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INVARIANT SUBDOMAINS OF THE RIEMANN SPHERE

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1. In this note we shall give answer to a special case of the following problem: let X be a complex manifold and Y be a subdomain of X. What can we say about Y when the automorphism of Y is always the restriction of an automorphism of X? For general case the circumstances are so complicated, even in the case that X is compact. So in this note we shall confine curselves to the case where X is the Riemann sphere, which we shall denote by P^1, and Y is simply connected. The converse of this problem was treated by several authors, see [1], [2] pp. 183–188. We shall prove

THEOREM *Let D be a simply connected subdomain of P^1. Suppose that the automorphism of D is always the restriction of an automorphism of P^1. Then D belongs to one of the following three kind of domains;*

(i) P^1;

(ii) $P-\{\alpha\}$, *α arbitrary;*

(iii) *the disks on P^1.*

2. Since D is simply connected, it is obvious that the case (i) or (ii) occurs corresponding to the cases: (i)′ the boundary ∂D of D is empty; (ii)′ ∂D contains only one point, respectively.

It is also obvious that the domains in the cases (i), (ii) satisfy the conditions of the theorem. So it is sufficient to prove the theorem under the condition that the boundary ∂D contains more than two points.

By the fundamental theorem of conformal transformation the domain D is holomorphically equivalent to the unit disk E. Let us denote the inverse of this equivalence by $\varphi(z)$, $z \in E$. Since ∂D contains more than two points, we can assume that D is the subdomain of the complex number plane C. Then, if we could prove that φ is a linear transformation, our proof will be completed, because the image of a disk under a linear transformation is a disk. It is easily verified that the group of automorphisms $A(D)$ of D is isomorphic to the group of automorphisms $A(E)$ of E. The isomorphism is defined by the correspondence $L \leftrightarrow \varphi^{-1} \circ L \circ \varphi$,

$L \in A(D)$. Put $\mu=\varphi^{-1}\circ L\circ\varphi$. Then

(1) $$L\circ\varphi=\varphi\circ\mu.$$

Let us denote by $\mathfrak{S}f(z)$ the *Schwartzian derivative*

$$\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

of a function $f(z)$. Since $L \in A(D)$ is, by assumption, a linear transformation, w know $\mathfrak{S}L\circ\varphi(z)=\mathfrak{S}\varphi(z)$. On the other hand direct calculation shows

$$\mathfrak{S}\varphi\circ\mu(z)=\mathfrak{S}\varphi\big(\mu(z)\big)\cdot\mu'(z)^2+\mathfrak{S}\mu(z).$$

Since $\mu \in A(E)$ and therefore is a linear transformation, we have $\mathfrak{S}\mu(z)\equiv 0$. Put ing $z=0$, we obtain

$$\mathfrak{S}\varphi(0)=\mathfrak{S}\varphi\big(\mu(0)\big)\cdot\mu'(0)^2$$

This identity holds for arbitrary $\mu \in A(E)$, especially for μ of the form: $\mu(z)$ $(z+\alpha)/(1+\bar{\alpha}z)$, $|\alpha|<1$. So

(2) $$\mathfrak{S}\varphi(0)=\mathfrak{S}\varphi(\alpha)\cdot(1-|\alpha|^2)^2.$$

Replacing α by z, we have

(3) $$\mathfrak{S}\varphi(z)=\mathfrak{S}\varphi(0)\cdot(1-|z|^2)^{-2}.$$

Since $\varphi'(z)$ is non-zero, the function $\mathfrak{S}\varphi(z)$ is holomorphic. This contradicts to th right hand side of (3), except when $\mathfrak{S}\varphi(0)=0$. Thus, $\mathfrak{S}\varphi(z)\equiv 0$; this means th $\varphi(z)$ is the restriction of a linear transformation.

References

[1] REMMERT, R. u. K. STEIN: *Eigentlich holomorphe Abbildungen*, Math. Zeitschr. **73**, 159–189 (1960).

[2] HEDTFELD, K. H.: *Starre einfach zusammenhängend Holomorphiegebiete*, Schriftenreihe Math. Inst. Univ Münster, H. **8** (1954).