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ON VARIANTS OF LIE TRIPLE SYSTEMS AND THEIR LIE ALGEBRAS

by

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In this paper we study from a more general point of view an algebraic construction which appeared in *E.* Cartan's classification of simple real Lie algebras (see, for example, [3] and [7]) and which has turned up again in connection with the relation of Lie algebras of type $\mathfrak{E}_6$ and the exceptional simple Jordan algebras ([8]).

One way to formulate the general situation is as follows: Let $\mathfrak{A}$ be a non associative algebra over a field Φ, and suppose that S is a *reflection* in $\mathfrak{A}$ in the sense that S is an automorphism of $\mathfrak{A}$ with $S^2 = I$ and $S \neq I$. Let $\mathfrak{A} = \mathfrak{A}_1(S) \oplus \mathfrak{A}_{-1}(S) \equiv \mathfrak{A}_1 \oplus \mathfrak{A}_{-1}$ be the associated decomposition of $\mathfrak{A}$. Now $\mathfrak{A}_1$ is a subalgebra and $\mathfrak{A}_{-1}$ is, in the obvious sense, a sub triple system of $\mathfrak{A}$. For any $\alpha \in \Phi$ a new algebra $(\mathfrak{A}, S, \alpha) = \mathfrak{B}$ can be built on the vector space $\mathfrak{A}$ as follows:

i) If $a, b \in \mathfrak{A}_{-1}$ the product $\{ab\}$ in $\mathfrak{B}$ is given by $\{ab\} = \alpha ab$.

ii) $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{B}$.

iii) $\mathfrak{A}_1$ acts on $\mathfrak{A}_{-1}$ in $\mathfrak{B}$ just as it does in $\mathfrak{A}$.

Examples show that $\mathfrak{B}$ is not generally isomorphic to $\mathfrak{A}$. This suggests the twin problems of determining criteria for the isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ and of classifying the algebras $\mathfrak{B}$ which can be obtained from $\mathfrak{A}$ by varying S and α. Note that if $\mathfrak{A}$ belongs to a class of algebras defined by multilinear identities, then $\mathfrak{B}$ is in the same class. In the sequel $\mathfrak{A}$ will be a Lie algebra, $\mathfrak{A}_{-1}$ consequently a Lie triple system. Rather than proceed from $\mathfrak{A}$ and S we will begin with a more basic Lie triple system construction. Results on Lie triple systems cited without reference may be found in [6].

1. Variants and Twists.

Let $\mathfrak{T}$ be a Lie triple system over a field Φ of characteristic 0, and suppose that the ternary operation in $\mathfrak{T}$ is $(a, b, c) \rightarrow [abc]$. For each $\alpha \in \Phi^*$ denote by $\mathfrak{T}^\alpha$ the triple system consisting of the vector space of $\mathfrak{T}$ and the operation

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$$\langle abc \rangle = \alpha[abc].$$

Since the class of Lie triple systems is defined by multilinear identities $\mathfrak{T}^\alpha$ is a Lie triple system, which will be called the α-variant of $\mathfrak{T}$.

THEOREM 1. a) $\mathfrak{T}$ and $\mathfrak{T}^\alpha$ have the same subalgebras, ideals, inner derivations, and the same automorphisms. b) if $\beta \in (\Phi^*)^2$ then $\mathfrak{T}^\beta \cong \mathfrak{T}^\alpha$. In particular, if a field $\Omega \supseteq \Phi(\alpha^{1/2})$ then $\mathfrak{T}$ and $\mathfrak{T}^\alpha$ are isomorphic over Ω, i. e., $\mathfrak{T}_\Omega \cong (\mathfrak{T}^\alpha)_\Omega$.

PROOF. Part a) is easily verified. It is worth noting, however, that if $D_{[ab]}: x \to [abx]$, then $D_{\langle ab \rangle} = \alpha D_{[ab]}$. Part b) follows from the observation that if $\gamma = \beta^{-2}$ then $x \to \beta x$ is an isomorphism of $\mathfrak{T}^\alpha$ and $\mathfrak{T}^{\alpha\gamma}$.

Let S be a reflection of a Lie algebra $\mathfrak{L}$, and let $\mathfrak{T} = \mathfrak{L}_{-1}(S)$. Then S induces a reflection in the ideal $\mathfrak{M} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$ and $[\mathfrak{T}\mathfrak{T}] = \mathfrak{M}_1(S)$. Conversely if $\mathfrak{T}_0$ is a triple system for which $\mathfrak{T}_0 \oplus [\mathfrak{T}_0\mathfrak{T}_0] = \mathfrak{L}$ then the unique endomorphism S_0 such that $\mathfrak{T}_0 = \mathfrak{L}_{-1}(S_0)$ is a reflection of $\mathfrak{L}$. Given a reflection S of $\mathfrak{L}$ and $\alpha \in \Phi^*$, define another algebra with product $\langle \, \rangle$ on the vector space of $\mathfrak{L}$ by setting:

i) $\langle ab \rangle \equiv \alpha[ab]$ for $a, b \in \mathfrak{T}$, $\mathfrak{T} = \mathfrak{L}_1(S)$.

ii) $\langle ab \rangle \equiv [ab]$ if both a and b are in $[\mathfrak{T}\mathfrak{T}]$ or if one is in $\mathfrak{T}$ and the other in $[\mathfrak{T}\mathfrak{T}]$. We will call this algebra the *α-twist* of $\mathfrak{L}$ with respect to S, and denote it by $(\mathfrak{L}, S, \alpha)$.

In the sequel we shall be concerned with semi-simple $\mathfrak{T}$. In this case any imbedding of $\mathfrak{T}$ in an enveloping algebra $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$ is equivalent to a *standard* imbedding, namely the semidirect sum of $\mathfrak{T}$ and its algebra $\mathcal{I}(\mathfrak{T})$ of inner derivations. This amounts to the assertion that for $a, b \in T$, $[ab] \to D_{[ab]}$ defines an isomorphism of $[\mathfrak{T}\mathfrak{T}]$ and $\mathcal{I}(\mathfrak{T})$. For such imbeddings $\mathfrak{L}$ is semi-simple and universal for $\mathfrak{T}$. Conversely if S is a reflection of a semi-simple algebra $\mathfrak{L}$ and $\mathfrak{T} = \mathfrak{L}_{-1}(S)$, then $\mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$, the imbedding of $\mathfrak{T}$ in $\mathfrak{L}$ is standard and $\mathfrak{T}$ is semi-simple. We will denote by $\mathcal{L}_s(\mathfrak{T})$ the standard enveloping algebra of $\mathfrak{T}$.

THEOREM 2. If $\mathfrak{T}$ is semi-simple then $\mathfrak{T}^\alpha$ is semi-simple and $\mathcal{L}_s(\mathfrak{T}^\alpha)$ is the α-twist of $\mathcal{L}_s(\mathfrak{T})$ with respect to the reflection S of $\mathfrak{T}$ such that $\mathfrak{T} = \mathfrak{L}_{-1}(S)$. Conversely if $\mathfrak{L}$ is semi-simple, S is a reflection of $\mathfrak{L}$ and $\mathfrak{T} = \mathfrak{L}_{-1}(S)$, then $(\mathfrak{L}, S, \alpha) = \mathcal{L}_s(\mathfrak{T}^\alpha)$.

PROOF. That $\mathfrak{T}$ semi-simple implies $\mathfrak{T}^\alpha$ semi-simple follows from part b) of theorem 1. If we take $\mathcal{L}_s(\mathfrak{T}) = \mathfrak{T} \oplus \mathcal{I}(\mathfrak{T})$, the semidirect sum, then $\mathcal{L}_s(\mathfrak{T}^\alpha) = \mathfrak{T} \oplus \mathcal{I}(\mathfrak{T}^\alpha) = \mathfrak{T} \oplus \mathcal{I}(\mathfrak{T})$ as a vector space. The product rules are: i) $\langle ab \rangle = D_{\langle ab \rangle} = D_{\alpha[ab]} = \alpha D_{[ab]} = \alpha[a, b]$ in $\mathcal{L}_s(\mathfrak{T})$ for $a, b \in \mathfrak{T}$. ii) $\langle aD \rangle = aD = [aD]$ for $a \in T$, $D \in \mathcal{I}(\mathfrak{T})$. iii) $\langle D_1 D_2 \rangle = D_1 D_2 - D_2 D_1 = [D_1 D_2]$. This shows that $\mathcal{L}_s(\mathfrak{T}^\alpha)$ is an α-twist of $\mathcal{L}_s(\mathfrak{T})$. Conversely if $\mathfrak{L}$ and S are given and $\mathfrak{M}$ is the associated α-twist, $\mathfrak{M}$ has the vector space decomposition $\mathfrak{M} = \mathfrak{T} \oplus \langle \mathfrak{T}\mathfrak{T} \rangle$ and $\langle \langle ab \rangle c \rangle = \alpha \langle [ab] c \rangle$

$=\alpha[[ab]c]$. Thus the product in $\mathfrak{M}$ induces the structure $\mathfrak{T}^{\alpha}$ on $\mathfrak{T}$. Consequently $\mathfrak{M}$ is semi-simple and $\mathfrak{M}=\mathcal{L}_s(\mathfrak{T}^{\alpha})$.

Assume that $\mathfrak{L}$ is semi-simple with reflection S, and that $\mathfrak{T}=\mathfrak{L}_{-1}(S)$. Since $\mathfrak{L}$ and $(\mathfrak{L}, S, \alpha)$ are universal for $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$ respectively, any isomorphism A of $\mathfrak{T}$ onto $\mathfrak{T}^{\alpha}$ uniquely extends to an isomorphism $\tilde{A}$ of $\mathfrak{L}$ and $(\mathfrak{L}, S, \alpha)$. The restriction of $\tilde{A}$ to $[\mathfrak{T}\mathfrak{T}]$ is an automorphism of $[\mathfrak{T}\mathfrak{T}]$ (and of $<\mathfrak{T}\mathfrak{T}>$). The following immediate consequences of theorems 1 and 2 will be useful.

Theorem 3. Let $\mathfrak{L}$ be semi-simple. If a field $\Omega\supseteq\Phi(\alpha^{1/2})$ then $(\mathfrak{L}, S, \alpha)_{\Omega}=\mathfrak{L}_{\Omega}$. If Φ is the real field then any twist of $\mathfrak{L}$ is isomorphic either to $\mathfrak{L}$ or to a twist $(\mathfrak{L}, S, -1)$.

2. The isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$ in the general case. The observations of the preceding section imply that the study of the variants of a semi-simple $\mathfrak{T}$ reduces to the case $\mathfrak{T}$ simple. For a Lie algebra $\mathfrak{L}$ let $\mathscr{T}(\mathfrak{L})$ denote the Lie triple system with product $[abc]=[[ab]c]$, and call $\mathscr{T}(\mathfrak{L})$ the *Lie triple system of* $\mathfrak{L}$. The simple Lie triple systems fall into two disjoint classes:

i) those with simple universal Lie algebras,

ii) those with a universal Lie algebra of the form $\mathfrak{L}_1\oplus\mathfrak{L}_2$ where the $\mathfrak{L}_i$ are isomorphic ideals. A necessary and sufficient condition that (ii) hold for $\mathfrak{T}$ is that $\mathfrak{T}\cong\mathscr{T}(\mathfrak{L}_i)$.

With respect to the question of the isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$ the elements of class (ii) behave quite simply, as we shall establish in theorem 4. To specify the situation exactly it is convenient to develop the notion of centrality for Lie triple systems and to classify the central simple systems, a result of possibly broader interest.

The *centroid* of a Lie triple system $\mathfrak{T}$ is the centralizer in Hom $\Phi(\mathfrak{T}, \mathfrak{T})$ of the space of *multiplications* $x\rightarrow\sum[xa_ib_i]$ in $\mathfrak{T}$. In case this centralizer is trivial, call $\mathfrak{T}$ *central.*

Lemma. If $\mathfrak{T}$ is simple its centroid is a field.

Proof. Let Γ be the centroid of $\mathfrak{T}$ and $\theta\in\Gamma$. From the identity $[xyz]=-[yxz]$ and the Jacobi identity we infer the identity

(1) $[xyz]\theta=[(x\theta)yz]=[x(y\theta)z]=[xy(z\theta)]$.

For θ, $\psi\in\Gamma$ this implies $[xyz]\theta\psi=[(x\theta)y(z\psi)]$ and hence $[xyz](\theta\psi-\psi\theta)=0$. Now if $\mathfrak{T}$ is simple $[\mathfrak{T}\mathfrak{T}\mathfrak{T}]=\mathfrak{T}$ and $\mathfrak{T}$ is irreducible with respect to its multiplications, and so Γ is a commutative division ring.

Lemma. If $\mathfrak{L}$ is a simple Lie algebra with centroid Γ_0 and $\mathfrak{T}=\mathscr{T}(\mathfrak{L})$ has centroid Γ, then $\Gamma=\Gamma_0$.

Proof. If $\theta\in\Gamma$ then (1) shows $[[xy]z]\theta=[[xy](z\theta)]$, which implies that θ

commutes with all algebra multiplications $ad\ u$, $u \in [\mathfrak{L}\mathfrak{L}]$. Since $[\mathfrak{L}\mathfrak{L}] = \mathfrak{L}$, $\theta \in \Gamma_0$. The converse is immediate.

We suppose that $\mathfrak{T}$ is simple, $\mathfrak{M} = \mathcal{L}_s(\mathfrak{T})$, and we investigate the relation between the centroids Γ and Γ_0 of $\mathfrak{T}$ and $\mathfrak{M}$ respectively. First we observe that Γ can be naturally regarded as a subfield of Γ_0. For $\theta \in \Gamma$ there is at most one linear transformation $\tilde{\theta}$ of M such that $\tilde{\theta} \mid T = \tilde{\theta}$ and

$$[x\,y]\tilde{\theta} = [(x\theta)\,y] = [x\,(y\theta)] \text{ for } x,\ y \in T.$$

To see that $\tilde{\theta}$ exists suppose that $\sum[x_i\,y_i] = 0$. Then since $\theta \in \Gamma$,

$$\sum[x_i\,y_i z] = 0 = \sum[(x_i\theta)\,y_i z] = \sum[x_i(y_i\theta)\,z].$$

But because $\mathfrak{M}$ is standard for $\mathfrak{T}$,

$$\sum[(x_i\theta)\,y_i] = 0 = \sum[x_i(y_i\theta)].$$

A direct verification shows that $\tilde{\theta} \in \Gamma_0$ and that $\theta \to \tilde{\theta}$ is a monomorphism. This map will be used to identify Γ with a subfield of Γ_0.

If Φ is algebraically closed then either $\Gamma_0 = \Gamma$ ($\mathfrak{M}$ simple) or $\Gamma_0 = \Gamma \oplus \Gamma$ ($\mathfrak{M} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$). Thus in the general case $(\Gamma_0 \colon \Gamma) \leq 2$ and Γ_0 is commutative. If $\mathfrak{T}$ is not the triple system of an algebra, then Γ_0 is a field which is at most quadratic over Γ. A precise description of the relation between Γ and Γ_0 is provided by the following theorem, which for simplicity is stated for $\mathfrak{T}$ central simple. The general result follows by considering a simple $\mathfrak{T}$ with centroid Γ as a Γ-triple system.

THEOREM 4. Let $\mathfrak{T}$ be central simple and suppose that $\mathfrak{M} = \mathcal{L}_s(\mathfrak{T})$ has centroid Γ_0. Then i) $\Gamma_0 = \Phi$ if and only if $\mathfrak{T}$ is not a variant of a triple system of an algebra, ii) $\Gamma_0 = \Phi(\alpha^{1/2})$ for some $\alpha \notin \Phi^2$ if and only if $\mathfrak{T}$ is the non-isomorphic α-variant of a triple system of an algebra, iii) $\Gamma_0 = \Phi \oplus \Phi$ if and only if $\mathfrak{T}$ is the triple system of an algebra.

PROOF. If $\mathfrak{T} = \mathfrak{T}_1^\alpha$ where $\mathfrak{T}_1 = \mathfrak{T}(\mathfrak{L})$, then $\mathfrak{T}_\Omega \cong (\mathfrak{T}_1)_\Omega$ for Ω the algebraic closure of Φ, and so $(\Gamma_0 \colon \Phi) = 2$. Suppose now that $(\Gamma_0 \colon \Phi) = 2$. Choose a generator ψ_0 of Γ_0. Since the elements of Φ are precisely those elements of Γ_0 for which $\mathfrak{T}$ is invariant, and since $\mathfrak{T}\psi_0$ is an ideal of $\mathfrak{T}(\mathfrak{M})$, it follows that $\mathfrak{T} \oplus \mathfrak{T}\psi = \mathfrak{M}$. Let (E_-, E_+) be the pair of projections associated with the decomposition $\mathfrak{M} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$, and let $\psi_- = \psi_0 E_-$, $\psi_+ = \psi_0 E_+$. Define a linear map ψ in $\mathfrak{M}$ by setting

$$\psi \mid \mathfrak{T} = \psi_+,\ \psi \mid [\mathfrak{T}\mathfrak{T}] = \psi_-.$$

That $\psi \in \Gamma_0$ follows readily from the basic formulas

$$[x\,y]\psi_- = [(x\psi_+)\,y],\ [x\,y]\psi_+ = [(x\psi_-)\,y]$$

which hold for $x,\ y \in \mathfrak{T}$, and result from a comparison of $[x\,y]\psi_0$ with $[(x\psi_0)\,y]$.

Because $\mathfrak{T}\psi \subseteq [\mathfrak{T}\mathfrak{T}]$ and $\psi \in \Gamma_0$, $\mathfrak{T}\psi^2 \subseteq \mathfrak{T}$, consequently $\psi^2 \in \Phi$. Since ψ is a monomorphism on $\mathfrak{T}$ and since in no case does Γ_0 have nilpotent elements other than zero, $\psi^2 = \alpha \neq 0$. In particular ψ is non singular and maps $\mathfrak{T}$ onto $[\mathfrak{T}\mathfrak{T}]$. Observe also that

$$[(x\psi)(y\psi)(z\psi)] = [x\,yz]\psi^3 = \alpha[x\,yz]\psi.$$

This amounts to the assertion that ψ is an isomorphism of $\mathfrak{T}^\alpha$ and $\mathfrak{T}([\mathfrak{T}\mathfrak{T}])$. Of course $\mathfrak{T}([\mathfrak{T}\mathfrak{T}]) \cong \mathfrak{T}(\mathfrak{L})$. We have proved therefore that $(\Gamma_0 : \Phi) = 2$ if and only if $\mathfrak{T}$ is a variant of the triple system of an algebra.

To distinguish cases ii) and iii) note that $\Gamma_0 = \Phi[\psi]$ and that Γ_0 is a field in precisely the situation $\psi^2 \notin \Phi^2$. If $\psi^2 \in (\Phi^*)^2$ then $\mathfrak{T}^\alpha \cong \mathfrak{T} \cong T(\mathfrak{L})$, $\mathfrak{M} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ where $\mathfrak{L}_i$ is an ideal isomorphic to $\mathfrak{L}$, and $\Gamma_0 = \Phi \oplus \Phi$.

The following isomorphism theorem for central simple triple systems $\mathfrak{T} = \mathfrak{T}(\mathfrak{L})$ is an immediate consequence of theorem 4 since if $\mathfrak{T}^\alpha \cong \mathfrak{T} = \mathfrak{T}(\mathfrak{L})$ then iii) holds, and so $\alpha \in (\Phi^*)^2$.

Theorem 5. Let $\mathfrak{L}$ be a central simple Lie algebra and let $\mathfrak{T} = \mathfrak{T}(\mathfrak{L})$. A necessary and sufficient condition that $\mathfrak{T}^\alpha \cong \mathfrak{T}$ is that $\alpha \in (\Phi^*)^2$. If $\alpha \notin (\Phi^*)^2$ then $\mathfrak{M} = \mathcal{L}_s(\mathfrak{T}^\alpha)$ is a simple Lie algebra with centroid $\Gamma_0 = \Phi(\alpha^{1/2})$.

It remains to investigate what can be said about the isomorphisms of $\mathfrak{T}$ and $\mathfrak{T}^\alpha$ in the general case of an arbitrary field (always of characteristic 0), and a simple $\mathfrak{T}$ which is not the triple system of a Lie algebra. In this case $\mathcal{L}_s(\mathfrak{T})$ is simple and is central if $\mathfrak{T}$ is central.

Recall that if $\mathcal{L}_s(\mathfrak{T}) = \mathfrak{L} = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$, the subalgebra $[\mathfrak{T}\mathfrak{T}]$ acts on $\mathfrak{T}$ and may be identified thereby with the derivation algebra of $\mathfrak{T}$. In [5] it was shown that either $\mathfrak{T}$ is irreducible with respect to (the action of) $[\mathfrak{T}\mathfrak{T}]$ or else $\mathfrak{T} = \mathfrak{T}_1 \oplus \mathfrak{T}_2$, where $\mathfrak{T}_i$ is $[\mathfrak{T}\mathfrak{T}]$ -irreducible, $(\mathfrak{T}_1 : \Phi) = (\mathfrak{T}_2 : \Phi)$, and $[\mathfrak{T}_i\mathfrak{T}_i] = 0$. Our first result concerns the latter case.

Theorem 6. If $\mathfrak{T}$ is a simple Lie triple system which is reducible with respect to its derivation algebra, then $\mathfrak{T}^\alpha \cong \mathfrak{T}$ for every $\alpha \in \Phi^*$.

Proof. Let $\mathfrak{T} = \mathfrak{T}_1 \oplus \mathfrak{T}_2$ be the decomposition of $\mathfrak{T}$ cited above. Let A be the linear map of $\mathfrak{T}$ defined by

$x_1 A = \alpha x_1$, $x_1 \in \mathfrak{T}_1$ and $x_2 A = x_2$, $x_2 \in \mathfrak{T}_2$. Let $\langle\ \ \rangle$ denote the $\mathfrak{T}^\alpha$ product in the space $\mathfrak{T}$. For each $x \in \mathfrak{T}$ let x_i be the $\mathfrak{T}_i$ component of x. Then

$$\langle x\,yz \rangle = \langle x\,yz_1 \rangle + \langle x\,yz_2 \rangle = \alpha[x\,yz_1] + \alpha[x\,yz_2]$$

and $$\langle x\,yz \rangle A = \alpha^2[x\,yz_1] + \alpha[x\,yz_2] = \alpha[x\,y(zA)].$$

On the other hand

$$[(xA)\,(yA)\,(zA)] = [(\alpha x_1 + x_2)\,(\alpha y_1 + y_2)(zA)]$$
$$= [(\alpha x_1)\,y_2(zA)] + [x_2(\alpha y_1)(zA)]$$
$$= \alpha[x\,y(zA)].$$

There is at least one other condition which implies the isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$ and which is independent of the structure of Φ. To formulate this condition in strict Lie triple system theoretic terms requires a detailed investigation of suitably defined "split" systems. Such a digression can be avoided by appealing to the structure theory for $\mathfrak{L} = \mathscr{L}_s(\mathfrak{T}) = \mathfrak{T} \oplus [\mathfrak{T}\mathfrak{T}]$.

Let $\mathfrak{L}$ be a simple, split Lie algebra and let $\{e_{\lambda}, f_{\lambda}, h_{\lambda}\}$, $\lambda \in \pi$, be canonical generators for $\mathfrak{L}$ associated with a splitting Cartan subalgebra $\mathfrak{H}$ and the fundamental system of roots π ([5] p. 121). Let S be a reflection of $\mathfrak{L}$ and let $\mathfrak{T} = \mathfrak{L}_{-1}(S)$. Suppose further that S is a *canonical inner automorphism* of $\mathfrak{L}$ with respect to π in the sense that:

i) $\mathfrak{H} \subset \mathfrak{L}_1(S)$ and ii) $e_{\lambda}S = \pm\, e_{\lambda}$.

In these circumstances $\mathfrak{T}$ will be called *split*.

THEOREM 7. If $\mathfrak{T}$ is a simple, split Lie triple system then $\mathfrak{T}^{\alpha} \cong \mathfrak{T}$ for every $\alpha \in \Phi^*$.

PROOF. Let $\mathfrak{L} = \mathscr{L}_s(\mathfrak{T})$, let $\mathfrak{H}$ and S be the Cartan subalgebra and reflection of $\mathfrak{L}$ associated with $\mathfrak{T}$. Since $hS = h$ for $h \in \mathfrak{H}$, $f_{\lambda}\,S = -f_{\lambda}$ if and only if $e_{\lambda}\,S = -\,e_{\lambda}$. Now consider $\mathfrak{M} = \mathscr{L}_s(\mathfrak{T}^{\alpha})$. Clearly $\mathfrak{M}$ is a split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{H}$, fundamental root system π, and a generating system $\{e_{\lambda}, f_{\lambda}, h_{\lambda}\}$, $\lambda \in \pi$. Note that $ad_{\mathfrak{L}}h = ad_{\mathfrak{M}}h$ for $h \in \mathfrak{H}$ and hence that the Killing forms of $\mathfrak{L}$ and $\mathfrak{M}$ have the same restriction to $\mathfrak{H}$. This implies that the Cartan matrices associated with $(\mathfrak{L}, \mathfrak{H}, \pi)$ and $(\mathfrak{M}, \mathfrak{H}, \pi)$ coincide. Set $e'_{\lambda} = e_{\lambda}$, $h'_{\lambda} = h_{\lambda}$ and set $f'_{\lambda} = f_{\lambda}$ if $f_{\lambda} \in [\mathfrak{T}\mathfrak{T}]$, $f'_{\lambda} = \alpha^{-1}f_{\lambda}$ if $f_{\lambda} \in \mathfrak{T}$. The system $\{e'_{\lambda}, f'_{\lambda}, h'_{\lambda}\}$, $\lambda \in \pi$, canonically generates $\mathfrak{M}$; for in $\mathfrak{M}$ $\langle e'_{\lambda} f'_{\lambda} \rangle = [e'_{\lambda} f'_{\lambda}] = e_{\lambda} f_{\lambda}] = h_{\lambda} = h'_{\lambda}$ if $f_{\lambda} \in \mathfrak{T}$, and in all other cases products of generators in $\mathfrak{L}$ automatically coincide with the corresponding products in $\mathfrak{M}$.

In these circumstances it follows ([5] p. 127) that the map $e_{\lambda} \to e'_{\lambda}$, $f_{\lambda} \to f'_{\lambda}$, $h_{\lambda} \to h'_{\lambda}$ has a unique extension to an isomorphism of $\mathfrak{L}$ and $\mathfrak{M}$. Since the space $\mathfrak{T}$ is invariant under this map, it induces an isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$.

Theorems 6 and 7 give sufficient conditions for the isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^{\alpha}$. *A* strengthening of the converse to theorem 6 provides useful sufficient conditions for non-isomorphism.

THEOREM 8. Let $\mathfrak{T}$ be a simple Lie triple system. If (i) $\mathfrak{T}$ is absolutely irreducible with respect to its derivation algebra $\mathscr{D}(\mathfrak{T})$ and (ii) every automorphism of $\mathfrak{D}(\mathfrak{T})$ is induced by an automorphism of $\mathfrak{T}$, then $\mathfrak{T}^\alpha \cong \mathfrak{T}$ if and only if $\alpha \in (\Phi^*)^2$.

Proof. By (i) $\mathfrak{T}$ is central and the centralizer of $\mathfrak{D}(\mathfrak{T})$ is the set of scalar multiplications. Suppose that A_0 is an isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^\alpha$. Then A_0 extends uniquely to an isomorphism $\tilde{A}_0$ of $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{T}^\alpha)$. Since the latter coincide $\tilde{A}_0$ is the automorphism of $D(T)$ given by

$$D\tilde{A}_0 = A_0^{-1} D A_0$$

By (ii) there is an automorphism A of $\mathfrak{T}$ which also induces $\tilde{A}_0$ on $\mathfrak{D}(\mathfrak{T})$. Thus $\tilde{A}_0 \tilde{A}^{-1}$ is in the centralizer of $\mathfrak{D}(\mathfrak{T})$. But then $A_0 = \gamma\, A$ for some $\gamma \in \Phi^*$, and multiplication by γ is an isomorphism of T and T^α. It follows directly that $\gamma^2 = \alpha^{-1}$.

Since theorem 5 can be obtained from theorem 8, the hypotheses of the latter are realizable. That their scope is broader is seen by the following typical application.

Let $\mathfrak{S}$ be the Lie algebra of skew symmetric matrices in Φ_n for n sufficiently large. Let $\mathfrak{T}$ be the Lie triple system of symmetric matrices of trace 0. Then $\mathfrak{T}$ is simple, $\mathfrak{S}$ is the derivation algebra of $\mathfrak{T}$ and $\Phi_n' = \mathfrak{T} \oplus \mathfrak{S}$ is the universal Lie algebra of $\mathfrak{T}$. The automorphisms of $\mathfrak{T}$ and $\mathfrak{S}$ are those induced by the (inner) automorphisms of Φ_n ([5, page 308]), hence in both cases are those induced by matrices commuting with transposition. This implies condition (ii). Condition (i) follows from the observation that $\mathfrak{T}$ is irreducible with respect to $\mathfrak{S}$ in case Φ is algebraically closed.

3. The isomorphism of twists: classification problems. Suppose again that $\mathfrak{T}$ is simple. The isomorphism of $\mathfrak{T}$ and $\mathfrak{T}^\alpha$ entails that of $\mathcal{L}_s(\mathfrak{T})$ and $\mathcal{L}_s(\mathfrak{T}^\alpha)$ but the converse is false. It will be convenient to consider $\mathfrak{T}$ over its centroid, and therefore to suppose that $\mathfrak{T}$ is central. In case $\mathfrak{T} = \mathfrak{T}(\mathfrak{L})$ for some central simple Lie algebra $\mathfrak{L}$, it follows from the structure of $\mathcal{L}_s(\mathfrak{T})$ that the simple Lie triple systems with universal algebra $\mathcal{L}_s(\mathfrak{T})$ are all isomorphic to $\mathfrak{T}$. The situation is different for $\mathfrak{T}^\alpha$, however, and this is a point of considerable interest.

Theorem 9. If i) $\mathfrak{L}$ is central simple, ii) $\alpha \notin \Phi^2$, $\Omega = \Phi(\psi)$ where $\psi^2 = \alpha$, iii) $\mathfrak{T} = \mathfrak{T}(\mathfrak{L})$, $\mathfrak{M} = \mathcal{L}_s(\mathfrak{T}^\alpha)$, and iv) $\mathfrak{M} = \mathcal{L}_s(\mathfrak{T}_0)$; then either a) the reflection S_0 of $\mathfrak{M}$ associated with $\mathfrak{T}_0$ commutes with ψ and Ω is the centroid of $\mathfrak{T}_0$ or b) S_0 anticommutes with ψ, $\mathfrak{T}_0 = (\mathfrak{T}(\mathfrak{L}_0))^\alpha$ for some central simple Lie algebra $\mathfrak{L}_0$ with $(\mathfrak{L}_0)_\Omega \cong \mathfrak{M}$ over $\Omega \cong \mathfrak{L}_\Omega$, and every such $\mathfrak{L}_0$ arises in this way.

Proof. By theorem 5 $\mathfrak{T}^\alpha \ncong \mathfrak{T}$ and Ω is the centroid of $\mathfrak{M}$. If S_0 commutes with (multiplication by) ψ then $\mathfrak{L}_{-1}(S_0) = \mathfrak{T}_0$ is invariant with respect to ψ so that ψ is in the centroid of $\mathfrak{T}_0$, which is therefore Ω. On the other hand since $(S_0\, \psi\, S_0^{-1})^2$

$=\alpha$, if S_0 does not commute with ψ it anti-commutes with ψ. In this case ψ is not in the centroid of $\mathfrak{T}_0$, which is therefore Φ. By theorem 4 $\mathfrak{L}_0=(\mathcal{T}(\mathfrak{L}_0))^\alpha$, and since $\mathfrak{T}_0$ is central so is $\mathfrak{L}_0$. It may now be readily verified that for x and y in either $\mathfrak{L}$ or $\mathfrak{L}_0$

$$x+y\psi\rightarrow x\otimes 1+y\otimes\psi$$

defines a unique Ω-isomorphism of $\mathfrak{M}$ onto $\mathfrak{L}_\Omega$.

Finally, if $(\mathfrak{L}_1)_\Omega\cong\mathfrak{M}$ over Ω then we may regard $\mathfrak{L}_1$ as a Φ-subalgebra of $\mathfrak{M}$ for which $\mathfrak{L}_1\oplus\mathfrak{L}_1\psi=\mathfrak{M}$. But then $\mathfrak{L}_1\psi\cong(\mathcal{T}(\mathfrak{L}_1))^\alpha$ and $\mathfrak{M}=\mathcal{L}_s(\mathfrak{L}_1\psi)$.

Theorem 9 implies that if $\mathfrak{M}$ is a simple Ω-algebra and $\Omega=\Phi(\psi)$ where $\psi^2=\alpha$ is in Φ, then the Φ-algebras $\mathfrak{L}_0$ of type $\mathfrak{M}$, i.e., $\mathfrak{L}_\Omega\cong\mathfrak{M}$, all occur as the spaces $\mathfrak{M}_1(S_0)$ where S_0 is a Φ-reflection in $\mathfrak{M}$ which anti-commutes with ψ. We wish to discover under what circumstances any two algebras of type $\mathfrak{M}$ are α-twists of one another.

In section 4 it will develop that in case Φ is the real field the work on the classification of real simple Lie algebras does not appear to provide a ready answer. The following theorem delineates the general problem in terms of reflections.

THEOREM 10. In the setting of theorem 9 suppose that S_0 is a reflection of $\mathfrak{M}$ anti-commuting with ψ, and that $\mathfrak{L}_0=\mathfrak{M}_1(S_0)$. Then $\mathfrak{L}_0$ is isomorphic to an α-twist of $\mathfrak{L}$ if and only if S_0 is conjugate in the automorphism group of ($\mathfrak{M}$ over Ω) to a reflection commuting with S, where $\mathfrak{L}=\mathfrak{M}_1(S)$.

PROOF. In general two simple Lie triple systems $\mathfrak{T}$ and $\mathfrak{T}_0$ imbedded in a common universal algebra $\mathfrak{M}$ are isomorphic if and only if the associated reflections are conjugate in the automorphism group $\mathcal{A}(\mathfrak{M})$ of $\mathfrak{M}$. From this and the hypotheses of theorem 9 it follows that $\mathfrak{M}_1(S_0)\cong\mathfrak{M}_1(S)$ if and only if S and S_0 are conjugate in $\mathcal{A}(\mathfrak{M})$.

Suppose first that $\mathfrak{L}_0$ is isomorphic to a twist $(\mathfrak{L}, B, \alpha)$ of $\mathfrak{L}$. Let $\mathfrak{U}=\mathfrak{L}_{-1}(B)$. Then the map whose restriction to $\mathfrak{U}$ is ψ and to $[\mathfrak{U}\mathfrak{U}]$ is the identity is an isomorphism of $(\mathfrak{L}, B, \alpha)$ and $\mathfrak{U}\psi\oplus[\mathfrak{U}\mathfrak{U}]\equiv\tilde{\mathfrak{L}}_0$, where $\tilde{\mathfrak{L}}_0\cong\mathfrak{L}_0$. Now $\tilde{\mathfrak{L}}_0\oplus\tilde{\mathfrak{L}}_0\psi=\mathfrak{M}=\mathcal{L}_s[\mathcal{T}(\mathfrak{L}_0)^\alpha]$ and if $\mathfrak{L}_0=\mathfrak{M}_1(\tilde{S}_0)$ then $\tilde{\mathfrak{L}}_0$ is S invariant. Thus $\tilde{S}_0S=S\tilde{S}_0$. Finally, $\mathfrak{M}_1(S_0)=\mathfrak{L}_0\cong\tilde{\mathfrak{L}}_0=\mathfrak{M}_1(\tilde{S}_0)$ so that S_0 and S_0 are conjugate in $\mathcal{A}(\mathfrak{M})$. It is readily seen that the conjugating element can be taken to commute with ψ.

To establish the converse suppose that S_0 is conjugate in $\mathcal{A}(\mathfrak{M})$ to $\tilde{S}_0$, a reflection commuting with S. It is sufficient to show that $\tilde{\mathfrak{L}}_0\equiv\mathfrak{M}_1(\tilde{S}_0)$ is isomorphic to a twist of $\mathfrak{L}$. Since $\tilde{S}_0$ induces a reflection in $\mathfrak{L}$ and ψ exchanges $\mathfrak{M}_{-1}(\tilde{S}_0)$ and $\mathfrak{M}_1(\tilde{S}_0)$, $\mathfrak{L}=\mathfrak{U}\oplus[\mathfrak{U}\mathfrak{U}]$ and $\mathfrak{L}\psi=\mathfrak{U}\psi\oplus[\mathfrak{U}\mathfrak{U}]\psi$, where $[\mathfrak{U}\mathfrak{U}]=\mathfrak{M}_1(\tilde{S}_0)\cap\mathfrak{L}$ and $\mathfrak{U}\psi=\mathfrak{M}_1(\tilde{S}_0)\cap\mathfrak{L}\psi$. Therefore $\mathfrak{M}_1(\tilde{S}_0)=\mathfrak{U}\psi\oplus[\mathfrak{U}\mathfrak{U}]$, which we have already shown is isomorphic to a twist of $\mathfrak{L}$.

In the remaining case, $\mathfrak{T}$ central simple and $\mathfrak{L}=\mathscr{L}_s(\mathfrak{T})$ central simple, examples show that $\mathfrak{T}^\alpha$ and $\mathfrak{T}$ generally have non-isomorphic universal algebras if $\mathfrak{T}^\alpha \not\cong \mathfrak{T}$. An exception will be exhibited in the proof of theorem 13.

4. The real field. In this case since only (-1) -variants and twists need be considered they will be called simply variants and twists, and the denotation of a twist will be abbreviated to $(\mathfrak{L}, S)$. In terms of theorem 9 $\Omega=\Phi((-1)^{1/2})$ is the complex field and so $x\psi$ identifies with ix. The classical results in the classification of simple real Lie algebras can be stated in terms of the present discussion. The only preliminaries required concern the signature $\sigma(\mathfrak{L})$ of the Killing form of a Lie algebra $\mathfrak{L}$.

Let $\mathfrak{L}$ be central simple, S a reflection of $\mathfrak{L}$, and $\mathfrak{L}=\mathfrak{T}\oplus[\mathfrak{T}\mathfrak{T}]$ the associated decomposition. The following are easily verified.

i) $\mathfrak{T}$ and $[\mathfrak{T}\mathfrak{T}]$ are orthogonal complements with respect to the Killing form.

ii) If $\sigma(\mathfrak{T})$ denotes the signature of the restriction of the Killing form to $\mathfrak{T}$, then $\sigma(\mathfrak{L}, S)=\sigma(\mathfrak{L})-2\sigma(\mathfrak{T})$.

The first result goes back to Weyl (see [5] p. 147 or [7]). A sketch of a modified proof is included because the twist point of view suggests a more transparent argument.

THEOREM 11. If $\mathfrak{L}$ is a split simple real Lie algebra then there is a twist of $\mathfrak{L}$ which is compact.

PROOF. Let $\{e_\lambda, f_\lambda, h_\lambda\}$ be a canonical set of generators for $\mathfrak{L}$ associated with the Cartan subalgebra $\mathfrak{H}$ and fundamental root system π. Let (x, y) denote the Killing form of $\mathfrak{L}$. This form is positive definite on $\mathfrak{H}$. Otherwise pairs of generators are orthogonal with respect to the form except for the relation $(e_\lambda, f_\lambda)<0$.

Let $\mathfrak{L}_0=(\mathfrak{L}, S)$ and let $(x, y)_0$ denote the Killing form of $\mathfrak{L}_0$. Then $(x, x)_0=(x, xS)$ so that $\mathfrak{L}_0$ is compact if and only if S is negative definite with respect to the Killing form on $\mathfrak{L}$. On the other hand since $\{f_\lambda, e_\lambda, -h_\lambda\}$, $\lambda\in\pi$, is a canonical set of generators, there is a unique automorphism S of $\mathfrak{L}$ such that

$$e_\lambda S=f_\lambda,\ f_\lambda S=e_\lambda,\ h_\lambda S=-h_\lambda.$$

Clearly S is negative definite on the space with basis $\{e_\lambda, f_\lambda, h_\lambda\}$, $\lambda\in\pi$, and it is readily verified by induction on the degree of the monomials in a canonical basis determined by the generators that S is negative definite.

The second result, due to Cartan and often called the Cartan decomposition theorem, concerns twists of compact algebras. According to theorem 9 the classification of all central simple real Lie algebras with isomorphic complexifications

($\mathfrak{M}$ over Ω) reduces to the determination of the conjugacy classes of those reflections of $\mathfrak{M}$ (over Φ) which are semi-linear as ($\mathfrak{M}$ over Ω) mappings and have complex conjugation as associated automorphism of Ω. Call such reflections c-reflections in $\mathfrak{M}$.

Now let $\mathfrak{L}_0$ be compact and suppose $\mathfrak{L}_\Omega \cong (\mathfrak{L}_0)_\Omega \cong (\mathfrak{M}$ over $\Omega)$. Then there are c-reflections S and S_0 such that $\mathfrak{L} \cong \mathfrak{M}_1(S)$ and $\mathfrak{L}_0 \cong \mathfrak{M}_1(S_0)$. By theorem 10 $\mathfrak{L}$ is a twist of $\mathfrak{L}_0$ if and only if S is conjugate in $\mathcal{A}(\mathfrak{M}$ over $\Omega)$ to a c-reflection commuting with S_0. Cartan's theorem translates to the following:

THEOREM 12. If $\mathfrak{N}$ is a complex simple Lie algebra, if S_0 is a c-reflection in $\mathfrak{N}$ such that $\mathfrak{N}_1(S_0) = \mathfrak{L}_0$ is compact, and if S is any c-reflection in $\mathfrak{N}$, then S is conjugate in $\mathcal{A}(\mathfrak{N})$ to a c-reflection commuting with S_0.

COROLLARY. Every central simple real Lie algebra is a twist of a compact algebra.

Discussions of this theorem may be found in [1], [7], and in chapter III of [5].

Considerations of Killing form signature shaw that no twist of a compact algebra is isomorphic to it and that no variant of a compact Lie triple system is isomorphic to it. Thus compact and split Lie triple systems are extremes in this respect. An immediate question raised by the corollary to theorem 12 is the following: is every central simple real Lie algebra a twist of every other algebra with an isomorphic complexification? In particular can every algebra be so obtained from the split algebra?

On this point our conclusions are limited. Gantmacher [2] has found canonical representatives of the conjugacy classes of c-reflections for all simple complex algebras. It is an easy consequence of theorem 10 that if a commutative set of representatives exist then our first question has an affirmative answer. Furthermore it is implicit in their definition that Gantmacher's canonical inner automorphisms all commute. In the case of $\mathfrak{E}_6$ these canonical inner automorphisms can be adjusted to commute with the canonical outer automorphism. For $\mathfrak{A}_n$ and $\mathfrak{D}_n$ this cannot be done. Our final result asserts the favorable general cases and cites a specific exceptional case.

THEOREM 13. With the possible exception of algebras of types $\mathfrak{A}_n$(n odd, $n > 1$) and $\mathfrak{D}_n$, any two central simple real Lie algebras with isomorphic complexifications are twists of one another. The algebras of type $\mathfrak{A}_3$ with signatures -5 and -3 are not twists of one another.

PROOF. Let $\mathfrak{D}_2$ be the algebra of 2 by 2 matrices over the quaternion division algebra $\mathfrak{D}$. The derived Lie algebra $\mathfrak{D}_2'$ is of type $\mathfrak{A}_3$ and has signature -5. The automorphisms of $\mathfrak{D}_2'$ are explicitly described by Jacobson in [4]. It turns out that the reflections of $\mathfrak{D}_2'$ divide into four conjugacy classes represented by:

S_1: $X \to PXP^{-1}$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

S_2: $X \to QXQ^{-1}$ where $Q = \mathrm{i}I$.

S_3: $X \to -\bar{X}$ where $x \to \bar{x}$ denotes quarternionic conjugation.

S_4: $X \to -P\bar{X}'P$

S_5: $X \to Q\bar{X}'Q$

Denote by $\mathfrak{T}_i$ the triple system of S_i, let $\mathfrak{L}_i = [\mathfrak{T}_i\mathfrak{T}_i]$, and let (n_i, m_i) be the respective dimensions in the i^{th} decomposition of $\mathfrak{D}_2'$. Computation reveals that

$$n_1 = 8 = n_2,\ n_3 = 5 = n_4,\ n_5 = 9.$$

Moreover $\{S_i\}$ is a commutative set and the S_3 -twist is the compact form of $\mathfrak{A}_3$. Thus the signatures of possible twists of $\mathfrak{D}_2'$ may be computed by considering the possible $\mathfrak{T}_3 \cap \mathfrak{T}_i$ dimensions. It turns out that this technique leads to the complete identification of the S_i -twists, as follows:

The S_1 twist has signature -5 (and is therefore $\mathfrak{D}_2'$)

The S_2 twist has signature 3.

The S_3 twist has signature -15.

The S_4 twist has signature 1.

The S_5 twist has signature 1.

To illustrate the method consider the S_5 case. Possible twist signatures are 1 and -3. But if the twist had signature -3 then $\mathfrak{L}_3$ would have a reflection S_5^* with an associated 5 dimensional triple system. However $\mathfrak{L}_3$ is an algebra of type $\mathfrak{C}_2$ and these algebras have only (4, 6) and (6, 4) dimensional triple system decompositions.

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