate treatment, $\bar{x}_{22}(\varphi_i) > \hat{x}_2(\varphi_i) > \hat{x}_2(\varphi_i) \geqq \bar{x}_{22}(\varphi_i) > \hat{x}_1(\varphi_i)$, and $\bar{x}_{22}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$. We note that

$$\text{(3.13)}\quad G'_{2n}(u, \varphi_i) = c_0 + L'(u, \varphi_i) + L'_1(u, \varphi_i) + f'_{n-2,2}(u, \varphi_i) \qquad u < \hat{x}_1(\varphi_i)$$
$$= c_1 + L'_1(u, \varphi_i) + V'(u, \varphi_i) + f'_{n-2,2}(u, \varphi_i) \qquad \hat{x}_1(\varphi_i) < u < \hat{x}_2(\varphi_i)$$
$$= c_2 + V'_1(u, \varphi_i) + f'_{n-2,2}(u, \varphi_i) \qquad u > \hat{x}_2(\varphi_i)$$

The explict form of $f_2(x, \varphi_i)$ and $f'_2(x, \varphi_i)$ now may be determined by use of (2.5), (2.9), (3.5), (3.6), and (3.7)

Case (a) $\bar{x}_{22}(\varphi_i) > \hat{x}_2(\varphi_i)$ $(G'_{22}(\hat{x}_2(\varphi_i), \varphi_i) < 0)$ $\qquad i = 1, 2, \cdots, m$

$$\text{(3.14)}\quad f_2(x, \varphi_i) = c_0(\hat{x}_1(\varphi_i) - x) + c_1(\hat{x}_2(\varphi_i) - \hat{x}_1(\varphi_i)) + c_2(\bar{x}_{22}(\varphi_i) - \hat{x}_2(\varphi_i))$$
$$+ L(\hat{x}_1(\varphi_i), \varphi_i) - V(\hat{x}_1(\varphi_i), \varphi_i) + L_1(\hat{x}_2(\varphi_i), \varphi_i) - V_1(\hat{x}_2(\varphi_i), \varphi_i)$$
$$+ V(\hat{x}_2(\varphi_i), \varphi_i) + V_1(\bar{x}_{22}(\varphi_i), \varphi_i) + f_{0,2}(\bar{x}_{22}(\varphi_i), \varphi_i) \qquad x < \hat{x}_1(\varphi_i)$$
$$= c_1(\hat{x}_2(\varphi_i) - x) + c_2(\bar{x}_{22}(\varphi_i) - \hat{x}_2(\varphi_i)) + L(x, \varphi_i) - V(x, \varphi_i)$$
$$+ L_1(\hat{x}_2(\varphi_i), \varphi_i) - V_1(\hat{x}_2(\varphi_i), \varphi_i) + V(\hat{x}_2(\varphi_i), \varphi_i) + V_1(\bar{x}_{22}(\varphi_i), \varphi_i)$$
$$+ f_{02}(\bar{x}_{22}(\varphi_i), \varphi_i) \qquad x_1(\varphi_i) \leqq x < \hat{x}_2(\varphi_i)$$
$$= c_2(\bar{x}_{22}(\varphi_i) - x) + L(x, \varphi_i) + L_1(x, \varphi_i) - V_1(x, \varphi_i)$$
$$+ V_1(\bar{x}_{22}(\varphi_i), \varphi_i) + f_{0,2}(\bar{x}_{22}(\varphi_i), \varphi_i) \qquad \hat{x}_2(\varphi_i) \leqq x < \bar{x}_{22}(\varphi_i)$$
$$= L(x, \varphi_i) + L_1(x, \varphi_i) + f_{0,2}(x, \varphi_i) \qquad x > \bar{x}_{22}(\varphi_i)$$

$$\text{(3.15)}\quad f'_2(x, \varphi_i) = -c_0 \qquad x < \hat{x}_1(\varphi_i)$$
$$= -c_1 + L'(x, \varphi_i) - V'(x_1\varphi_i) \qquad \hat{x}_1(\varphi_i) < x < \hat{x}_2(\varphi_i)$$

$$= -c_2 + L'(x, \varphi_i) + L_1'(x, \varphi_i) - V_1'(x, \varphi_i) \qquad \hat{x}_2(\varphi_i) < x < \bar{x}_{22}(\varphi_i)$$

$$= L'(x, \varphi_i) + L_1'(x, \varphi_i) + f_{0,2}'(x, \varphi_i) \qquad x > \bar{x}_{22}(\varphi_i).$$

Case (b) $\hat{x}_2(\varphi_i) \geqq \bar{x}_{22}(\varphi_i) > \hat{x}_1(\varphi_i) (G_{22}'(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0, \ G_{22}'(\hat{x}_1(\varphi_i), \varphi_i) < 0)$

$$(3.16) \quad f_2(x, \varphi_i) = c_0(\hat{x}_1(\varphi_i) - x) + c_1(\bar{x}_{22}(\varphi_i) - \hat{x}_1(\varphi_i)) + L(\hat{x}_1(\varphi_i), \varphi_i) - V(\hat{x}_1(\varphi_i), \varphi_i) + L_1(\bar{x}_{22}(\varphi_i), \varphi_i) + V(\bar{x}_{22}(\varphi_i), \varphi_i) + f_{0,2}(\bar{x}_{22}(\varphi_i), \varphi_i), \qquad x < \hat{x}_1(\varphi_i)$$

$$= c_1(\bar{x}_{22}(\varphi_i) - x) + L(x, \varphi_i) - V(x, \varphi_i) + L_1(\bar{x}_{22}(\varphi_i), \varphi_i) + V(\bar{x}_{22}(\varphi_i), \varphi_i) + f_{0,2}(\bar{x}_{22}(\varphi_i), \varphi_i) \qquad \hat{x}_1(\varphi_i) \leqq x < \bar{x}_{22}(\varphi_i)$$

$$= L(x, \varphi_i) + L_1(x, \varphi_i) + f_{0,2}(x, \varphi_i) \qquad x \geqq \bar{x}_{22}(\varphi_i)$$

$$(3.17) \quad f_2'(x, \varphi_i) = -c_0 \qquad x < \hat{x}_1(\varphi_i)$$

$$= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) \qquad \hat{x}_1(\varphi_1) < x < \bar{x}_{22}(\varphi_i)$$

$$= L'(x, \varphi_i) + L_1'(x, \varphi_i) + f_{02}'(x, \varphi_i) \qquad x > \bar{x}_{22}(\varphi_i)$$

Case (c) $\bar{x}_{22}(\varphi_i) \leqq \hat{x}_1(\varphi_i) \ (G_{22}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0)$

$$(3.18) \quad f_2(x, \varphi_i) = c_0(\bar{x}_{22}(\varphi_i) - x) + L(\bar{x}_{22}(\varphi_i), \varphi_i) + L_1(\bar{x}_{22}(\varphi_i), \varphi_i) + f_{0,2}(\bar{x}_{22}(\varphi_i), \varphi_i) \qquad x < \bar{x}_{22}(\varphi_i)$$

$$= L(x, \varphi_i) + L_1(x, \varphi_i) + f_{02}(x, \varphi_i) \qquad x \geqq \bar{x}_{22}(\varphi_i)$$

$$(3.19) \quad f_2'(x, \varphi_i) = -c_0 \qquad x < \bar{x}_{22}(\varphi_i)$$

$$= L'(x, \varphi_i) + L_1'(x, \varphi_i) + f_{02}'(x, \varphi_i) \qquad x > \bar{x}_{22}(\varphi_i)$$

It follows from (3.14), (3.16), and (3.18) that (i) is valid for $n=2$. The expression (3.15), (3.16), and (3.19) with (2.1b), verifies part (ii). If the theorem is true for the integer $n-1$, then we have from (1.8), (2.1a) (3.7), (3.8), and the inductive assumption

$$(3.20) \quad \lim_{u\to\infty} G_{2n}'(u, \varphi_i) = c_2 + \lim_{u\to\infty} \Lambda'^{(2)}(u, \varphi_i) + \lim_{u\to\infty} V_1'(u, \varphi_i) + \lim_{u\to\infty} f_{n-2,2}'(u, \varphi_i)$$

$$= c_2 - \alpha c_1 + \alpha \sum_{i_1=1}^{m} p_{i i_1} \lim_{u\to\infty} \int_0^\infty [c_1 + V'(u-t, \varphi_{i_1}) + f_{n-2,1}'(u-t, \varphi_{i_1})] \varphi_i(t)\, dt > c_2 - \alpha c_1 + \alpha c_1 = c_2 > 0$$

$$(3.21) \quad \lim_{u\to-\infty} G_{2n}'(u, \varphi_i) = c_2 + \lim_{u\to-\infty} \Lambda'^{(2)}(u, \varphi_i) - \alpha v + \lim_{u\to-\infty} f_{n-2,2}'(u, \varphi_i)$$

$$= c_0 + \lim_{u\to-\infty} L'(u, \varphi_i) + \lim_{u\to-\infty} L_1'(u, \varphi_i) + \lim_{u\to-\infty} f_{n-2,2}'(u, \varphi_i)$$

$$< c_1 - v + \alpha(c_1 - c_0 - v) < 0$$

There exists a unique finite $\bar{x}_{2n}(\varphi_i)$ such that $G'_{2n}(x, \varphi_i) = 0$. Uniqueness follows from (2. 1c, d). From the same analysis as in the case $n = 2$, we get the theorem. $f'_n(x, \varphi_i)$ is given by

Case (a) $\bar{x}_{2n}(\varphi_i) > \hat{x}_2(\varphi_i)$

(3.22)
$$\begin{aligned} f'_n(x, \varphi_i) &= -c_0 & x &< \hat{x}_1(\varphi_i) \\ &= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & \hat{x}_1(\varphi_i) &< x < \hat{x}_2(\varphi_i) \\ &= -c_2 + L'(x, \varphi_i) + L'_1(x, \varphi_i) - V'_1(x, \varphi_i) & \hat{x}_2(\varphi_i) &< x < \bar{x}_{2n}(\varphi_i) \\ &= L'(x, \varphi_i) + L'_1(x, \varphi_i) + f'_{n-2,2}(x, \varphi_i) & x &> \bar{x}_{2n}(\varphi_i) \end{aligned}$$

Case (b) $\hat{x}_2(\varphi_i) \geqq \bar{x}_{2n}(\varphi_i) > \hat{x}_1(\varphi_i)$

(3.23)
$$\begin{aligned} f'_n(x, \varphi_i) &= -c_0 & x &< \hat{x}_1(\varphi_i) \\ &= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & \hat{x}_1(\varphi_i) &< x < \bar{x}_{2n}(\varphi_i) \\ &= L'(x, \varphi_i) + L'_1(x, \varphi_i) + f'_{n-2,2}(x, \varphi_i) & x &> \bar{x}_{2n}(\varphi_i) \end{aligned}$$

Case (c) $\bar{x}_{2n}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$

(3.24)
$$\begin{aligned} f'_n(x, \varphi_i) &= -c_0 & x &< \bar{x}_{2n}(\varphi_i) \\ &= L'(x, \varphi_i) + L'_1(x, \varphi_i) + f'_{n-2,2}(x, \varphi_i) & x &> \bar{x}_{2n}(\varphi_i) \end{aligned}$$

The proof is complete.

In order to advance the further discussion, let us define

(3.25)
$$\begin{aligned} g^{(1)}(x, \varphi_i) &= -c_0 & x &< \hat{x}_1(\varphi_i) \\ &= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & x &> \hat{x}_1(\varphi_i) \end{aligned}$$

(3.26)
$$\begin{aligned} g^{(2)}(x, \varphi_i) &= -c_0 & x &< \hat{x}_1(\varphi_i) \\ &= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) & \hat{x}_1(\varphi_i) &< x < \hat{x}_2(\varphi_i) \\ &= -c_2 + L'(x, \varphi_i) + L'_1(x, \varphi_i) - V'_1(x, \varphi_i) & x &> \hat{x}_2(\varphi_i) \end{aligned}$$

Moreover, using (3.25) and (3.26), let us define the next equations

(3.27)
$$F_2^{(0)}(x, \varphi_i) = c_0(1 - \alpha^2) + L'(x, \varphi_i) + L'_1(x, \varphi_i) = 0$$

(3.28)
$$F_2^{(1)}(x, \varphi_i) = c_1 + L'_1(x, \varphi_i) + V'(x, \varphi_i) + g_2^{(1)}(x, \varphi_i) = 0$$

and

(3.29)
$$F_2^{(2)}(x, \varphi_i) = c_2 + V'_1(x, \varphi_i) + g_2^{(2)}(x, \varphi_i) = 0,$$

where $g_2^{(1)}(x, \varphi_i)$ and $g_2^{(2)}(x, \varphi_i)$ are defined by $g^{(1)}(x, \varphi_i)$ and $g^{(2)}(x, \varphi_i)$, respectively, in the identical way as $L_l(x, \varphi_i)$ is defined by $L(x, \varphi_i)$ in (1.11). It is clear that there exist a unique finite $\tilde{x}_2^{(0)}$, $\tilde{x}_2^{(1)}(\varphi_i)$, and $\tilde{x}_2^{(2)}(\varphi_i)$ determined by (3.27), (3.28), and (3.29), respectively. From (3.22), (3.23), (3.24), (3.25) and (3.26) we have

$$(3.30)\quad f'_n(x, \varphi_i) \geqq g^{(2)}(x, \varphi_i) \geqq g^{(1)}(x, \varphi_i) \geqq -c_0 \text{ for all } x \ (n \geqq 2)$$

THEOREM 3.2. *If conditions of Theorem* 3.1 *are satisfied, then for* $n \geqq 4$

(i) $$f'_{n-2}(x, \varphi_i) \geqq f'_n(x, \varphi_i) \quad for\ x < \bar{x}_{2n}(\varphi_i) \qquad i = 1, 2, \cdots, m.$$

(ii) *Case I* $F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) < 0$, *then*

$$\bar{x}_{2n}(\varphi_i) \leqq \bar{x}_{2,n+2}(\varphi_i) \leqq \tilde{x}_2^{(2)}(\varphi_i);$$

*there also exist integers j and l snch that* $j \geqq l$, $j \geqq 5$, $l \geqq 4$, *and* $\bar{x}_{2l}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{2,l+2}(\varphi_i)$.

*Case II If* $F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) < 0 \leqq F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i)$, *then we have*

$$\bar{x}_{2n}(\varphi_i) \leqq \bar{x}_{2,n+2}(\varphi_i) \leqq \tilde{x}_2^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i);$$

*there also exists an integer* $h \geqq 4$ *and* $\bar{x}_{2h}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{2,h+2}(\varphi_i)$,

Case III If $0 \leqq F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i)$, then we have

$$\bar{x}_{2n}(\varphi_i) \leqq \bar{x}_{2,n+2}(\varphi_i) \leqq \tilde{x}_2^{(0)}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$$

Proof (by indution).

Case 1 $F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. Then we have $\hat{x}_2(\varphi_i) < \tilde{x}_2^{(2)}(\varphi_i)$. Since

$$(3.31)\quad G'_{2n}(\tilde{x}_2^{(2)}(\varphi_i), \varphi_i) = c_1 + V'_1(\tilde{x}_2^{(2)}(\varphi_i), \varphi_i) + f'_{n-2,2}(\tilde{x}_2^{(2)}(\varphi_i), \varphi_i) \geqq c_2 + V'_1(\tilde{x}_2^{(2)}(\varphi_i), \varphi_i) + g_2^{(2)}(\tilde{x}_2^{(2)}(\varphi_i), \varphi_i) = 0$$

it follows that $\bar{x}_{2n}(\varphi_i) \leqq \tilde{x}_2^{(2)}(\varphi_i)$ for $n \geqq 4$. Since

$$(3.32)\quad G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) = c_2 + V'_1(\hat{x}_2(\varphi_i), \varphi_i) + f'_{n-2,2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq c_2 + V'_1(\hat{x}_2(\varphi_i), \varphi_i) + g_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) < 0 \qquad \text{for } n \geqq 4$$

and

$$(3.33)\quad G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + L'_1(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) + f'_{n-2,2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq c_1 + L'_1(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) + g_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) = F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) < 0 \qquad \text{for } n \geqq 4$$

there are three posibilities separate treatment, Case 1. A $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. Case 1. B $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Case 1. C $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. If $f'_n(x, \varphi_i) \leqq f'_{n-2}(x, \varphi_i)$ for all $x < \bar{x}_{2n}(\varphi_i)$, again following subcases are possible.

Case 1. A. $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) < 0$, Then we have $G'_{2,2n+2}(\hat{x}_2(\varphi_i), \varphi_i) < 0$ by (3.8)
Case 1. B. $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$. $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Since

$$G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) < 0$$

and

$$G'_{2,n+2}(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + L'_1(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i)$$
$$+ f'_{n-2,2}(\hat{x}_1(\varphi_i), \varphi_i) \leqq G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$$

Two subcases are possible. Case 1. B. $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$, $G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. Case 1. B$_2$. $G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) < 0$, $G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$. Case 1. C. $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. Since

$$G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) < 0$$

and

$$G'_{2,n+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) < 0$$

Three subcases are possible. Case 1.C$_1$ $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. Case 1.C$_2$ $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,n+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,n+2}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Case 1. C$_3$ $G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,n+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. It is noticed that if $\bar{x}_{2,n+2}(\varphi_i) \geqq \bar{x}_{2n}(\varphi_i)$, the conditions of subcases will become simple. Assuming that the theorem holds for $n = 2r$ $(r = 2, 3, \ldots)$, then six subcases are possible (Fig. 2)

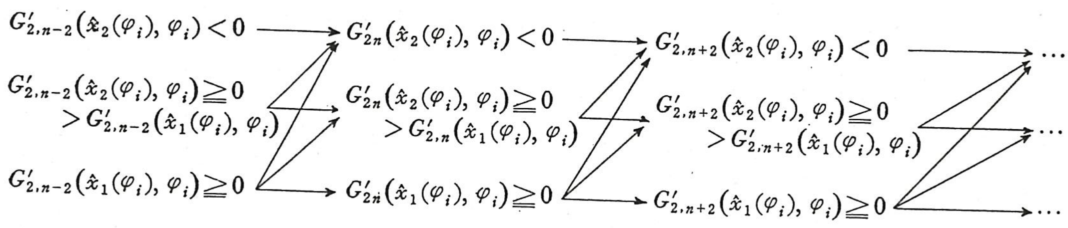


Fig. 2.

Case 1. A. $\hat{x}_1(\varphi_i) < \hat{x}_2(\varphi_i) < \bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(2)}(\varphi_i)$
$(G'_{2,2r}(\hat{x}_2(\varphi_i), \varphi_i) < 0)$

Case 1. B$_1$. $\hat{x}_1(\varphi_i) < \bar{x}_{2,2r}(\varphi_i) \leqq \hat{x}_2(\varphi_i) < \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(2)}(\varphi_i)$
$(G'_{2,2r}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0, G'_{2,2r}(\hat{x}_1(\varphi_i), \varphi_i) < 0)$

Case 1. B$_2$. $\hat{x}_1(\varphi_i) < \bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,2r+2}(\varphi_i) \leqq \hat{x}_2(\varphi_i) < \tilde{x}_2^{(2)}(\varphi_i)$
$(G'_{2,2r}(\hat{x}_1(\varphi_i), \varphi_i) < 0, G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0)$

Case 1. C$_1$. $\bar{x}_{2,2r}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \hat{x}_2(\varphi_i) < \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(2)}(\varphi_i)$

$$(G'_{2,2r}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0,\ G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) < 0)$$

Case 1. $C_2$. $\bar{x}_{2,2r}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{2,2r+2}(\varphi_i) \leqq \hat{x}_2(\varphi_i) < \tilde{x}_2^{(2)}(\varphi_i)$

$$(G'_{2,2r}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0,\ G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0,\ G_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) < 0).$$

Case 1. $C_3$. $\bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,2r+2}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \hat{x}_2(\varphi_i) < \tilde{x}_2^{(2)}(\varphi_i)$

$$(G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0).$$

Since we have

(3.35)
$$-c_1 + L'(x, \varphi_i)\ V'(x, \varphi_i) \geqq -c_0 \qquad x \geqq \hat{x}_1(\varphi_i)$$
$$c_1 - c_1 + L'_1(x, \varphi_i) - V'_1(x, \varphi_i) + V'(x, \varphi_i) \geqq 0 \qquad x \geqq \hat{x}_2(\varphi_i)$$
$$c_2 + V'_1(x, \varphi_i) + f'_{n-2,2}(x, \varphi_i) \geqq 0 \qquad x \geqq \bar{x}_{2n}(\varphi_i)$$

it follows from (3.22), (3.23), and (3.24) that

(3.26)
$$f'_{2r}(x, \varphi_i) \geqq f'_{2r+2}(x, \varphi_i) \qquad x < \bar{x}_{2,2r+2}(\varphi_i)$$

Again subcases are possible, according to the sign of $G'_{2,2r+4}(\hat{x}_1(\varphi_i), \varphi_i)$ and $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i)$ (Fig. 2)

(1) $G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) < 0$, $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. (2) $G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) < 0$, $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. (3) $G'_{2,2r+2}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) < 0$, $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+4}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. (4) $G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i) < 0$. (5) $G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+4}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+4}(\hat{x}_1(\varphi_i), \varphi_i) < 0$. (6) $G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G'_{2,2r+4}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. In the case (2), (4), and (5) it is clear that $\bar{x}_{2,2r+2}(\varphi) \leqq \bar{x}_{2,2r+4}(\varphi_i)$. Since

$$G'_{2,2r+4}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) = G'_{2,2r+4}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) - G'_{2,2r+2}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i)$$
$$= f'_{2r+2,2}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) - f'_{2r,2}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) \leqq 0$$

by (3.13), it follows that $\bar{x}_{2,2r+2}(\varphi_i) \leqq \bar{x}_{2,2r+4}(\varphi_i)$.

Case II. $F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) < 0 \leqq F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i)$. Then we have $\hat{x}_1(\varphi_i) < \tilde{x}_2^{(1)}(\varphi_i)$ and $\hat{x}_2(\varphi_i) \geqq \tilde{x}_2^{(2)}(\varphi_i)$. On the other side

(3.36)
$$F_2^{(1)}(\hat{x}_2(\varphi_i), \varphi_i) = c_1 + L'_1(\hat{x}_2(\varphi_i), \varphi_i) + V'(\hat{x}_2(\varphi_i), \varphi_i) + g_2^{(1)}(\hat{x}_2(\varphi_i), \varphi_i)$$
$$= c_2 + V'_1(\hat{x}_2(\varphi_i), \varphi_i) + g_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i)$$

by (3.4), (3.25) and (3.26)

$$= F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0$$

(3.37)
$$G'_{2n}(\hat{x}_2(\varphi_i), \varphi_i) = c_2 + V'_1(\hat{x}_2(\varphi_i), \varphi_i) + f'_{n-2,2}(\hat{x}_2(\varphi_i), \varphi_i)$$
$$\geqq c_2 + V'_1(\hat{x}_2(\varphi_i), \varphi_i) + g_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i)$$
$$= F_2^{(2)}(\hat{x}_2(\varphi_i), \varphi_i) \geqq 0 \qquad \text{for } n \geqq 4,$$

it follows that $\hat{x}_1(\varphi_i) < \tilde{x}_2^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i)$ and $\bar{x}_{2n}(\varphi_i) \leqq \hat{x}_2(\varphi_i)$. Since

$$(3.38) \quad F_2^{(1)}(\bar{x}_{2n}(\varphi_i), \varphi_i) = c_1 + L_1'(\bar{x}_{2n}(\varphi_i), \varphi_i) + V'(\bar{x}_{2n}(\varphi_i), \varphi_i) + g_2^{(1)}(\bar{x}_{2n}(\varphi_i), \varphi_i) \leqq G_{2n}'(\bar{x}_{2n}(\varphi_i), \varphi_i) = 0,$$

we have $\bar{x}_{2n}(\varphi_i) \leqq \tilde{x}_2^{(1)}(\varphi_i)$ for $n \geqq 4$. If $\bar{x}_{2n}(\varphi_i) < \hat{x}_1(\varphi_i)$, clearly $\bar{x}_{2n}(\varphi_i) < \tilde{x}_2^{(1)}(\varphi_i)$, Since

$$(3.39) \quad G_{2n}'(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + L_1'(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) + f_{n-2,2}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) < 0 \qquad \text{for } n \geqq 4.$$

two subcases are possible. Case II. A. $G_{2n}'(\hat{x}_1(\varphi_i), \varphi_i) < 0$. Case II. B $G_{2n}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. If $f_n'(x, \varphi_i) \leqq f_{n-2}'(x, \varphi_i)$ for all $x < \bar{x}_{2n}(\varphi_i)$, again three subcases are possible, according to the sign of $G_{2,n+2}'(\hat{x}_1(\varphi_i), \varphi_i)$. Assumiag that the theorem holds for the $n = 2r (r = 2, 3, \dots)$, then the following subcases possible (Fig. 2).

Case II. A. $\hat{x}_1(\varphi_i) < \bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i)$ $(G_{2,2r}'(\hat{x}_1(\varphi_i), \varphi_i) < 0)$

Case II. B₁. $\bar{x}_{2,2r}(\varphi_i) < \hat{x}_1(\varphi_i) < \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i)$ $(G_{2,2r}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0,\ G_{2,2r+2}'(\hat{x}_1(\varphi_i), \varphi_i) < 0)$

Case II. B₂. $\bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,r+2}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \tilde{x}_2^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i)$ $(G_{2,2r+2}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0)$

Using (3.35), we have from (3.23) and (3.24)

$$f_{2r}'(x, \varphi_i) \geqq f_{2r+2}'(x, \varphi_i) \qquad \text{for } x < \bar{x}_{2,2r+2}(\varphi_i)$$

Again subcases possible, according to the sign of $G_{2,2r+4}'(\hat{x}_1(\varphi_i), \varphi_i)$ (Fig. 2)

(1) $G_{2,2r+2}'(\hat{x}_1(\varphi_i), \varphi_i) < 0$, $G_{2,2r+4}'(\hat{x}_1(\varphi_i), \varphi_i) < 0$. (2) $G_{2,2r+2}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G_{2,2r+4}'(\hat{x}_1(\varphi_i), \varphi_i) < 0$. (3) $G_{2,2r+2}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$, $G_{2,2r+4}'(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0$. In the case (2), it is clear that $\bar{x}_{2,2r+2}(\varphi_i) < \bar{x}_{2,2r+4}(\varphi_i)$. In the other case, we have

$$G_{2,2r+4}'(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) = G_{2,2r+4}'(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) - G_{2,2r+2}'(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) = f_{2r+2,2}'(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) - f_{2r,2}'(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) \leqq 0;$$

hence $\bar{x}_{2,2r+2}(\varphi_i) \leqq \bar{x}_{2,2r+4}(\varphi_i)$.

Case III $0 \leqq F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i)$. Then we have $\hat{x}_1(\varphi_i) \geqq \tilde{x}_2^{(1)}(\varphi_i)$. Since

$$\begin{aligned} F_2^{(0)}(\hat{x}_1(\varphi_i), \varphi_i) &= c_0(1-\alpha^2) + L'(\hat{x}_1(\varphi_i), \varphi_i) + L_1'(\hat{x}_1(\varphi_i), \varphi_i) \\ &= c_1 + L_1'(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) - \alpha^2 c_0 && \text{by (2.4)} \\ &= F_2^{(1)}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0 && \text{by (3.25)} \end{aligned}$$

and

$$G'_{2n}(\hat{x}_1(\varphi_i), \varphi_i) = c_1 + L'_1(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) + f'_{n-2,2}(\hat{x}_1(\varphi_i), \varphi_i)$$
$$\geqq c_1 + L'_1(\hat{x}_1(\varphi_i), \varphi_i) + V'(\hat{x}_1(\varphi_i), \varphi_i) + g_2^{(2)}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0 \qquad \text{for } n \geqq 4,$$

it follows that $\hat{x}_1(\varphi_i) \geqq \tilde{x}_2^{(0)}$, $\hat{x}_1(\varphi_i) \geqq \bar{x}_{2n}(\varphi_i)$. If we assume the validity of Theorem for the $n = 2r$ $(r = 2, 3, \cdots)$, the following only situation occur (Fig. 2).

Case III. A. $\bar{x}_{2,2r}(\varphi_i) \leqq \bar{x}_{2,2r+2}(\varphi_i) \leqq \tilde{x}_2^{(0)}(\varphi_i) < \hat{x}_1(\varphi_i)$ $\quad (G'_{2,2r+2}(\hat{x}_1(\varphi_i), \varphi_i) \geqq 0)$
Then it follows from (3.25) that

$$(3.40) \qquad f'_{2r}(x, \varphi_i) \geqq f'_{2r+2}(x, \varphi_i) \qquad \text{for } x < \bar{x}_{2,2r+2}(\varphi_i)$$

Using (3.40), we have $G'_{2,2r+4}(\bar{x}_{2,2r+2}(\varphi_i), \varphi_i) \leqq 0$; hence $\bar{x}_{2,2r+2}(\varphi_i) \leqq \bar{x}_{2,2r+4}(\varphi_i)$. The relation

$$G'_{2n}(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) = c_0 + L'(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) + L'_1(\tilde{x}^{0}_{(2)}(\varphi_i), \varphi_i) + f'_{n-2,2}(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i)$$
$$\geqq c_0 + L'(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) + L'_1(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) + g_2^{(1)}(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) \qquad \text{by (3.30)}$$
$$= F_2^{(0)}(\tilde{x}_2^{(0)}(\varphi_i), \varphi_i) = 0 \qquad \text{by (3.25)}$$

implies $\bar{x}_{2n}(\varphi_i) \leqq \tilde{x}_2^{(0)}(\varphi_i)$ for $n \geqq 4$. The proof for the case $n = 2r+1$ $(r = 2, 3, \cdots)$ in the case I, case II, and case III follows from the same method as the above argument. The induction step in (I) and (II) from $n-1$ to $n$ is finished. To complete the proof it is necessary to verify all the property of (I) and (II) for $n = 4, 5$. It follows from (3.22), (3.23), and (3.24) that we have in both case, $\bar{x}_{2j}(\varphi_i) \geqq \bar{x}_{2,j+2}(\varphi_i)$ and $\bar{x}_{2j}(\varphi_i) < \bar{x}_{2,j+2}(\varphi_i)$ $(j = 4, 5)$

$$(3.41) \qquad f'_2(x, \varphi_i) \geqq f'_4(x, \varphi_i) \text{ for } x \leqq \bar{x}_{24}(\varphi_i)$$
$$f'_2(x, \varphi_i) \geqq f'_5(x, \varphi_i) \text{ for } x \leqq \bar{x}_{25}(\varphi_i) \qquad i = 1, 2, \cdots, m.$$

Using the above relation, we obtain the theorem for $n = 4$ and 5 from the same analysis as in the general case and we omit the details.

## §4. Optimal policy under the general modes of delivery

In this section we analyze the dynamic model for the case $k (\geqq 3)$. We obtain from (1.11)

$$(4.1) \qquad f_n(x, \varphi_i) = \min_{u_k \geqq u_{k-1} > \cdots \geqq u_0 \geqq x} \{ \sum_{j=1}^{k} [(c_{j-1} - c_j)(u_{j-1} - x) + L_{j-1}(u_{j-1}, \varphi_i)$$
$$- V_{j-1}(u_{j-1}, \varphi_i) + V_{j-2}(u_{j-1}, \varphi_i)] + c_k(u_k - x) + V_{k-1}(u_k, \varphi_i)$$

$$+f_{n-k,k}(u_k,\varphi_i)\} \qquad n\geqq k$$

$$f_n(x,\varphi_i)=\min_{u_n\geqq u_{n-1}\leqq\cdots\geqq u_0\geqq x}\{\sum_{j=1}^{n}[(c_{j-1}-c_j)(u_{j-1}-x)+L_{j-1}(u_{j-1},\varphi_i)$$
$$-V_{j-1}(u_{j-1}\ \varphi_i)+V_{j-2}(u_{j-1},\varphi_i)]+c_n(u_n-x)+V_{n-1}(u_n,\varphi_i)$$
$$-W_n(u_n,\varphi_i)\} \qquad n<k$$

$$i=1,2,\cdots,m.$$

If we use repeately the technique which was applied to reduced (3.1) to (3.6) in §3, then it will be inductively seen that (4.1) may be reduced to (4.2)

(4.2)
$$f_n(x,\varphi_i)=\sum_{l=1}^{k}\tilde{L}^{(l)}(x,\varphi_i)+\min_{u_k\geqq x}\{c_k(u_k-x)+\Lambda^{(k)}(u_k,\varphi_i)+V_{k-1}(u_k,\varphi_i)$$
$$+f_{n-k,k}(u_k,\varphi_i) \qquad n\geqq k$$

$$f_n(x,\varphi_i)=\sum_{l=1}^{n}\tilde{L}^{(l)}(x,\varphi_i)+\min_{u_n\geqq x}\{c_n(u_n-x)+\Lambda^{(n)}(u_n,\varphi_i)+V_{n-1}(u_n,\varphi_i)$$
$$-W_n(u_n,\varphi_i)\} \qquad n<k$$

where $\tilde{L}^{(j)}(x,\varphi_i)$ and $\Lambda^{(j)}(y,\varphi_i)$ are given by

(4.3)
$$\tilde{L}^{(j)}(x,\varphi_i)=\min_{u_{j-1}\geqq x}\{(c_{j-1}-c_j)(u_{j-1}-x)+\Lambda^{(j-1)}(u_{j-1},\varphi_i)$$
$$+L_{j-1}(u_{j-1},\varphi_i)-V_{j-1}(u_{j-1},\varphi_i)+V_{j-2}(u_{j-1},\varphi_i)\}$$

$$i=1,2,\cdots,m,$$
$$j=1,2,\cdots,k.$$

(4.4)
$$\Lambda^{(j)}(y,\varphi_i)=\begin{cases}(c_{j-1}-c_j)(y-\hat{x}_j(\varphi_i))+(\Lambda^{(j-1)}(y,\varphi_i)+L_{j-1}(y,\varphi_i)\\ \quad -V_{j-1}(y,\varphi_i)+V_{j-2}(y,\varphi_i))-(\Lambda^{j-1}(\hat{x}_j(\varphi_i),\varphi_i)\\ \quad +L_{j-1}(\hat{x}_j(\varphi_i),\varphi_i)-V_{j-1}(\hat{x}_j(\varphi_i),\varphi_i)+V_{j-2}(\hat{x}_j(\varphi_i),\varphi_i)) & y<\hat{x}_j(\varphi_i)\\ 0 & y\geqq\hat{x}_j(\varphi_i)\end{cases}$$
$$i=1,2,\cdots,m;\ j=1,2,\cdots,k,$$

where $\Lambda^{(0)}(y,\varphi_i)=0$ and $\hat{x}_j(\varphi_i)$ is a unique root of the equation

(4.5)
$$M^{(j)}(y,\varphi_i)=c_{j-1}-c_j+\Lambda^{(j-1)\prime}(y,\varphi_i)+L'_{j-1}(y,\varphi_i)-V'_{j-1}(y,\varphi_i)$$
$$+V'_{j-2}(y,\varphi_i)=0 \qquad i=1,2,\cdots,\ j=1,2,\cdots,k.$$

Let us define

(4.6)
$$G_{kn}(y,\varphi_i)=c_k y+\Lambda^{(k)}(y,\varphi_i)+V_{k-1}(y,\varphi_i)+f_{n-k,k}(y,\varphi_i)$$

$$i=1, 2, \cdots, m.$$

At $x=\hat{x}_j(\varphi_i)$ $(j=1, 2, \cdots, k)$, the derivative $G'_{kn}(u, \varphi_i)$ are given by

$$G'_{kn}(\hat{x}_k(\varphi_i), \varphi_i)=c_k+V'_{k-1}(\hat{x}_k(\varphi_i), \varphi_i)+f'_{n-k,k}(\hat{x}_k(\varphi_i), \varphi_i)$$

$$G'_{kn}(\hat{x}_{k-1}(\varphi_i), \varphi_i)=c_{k-1}+L'_{k-1}(\hat{x}_{k-1}(\varphi_i), \varphi_i)+V'_{k-2}(\hat{x}_{k-1}(\varphi_i), \varphi_i) +f'_{n-k,k}(\hat{x}_{k-1}(\varphi_i), \varphi_i)$$

$$(4.7)\quad G'_{kn}(\hat{x}_{k-2}(\varphi_i), \varphi_i)=c_{k-2}+L'_{k-2}(\hat{x}_{k-2}(\varphi_i), \varphi_i)+L'_{k-1}(\hat{x}_{k-2}(\varphi_i), \varphi_i) + V'_{k-3}(\hat{x}_{k-2}(\varphi_i), \varphi_i)+f'_{n-k,k}(\hat{x}_{k-2}(\varphi_i), \varphi_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$G'_{kn}(\hat{x}_1(\varphi_i), \varphi_i)=c_1+\sum_{j=2}^{k}L'_{j-1}(\hat{x}_1(\varphi_i), \varphi_i)+V'(\hat{x}_1(\varphi_i), \varphi_i) +f'_{n-k,k}(\hat{x}_1(\varphi_i), \varphi_i)$$

Then the following theorm which is analogous to Theorem 3.1 is seen be true

THEOREM 4.1. *Let conditions of Theorem 2.1 be valid, if* $\hat{x}_k(\varphi_i)>\hat{x}_{k-1}(\varphi_i)>\cdots>\hat{x}_1(\varphi_i)$, *then the optimal ordering policy is of the following form for* $n\geqq k$

*Case* (1) *If* $\bar{x}_{kn}(\varphi_i)>\hat{x}_k(\varphi_i)$, *it is optimal to order*:

1. *for* $x<\hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i)-x$ *at* $c_0$, *amount* $\hat{x}_2(\varphi_i)-\hat{x}_1(\varphi_i)$ *at* $c_1$, $\cdots$, *amount* $\hat{x}_k(\varphi_i)-\hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$, *and amount* $\bar{x}_{kn}(\varphi_i)-\hat{x}_k(\varphi_i)$ $c_k$;
2. *for* $\hat{x}_1(\varphi_i)\leqq x<\hat{x}_2(\varphi_i)$, *amount* $\hat{x}_2(\varphi_i)-x$ *at* $c_1$, *amount* $\hat{x}_3(\varphi_i)-\hat{x}_2(\varphi_i)$ *at* $c_2$, $\cdots$, *amount* $\hat{x}_k(\varphi_i)-\hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$, *and amount* $\bar{x}_{kn}(\varphi_i)-\hat{x}_k(\varphi_i)$ *at* $c_k$;

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$k-1$. *for* $\hat{x}_{k-2}(\varphi_i)\leqq x<\hat{x}_{k-1}(\varphi_i)$ *amount* $\hat{x}_{k-1}(\varphi_i)-x$ *at* $c_{k-2}$, *amount* $\hat{x}_k(\varphi_i)-\hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$ *and amount* $\bar{x}_{kn}(\varphi_i)-\hat{x}_k(\varphi_i)$ *at* $c_k$;

$k$. *for* $\hat{x}_{k-1}(\varphi_i)\leqq x<\hat{x}_k(\varphi_i)$, *amount* $\hat{x}_k(\varphi_i)-x$ *at* $c_{k-1}$, *and amount* $\bar{x}_{kn}(\varphi_i)-\hat{x}_k(\varphi_i)$ *at* $c_k$;

$k+1$. *for* $\hat{x}_k(\varphi_i)\leqq x<\bar{x}_{kn}(\varphi_i)$, *amount* $\bar{x}_{kn}(\varphi_i)-x$ *at* $c_k$;

$k+2$. *for* $x\geqq\bar{x}_{kn}(\varphi_i)$, *none*,

*Case* (2) *If* $\hat{x}_k(\varphi_i)\geqq\bar{x}_{kn}(\varphi_i)>\hat{x}_{k-1}(\varphi_i)$, *it is optimal to order*:

1. *for* $x<\hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i)-x$ *at* $c_0$, *amount* $\hat{x}_2(\varphi_i)-\hat{x}_1(\varphi_i)$ *at* $c_1$, $\cdots$, *amount* $\hat{x}_{k-1}(\varphi_i)-\hat{x}_{k-2}(\varphi_i)$ *at* $c_{k-2}$, *and amount* $\bar{x}_{kn}(\varphi_i)-\hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$;
2. *for* $\hat{x}_1(\varphi_i)\leqq x<\hat{x}_2(\varphi_i)$, *amonnt* $\hat{x}_2(\varphi_i)-x$ *at* $c_1$, *amount* $\hat{x}_3(\varphi_i)-\hat{x}_2(\varphi_i)$ *at* $c_2$, $\cdots$, *amount* $\hat{x}_{k-1}(\varphi_i)-\hat{x}_{k-2}(\varphi_i)$ *at* $c_{k-2}$, *and amount* $\bar{x}_{kn}(\varphi_i)$

$-\hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$;

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$k-2$. *for* $\hat{x}_{k-3}(\varphi_i) \leqq x < \hat{x}_{k-2}(\varphi_i)$, *amount* $\hat{x}_{k-2}(\varphi_i) - x$ *at* $c_{k-3}$, *amount* $\hat{x}_{k-1}(\varphi_i) - \hat{x}_{k-2}(\varphi_i)$ *at* $c_{k-2}$ *and amount* $\bar{x}_{kn}(\varphi_i) - \hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$;

$k-1$. *for* $\hat{x}_{k-2}(\varphi_i) \leqq x < \hat{x}_{k-1}(\varphi_i)$, *amount* $\hat{x}_{k-1}(\varphi_i) - x$ *at* $c_{k-2}$, *and amount* $\bar{x}_{kn}(\varphi_i) - \hat{x}_{k-1}(\varphi_i)$ *at* $c_{k-1}$;

$k$. *for* $\hat{x}_{k-1}(\varphi_i) \leqq x < \bar{x}_{kn}(\varphi_i)$, *amount* $\bar{x}_{kn}(\varphi_i) - x$ *at* $c_{k-1}$;

$k+1$. *for* $k+1$. *for* $x \geqq \bar{x}_{kn}(\varphi_i)$, *none.*

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Case $(k-1)$ *If* $\hat{x}_3(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i) > \hat{x}_2(\varphi_i)$, *it is optimal to order:*

1. *for* $x < \hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i) - x$ *at* $c_0$, *amount* $\hat{x}_2(\varphi_i) - \hat{x}_1(\varphi_i)$ *at* $c_1$, *and amount* $\bar{x}_{kn}(\varphi_i) - \hat{x}_2(\varphi_i)$ *at* $c_2$;
2. *for* $\hat{x}_1(\varphi_i) \leqq x < \hat{x}_2(\varphi_i)$, *amount* $\hat{x}_2(\varphi_i) - x$ *at* $c_1$, *and amount* $\bar{x}_{kn}(\varphi_i) - \hat{x}_2(\varphi_i)$ *at* $c_2$;
3. *for* $\hat{x}_2(\varphi_i) < x < \bar{x}_{kn}(\varphi_i)$, *amount* $\bar{x}_{kn}(\varphi_i) - x$ *at* $c_2$;
4. *for* $x \geqq \bar{x}_{kn}(\varphi_i)$, *none.*

Case $(k)$ *If* $\hat{x}_2(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i) > \hat{x}_1(\varphi_i)$, *it is optimal to order:*

1. *for* $x < \hat{x}_1(\varphi_i)$, *amount* $\hat{x}_1(\varphi_i) - x$ *at* $c_0$, *and amount* $\bar{x}_{kn}(\varphi_i) - \hat{x}_1(\varphi_i)$ *at* $c_1$;
2. *for* $\hat{x}_1(\varphi_i) \leqq x < \bar{x}_{kn}(\varphi_i)$, *amount* $\bar{x}_{kn}(\varphi_i) - x$ *at* $c_1$;
3. *for* $x \geqq \bar{x}_k(\varphi_i)$, *none.*

Case $(k+1)$ *If* $\hat{x}_1(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i)$, *it is optimal to order:*

1. *for* $x < \bar{x}_{kn}(\varphi_i)$, *amount* $\bar{x}_{kn}(\varphi_i) - x$ *at* $c_0$;
2. *for* $x \geqq \bar{x}_{kn}(\varphi_i)$, *none.*

*Furthermore, the following properties hold:*

(i) $\bar{x}_{kn}(\varphi_i)$ *is a unique root of equation* $G'_{kn}(y, \varphi_i) = 0$ *where* $G'_{kn}(y, \varphi_i)$ *is given by*

$$G'_{kn}(y, \varphi_i) = c_0 + \sum_{j=1}^{k} L'_{j-1}(y, \varphi_i) + f'_{n-k,k}(y, \varphi_i) \qquad y < \hat{x}_1(\varphi_i)$$

$$= c_1 + \sum_{j=2}^{k} L'_{j-1}(y, \varphi_i) + V'(y, \varphi_i) + f'_{n-k,k}(y, \varphi_i) \qquad \hat{x}_1(\varphi_i) < y < \hat{x}_2(\varphi_i)$$

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$$= c_{k-1} + L'_{k-1}(y, \varphi_i) + V'_{k-2}(y, \varphi_i) + f'_{n-k,k}(y, \varphi_i) \qquad \hat{x}_{k-1}(\varphi_i) < y < \hat{x}_k(\varphi_i)$$

$$= c_k + V'_{k-1}(y, \varphi_i) + f'_{n-k,k}(y, \varphi_i) \qquad y > \hat{x}_k(\varphi_i)$$

(ii) Case (I) $\bar{x}_{kn}(\varphi_i) > \hat{x}_k(\varphi_i)$

$$f'_n(x, \varphi_i) = -c_0 \qquad x < \hat{x}_1(\varphi_i)$$

$$= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) \qquad \hat{x}_1(\varphi_i) < x < \hat{x}_2(\varphi_i)$$

$$= -c_2 + L'(x, \varphi_i) + L'_1(x, \varphi_i) - V'(x, \varphi_i)_1 \qquad \hat{x}_2(\varphi_i) < x < \hat{x}_3(\varphi_i)$$

$$= -c_3 + L'(x, \varphi_i) + L'_1(x, \varphi_i) + L'_2(x, \varphi_i) - V'_2(x, \varphi_i) \qquad \hat{x}_3(\varphi_i) < x < \hat{x}_4(\varphi_i)$$

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$$= -c_{k-1} + \sum_{j=1}^{k-1} L'_{j-1}(x, \varphi_i) - V'_{k-2}(x, \varphi_i) \qquad \hat{x}_{k-1}(\varphi_i) < x < \hat{x}_k(\varphi_i)$$

$$= -c_k + \sum_{j=1}^{k} L'_{j-1}(x, \varphi_i) - V'_{k-1}(x, \varphi_i) \qquad \hat{x}_k(\varphi_i) < x < \bar{x}_{kn}(\varphi_i)$$

$$= \sum_{j=1}^{k} L'_{j-1}(x, \varphi_i) + f'_{n-k,k}(x, \varphi_i) \qquad x > \bar{x}_{kn}(\varphi_i)$$

Case (2) $\hat{x}_k(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i) > \hat{x}_{k-1}(\varphi_i)$

$$f'_n(x, \varphi_i) = -c_0 \qquad x < \hat{x}_1(\varphi_i)$$

$$= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) \qquad \hat{x}_1(\varphi_i) < x < \hat{x}_2(\varphi_i)$$

$$= -c_2 + L'(x, \varphi_i) + L'_1(x, \varphi_i) - V'_1(x, \varphi_i) \qquad \hat{x}_2(\varphi_i) < x < \hat{x}_3(\varphi_i)$$

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$$= -c_{k-2} + \sum_{j=1}^{k-2} L'_{j-1}(x, \varphi_i) - V'_{k-3}(x, \varphi_i) \qquad \hat{x}_{k-2}(\varphi_i) < x < \hat{x}_{k-1}(\varphi_i)$$

$$= -c_{k-1} + \sum_{j=1}^{k-1} L'_{j-1}(x, \varphi_i) - V'_{k-2}(x, \varphi_i) \qquad \hat{x}_{k-1}(\varphi_i) < x < \bar{x}_{kn}(\varphi_i)$$

$$= \sum_{j=1}^{k} L'_{j-1}(x, \varphi_i) + f'_{n-k,k}(x, \varphi_i) \qquad x > \bar{x}_{kn}(\varphi_i)$$

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*Case* ($k$) $\hat{x}_2(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i) > \hat{x}_1(\varphi_i)$

$$f'_n(x, \varphi_i) = -c_0 \qquad x < \hat{x}_1(\varphi_i)$$

$$= -c_1 + L'(x, \varphi_i) - V'(x, \varphi_i) \qquad \hat{x}_1(\varphi_i) < x < \bar{x}_{kn}(\varphi_i)$$

$$= \sum_{j=1}^{k} L'_{j-1}(x, \varphi_i) + f'_{n-k,k}(x, \varphi_i) \qquad x > \bar{x}_{kn}(\varphi_i)$$

*Case* ($k+1$) $\hat{x}_1(\varphi_i) \geqq \bar{x}_{kn}(\varphi_i)$:

$$f'_n(x, \varphi_i) = -c_0 \qquad x < \bar{x}_{kn}(\varphi_i)$$

$$=\sum_{j=1}^{k}L'_{j-1}(x,\varphi_i)+f'_{n-k,k}(x,\varphi_i) \qquad x>\bar{x}_{kn}(\varphi_i)$$

(iii) $f_n(x,\varphi_i)$ is a convex function of $x$, and $f'_n(x,\varphi_i)\geqq g^{(k)}(x,\varphi_i)\geqq\cdots\geqq g^{(1)}(x,\varphi_i)$ $\geqq -c_0$ for all $x$, where $g^{(j)}(x,\varphi_i)$ $(j=1, 2, \cdots, k)$ are given as follows

$$g^{(1)}(x,\varphi_i)=-c_0 \qquad x<\hat{x}_1(\varphi_i)$$

$$=-c_1+L'(x,\varphi_i)-V'(x,\varphi_i) \qquad x>\hat{x}_1(\varphi_i)$$

$$g^{(2)}(x,\varphi_i)=-c_0 \qquad x<\hat{x}_1(\varphi_i)$$

$$=-c_1+L'(x,\varphi_i)-V'(x,\varphi_i) \qquad \hat{x}_1(\varphi_i)<x<\hat{x}_2(\varphi_i)$$

$$=-c_2+L'(x,\varphi_i)+L'_1(x,\varphi_i)-V'_1(\varphi,\varphi_i) \qquad x>\hat{x}_2(\varphi_i)$$

$$g^{(3)}(x,\varphi_i)=-c_0 \qquad x<\hat{x}_1(\varphi_i)$$

$$=-c_1+L'(x,\varphi_i)-V'(x,\varphi_i) \qquad \hat{x}_1(\varphi_i)<x<\hat{x}_2(\varphi_i)$$

$$=-c_2+L'(x,\varphi_i)+L'_1(x,\varphi_i)-V'_1(x,\varphi_i) \qquad \hat{x}_2(\varphi_i)<x<\hat{x}_2(\varphi_i)$$

$$=-c_3+L'(x,\varphi_i)+L'_1(x,\varphi_i)+L'_2(x,\varphi_i)-V'_2(x,\varphi_i)$$

$$x>\hat{x}_3(\varphi_i)$$

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$$g^{(k)}(x,\varphi_i)=-c_0 \qquad x<\hat{x}_1(\varphi_i)$$

$$=-c_1+L'(x,\varphi_i)-V'(x,\varphi_i) \qquad \hat{x}_1(\varphi_i)<x<\hat{x}_2(\varphi_i)$$

$$=-c_2+L'(x,\varphi_i)+L'_1(x,\varphi_i)-V'_1(x,\varphi_i) \qquad \hat{x}_2(\varphi_i)<x<\hat{x}_3(\varphi_i)$$

$$=-c_3+L'(x,\varphi_i)+L'_1(x,\varphi_i)+L'_2(x,\varphi_i)$$

$$-V'_2(x,\varphi_i) \qquad \hat{x}_3(\varphi_i)<x<\hat{x}_4(\varphi_i)$$

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$$=-c_{k-1}+\sum_{j=1}^{k-1}L'_{j-1}(x,\varphi_i)-V'_{k-2}(x,\varphi_i) \qquad \hat{x}_{k-1}(\varphi_i)<x<\hat{x}_k(\varphi_i)$$

$$=-c_k+\sum_{j=1}^{k}L'_{j-1}(x,\varphi_i)-V'_{k-1}(x,\varphi_i) \qquad x>\hat{x}_k(\varphi_i)$$

Proof is similar to that of Theorem 3.1

Let us dsignate by $\tilde{x}_k^{(j)}(\varphi_i)$ $(j=0, 1, \cdots, k)$ the unique roots of the equations

$$F_k^{(0)}(x,\varphi_i)=c_0(1-\alpha^k)+\sum_{j=1}^{k}L'_{j-1}(x,\varphi_i)=0,$$

$$F_k^{(1)}(x,\varphi_i)=c_1+\sum_{j=2}^{k}L'_{j-1}(x,\varphi_i)+V'(x,\varphi_i)+g_k^{(1)}(x,\varphi_i)=0,$$

$$F_k^{(2)}(x, \varphi_i)=c_2+\sum_{j=3}^{k}L'_{j-1}(x, \varphi_i)+V'_1(x, \varphi_i)+g_k^{(2)}(x, \varphi_i)=0,$$

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$$F_k^{(k-1)}(x, \varphi_i)=c_{k-1}+L'_{k-1}(x, \varphi_i)+V'_{k-2}(x, \varphi_i)+g_k^{(k-1)}(x, \varphi_i)=0,$$

$$F_k^{(k)}(x, \varphi_i)=c_k+V'_{k-1}(x, \varphi_i)+g_k^{(k)}(x, \varphi_i)=0,$$

respectively, where $g_k^{(j)}(x, \varphi_i)$ $(j=1, 2, \cdots, k)$ are defined by $g^{(j)}(x, \varphi_i)$ $(j=1, 2, \cdots, k)$, respectively, in the identical way as $L_l(\varphi, \varphi_i)$ is defined by $L(x, \varphi_i)$ in (1.11)

THEOREM 4.2. *If conditions of Theorem* 3.1 *are satistied, then for* $n \geqq 2k$

(i) $$f'_{n-k}(x, \varphi_i) \geqq f'_n(x, \varphi_i) \qquad \textit{for } x < \bar{x}_{kn}(\varphi_i)$$

(ii) *Case I If* $F_k^{(k)}(\hat{x}_k(\varphi_i), \varphi_i)<0$, *then*

$$\bar{x}_{kn}(\varphi_i) \leqq \bar{x}_{k,n+k}(\varphi_i) \leqq \tilde{x}_k^{(k)}(\varphi_i);$$

*there also exist integers* $j_{11}, j_{12}, \cdots, j_{1,k-1}$ *and* $j_{1k}$ *such that* $j_{11} \geqq j_{12} \geqq \cdots \geqq j_{1k}$, $j_{11} \geqq 3k-1$, $j_{12} \geqq 3k-2$, $\cdots$, $j_{1k} \geqq 2k$, $\bar{x}_{k,j_{11}}(\varphi_i) \leqq \hat{x}_k(\varphi) < \bar{x}_{k,j_{11}+k}(\varphi_i)$, $\cdots$, $\bar{x}_{k,j_{1k}}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{k,j_{1k}+k}(\varphi_i)$.

*Case* 2 *If* $F_k^{(k-1)}(\hat{x}_{k-1}(\varphi_i), \varphi_i)<0 \leqq F_k^{(k)}(\hat{x}_k(\varphi_i), \varphi_i)$, *then*

$$\bar{x}_{kn}(\varphi_i) \leqq \bar{x}_{k,n+k}(\varphi_i) \leqq \tilde{x}_k^{(k-1)}(\varphi_i) \leqq \hat{x}_k(\varphi_i);$$

*there also exist integers* $j_{22}, \cdots, j_{2,k-1}$ *and* $j_{2,k}$ *such that* $j_{22} \geqq j_{22} \geqq \cdots \geqq j_{2k}$, $j_{22} \geqq 3k-2$, $\cdots$, $j_{2k} \geqq 2k$, $\bar{x}_{k,j_{22}}(\varphi_i) \leqq \hat{x}_{k-1}(\varphi_i) < \bar{x}_{k,j_{22}+k}(\varphi_i)$, $\cdots$, $\bar{x}_{k,j_{2k}}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{k,j_{2k}+k}(\varphi_i)$.

*Case* 3 *If* $F_k^{(k-2)}(\hat{x}_{k-2}(\varphi_i), \varphi_i)<0 \leqq F_k^{(k-1)}(\hat{x}_{k-2}(\varphi_i), \varphi_i)$, *then*

$$\bar{x}_{kn}(\varphi_i) \leqq \bar{x}_{k,n+k}(\varphi_i) \leqq \tilde{x}_k^{(k-2)}(\varphi_i) \leqq \hat{x}_{k-1}(\varphi_i);$$

*there also exist integers* $j_{33}, \cdots, j_{3,k-1}$ *and* $j_{3k}$ *such that* $j_{33} \geqq j_{34} \geqq \cdots, \geqq j_{3k}$; $j_{33} \geqq 3k-3$, $\cdots$, $j_{3k} \geqq 2k$, $\bar{x}_{k,j_{33}}(\varphi_i) \leqq \hat{x}_{k-2}(\varphi_i) < \bar{x}_{k,j_{33}+k}(\varphi_i)$, $\cdots$, $\bar{x}_{k,j_{3k}}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{k,j_{3k}+k}(\varphi_i)$

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*Case* $k$ *If* $F_k^{(1)}(\hat{x}_1(\varphi_i), \varphi_i)<0 \leqq F_k^{(2)}(\hat{x}_2(\varphi_i), \varphi_i)$, *then*

$$\bar{x}_{kn}(\varphi_i) \leqq \bar{x}_{k,n+k}(\varphi_i) \leqq \tilde{x}_k^{(1)}(\varphi_i) \leqq \hat{x}_2(\varphi_i);$$

*there also exists an integer* $j_{kk}$ *such that* $j_{kk} \geqq 2k$ *and* $\bar{x}_{k,j_{kk}}(\varphi_i) \leqq \hat{x}_1(\varphi_i) < \bar{x}_{k,j_{kk}+k}(\varphi_i)$

*Case* $k+1$ *If* $F_k^{(1)}(\tilde{x}_1(\varphi_i), \varphi_i) \geqq 0$, *then*

$$\bar{x}_{kn}(\varphi_i) \leqq \bar{x}_{k,n+k}(\varphi_i) \leqq \tilde{x}_k^{(0)}(\varphi_i) \leqq \hat{x}_1(\varphi_i)$$

PROOF. By using (i), (ii), and (iii) of Theorem 4.1, this is proved in a similar fashion as the proof of Theorem 3.2.

We also remark that similar results as above will also be obtained, according to the definite relationship amoung $\hat{x}_1(\varphi_i)$, $\cdots$, $\hat{x}_{k-1}(\varphi_i)$ and $\hat{x}_k(\varphi_i)$.

### §5. Ackno wledgement.

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