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THE OPTIMALITY OF (S, s) POLICIES IN THE DYNAMIC INVENTORY PROBLEM WITH EMERGENCY AND NON-STATIONARY STOCHASTIC DEMANDS

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§1. Introduction.

In this paper we consider an n-period, one-commodity dynamic inventory model with non-stationary stochastic demands, and with k (constant) period-lag delivery of regular orders, and with k "emergency" orders, characterized by delivery lags $\mu_1=0$, $\mu_2=1$, $\cdots$, $\mu_k=k-1$. Each ordering cost is composed of a unit cost plus a reorder cost. Let $m^i_{r,n-j+1}$ ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$; $r=0, 1, \cdots, k-1$ for $n\geqq k$; $r=0, 1, \cdots, n-1$ for $n<k$) denote the emergency quantity of the time lag r for the period j, at the beginning of which the demand density is given by φ_i and let $m^i_{k,n-j+1}$ ($n\geqq k$) denote the regular quantity. The cumlative demand in each period is non-negative random variable whose distribution may change from period to period by a Markov transition law with matrix $P=\|p_{ij}\|$ ($i, j=1, 2, \cdots, m$) where $p_{ij}\geqq 0$ and $\sum_{j=1}^{n} p_{ij}=1$ for each i. It is assumed that the demand density does not change during one period. In other words, when the demand density is φ_i duning a period, one of the following period change to φ_j with probability p_{ij} ($i, j=1, 2, \cdots, m$).

The inventory period $I_1, I_2, \cdots, I_n$ are numbered from left to right. At the beginning of the jth period ($j=1, 2, \cdots, n$) two action have to be taken (i) placing k "emergency" orders, i.e., ones for immediate delivery and to be delivered at the end of $j+l-1$ period ($l=1, 2, \cdots, k-1$), (ii) issuing a regular order to be delivered at the end of the $j+k-1$ period. The delivery lag $\lambda=k$ is constant throughout the rest of the paper. In [8] and [9], we discussed the several properties of the optimal policy in the above mentioned dynamic model under an assumption that the ordering costs are linear. In this paper it is shown that if the suitable conditions on the costs are given, then the optimal policy in each purchasing period is always of the (S, s) type. We shall impose the following conditions on the model.

(1. 1) The interval in ordering is k-period ($k\geqq 1$).

(1. 2) There is backlogging of excess demand.

(1. 3) The known distribution function of demand is absolutely continuous with respect to the Lebesgue measure. The density will be denote by $\varphi_i(\xi)$ ($i=1, 2, \cdots, m$).

(1. 4) The holding cost function $h(\eta)$ and the penalty cost function $p(\eta)$ are twice differentiable, positive convex function for positive arguments. We assume that $h(0)=p(0)=0$.

(1. 5) There is credit function $v(\eta)$ defined by

$$v(\eta)=\begin{cases} v\eta & \eta\geqq 0, \\ 0 & \eta<0. \end{cases}$$

The reduced penalty cost, that is, the net penalty cost, is defined in the following way. If at the beginning of any one period the order of size z to be delivered at the end of the period, has been known and a demand ξ occures, then the net penalty cost for this period is

$$p(\xi-y)-v[\min(z, \xi-y)]$$

where y is the starting stock level of that period.

(1. 6) There is a concave, twice differentiable salvage gain function $w(\eta)$ that is increasing for $\eta>0$, and is zero for $\eta\leqq 0$.

(1. 7) The ordering cost function $c_k(\eta)$ for regular orders to be delivered k period later is given by

$$c_k(\eta)=\begin{cases} c_k\eta+K_k(\eta) & \eta>0, \\ 0 & \eta\leqq 0 \end{cases} \qquad K_k(\eta)=\begin{cases} K_k & \eta>0, \\ 0 & \eta\leqq 0. \end{cases}$$

The ordering cost function $c_j(\eta)$ for emergency orders to be delivered j period later is given by

$$c_j(\eta)=\begin{cases} c_j\eta+K_j(\eta) & \eta>0, \\ 0 & \eta\leqq 0, \end{cases} \qquad K_j(\eta)=\begin{cases} K_j & \eta>0 \\ 0 & \eta\leqq 0 \end{cases}$$

with $c_0>c_1>\cdots>c_k>0$, $K_0\geqq K_1\geqq\cdots\geqq K_k>0$ $\quad j=0, 1, \cdots, k-1$.

(1. 8) (a) $\alpha^j \lim_{\eta\to-\infty} w'(\eta)=\alpha^j \lim_{\eta\to\infty}\int_0^\eta w'(\eta-\xi)\varphi_i(\xi)d\xi<c_j<\alpha^{j-1}v$

(b) $w'(0)<v$ $\quad j=1, 2, \cdots, k;\ i=1, 2, \cdots, m$.

(1. 9) (a) $\lim_{\eta\to-\infty} L'(\eta, \varphi_i)+c_0-c_1+v<0$, $\quad$ (b) $\lim_{\eta\to\infty} L'(\eta, \varphi_i)-\alpha\lim_{\eta\to\infty} w'(\eta)>0$

$i=1, 2, \cdots, m$.

(1.10) There is a discount factor α, $0<\alpha\leqq 1$.

(1.11) $L(\eta;\varphi_i)-v\int_0^\eta(\eta-\xi)\varphi_i(\xi)d\xi$ is convex.

where $L(\eta,\varphi_i)$, the expected one-period loss arising from penalty and holding cost, is given by

$$(1.12)\quad L(\eta;\varphi_i)=\begin{cases}\int_0^\eta h(\eta-\xi)\varphi_i(\xi)d\xi+\int_\eta^\infty p(\xi-\eta)\varphi_i(\xi)d\xi & \eta>0\\ \int_0^\infty p(\xi-\eta)\varphi_i(\xi)d\xi & \eta\leqq 0\end{cases}\qquad i=1,2,\dots,m.$$

We shall assume that all integrals occuring in this paper exist and are finite, and that integration and differentiation where needed can be interchanged. This impose certain restrictions on the class of demand densities.

§2. Optimal Policy.

Let $f_n(x;\varphi_i)$ denote the total discount expected loss for an n-period inventory model, where the demand density in the first period is φ_i, x is the initial stock level, and an optimal ordering policy is used at each purchasing opportunity. From the principle of optimalty, we obtain for $n\geqq k$

$$f_n(x;\varphi_i)=\min_{\substack{m_l\geqq 0\\ l=0,1,\dots,k}}\{\sum_{j=0}^{k}(c_jm_j+K_j(m_j))$$

$$+\sum_{j=0}^{k-1}[L_j(x+\sum_{l=0}^{j}m_l;\varphi_i)+V_j(x+\sum_{l=0}^{j+1}m_l;\varphi_i)-V_j(x+\sum_{l=0}^{j}m_l;\varphi_i)]$$

$$+f_{n-k,k}(x+\sum_{l=0}^{k}m_l;\varphi_i)\}$$

$$=\min_{u_0\geqq x}\{c_0(u_0-x)+K_0(u_0-x)+L(u_0;\varphi_i)-V(u_0;\varphi_i)$$

$$+\min_{u_1\geqq u_0}\{c_1(u_1-u_0)+K_1(u_1-u_0)+L_1(u_1;\varphi_i)-V_1(u_1;\varphi_i)+V(u_1;\varphi_i)$$

$$+\min_{u_2\geqq u_1}\{\cdots+\min_{u_{k-1}\geqq u_{k-2}}\{c_{k-1}(u_{k-1}-u_{k-2})+K_{k-1}(u_{k-1}-u_{k-2})$$

$$+L_{k-1}(u_{k-1};\varphi_i)-V_{k-1}(u_{k-1};\varphi_i)+V_{k-2}(u_{k-1};\varphi_i)$$

$$+\min_{u_k\geqq u_{k-1}}\{c_k(u_k-u_{k-1})+K_k(u_k-u_{k-1})+V_{k-1}(u_k;\varphi_i)+f_{n-k,k}(u_k;\varphi_i)\}\cdots\}$$

$$=\min_{u_0\geqq x}\{-c_0x+K_0(u_0-x)+G_{0n}(u_0;\varphi_i)\}$$

where

$$G_{kn}(u_k;\varphi_i)=c_ku_k+V_{k-1}(u_k;\varphi_i)+f_{n-k,k}(u_k;\varphi_i),\qquad n\geqq k$$

$$G_{jn}(u_j;\varphi_i)=c_ju_j+L_j(u_j;\varphi_i)-V_j(u_j;\varphi_i)+V_{j-1}(u_j;\varphi_i)+g_{jn}(u_j;\varphi_i)$$

$$j=0,1,\cdots,k-1$$

$$g_{jn}(u_j;\varphi_i)=\min_{u_{j+1}\geqq u_j}\{-c_{j+1}u_j+K_{j+1}(u_{j+1}-u_j)+G_{j+1n}(u_{j+1};\varphi_i)\}$$

$$j=-1,0,1,\cdots,k-1$$

$$g_{-1,n}(u_{-1};\varphi_i)=f_n(x;\varphi_i),\quad u_{-1}=x$$

$$i_0=i,\ x+m_0=u_0,\ x+m_0+m_1=u_1,\ \cdots,\ x+m_0+m_1+\cdots+m_k=u_k$$

$$L_l(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty L_{l-1}(x-t;\varphi_j)\varphi_i(t)dt,$$

$$V_l(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty V_{l-1}(x-t;\varphi_j)\varphi_i(t)dt, \qquad l\geqq1,$$

$$W_l(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^x W_{l-1}(x-t;\varphi_j)\varphi_i(t)dt,$$

$$f_{n-k,l}(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty f_{n-k,l-1}(x-t;\varphi_j)\varphi_i(t)dt$$

$$f_{n-k,0}(x;\varphi_i)=f_{n-k}(x;\varphi_i)$$

$$V_{-1}(x;\varphi_i)=0,\ V_0(x;\varphi_i)=V(x;\varphi_i),\ L_0(x;\varphi_i)=L(x;\varphi_i)$$

$$f_0(x;\varphi_i)=-W_0(x;\varphi_i)=-w(x)$$

$$V(x;\varphi_i)=\begin{cases}-vx+v\int_0^x(x-t)\varphi_i(t)dt & x\geqq0\\ -vx & x<0\end{cases}$$

It is noticed that $f_{0k}(x,\varphi_i)=-W_k(x,\varphi_i)$ for $x>0$. From the method as in the $n\geqq k$, we have

$$\begin{aligned}(2.\ 3)\quad f_n(x,\varphi_i)=&\min_{u_0\geqq x}\{c_0(u_0-x)+K_0(u_0-x)+L(u_0;\varphi_i)-V(u_0;\varphi_i)\\ &+\min_{u_1\geqq u_0}\{c_1(u_1-u_0)+K_1(u_1-u_0)+L_1(u_1;\varphi_i)-V(u_1;\varphi_i)+V_1(u_1;\varphi_i)\\ &+\min_{u_2\geqq u_1}\{\cdots+\min_{u_{n-1}\geqq u_{n-2}}\{c_{n-1}(u_{n-1}-u_{n-2})+K_{n-1}(u_{n-1}-u_{n-2})\\ &+L_{n-1}(u_{n-1};\varphi_i)-V_{n-1}(u_{n-1};\varphi_i)+V_{n-2}(u_{n-1};\varphi_i)\\ &+\min_{u_n\geqq u_{n-1}}\{c_n(u_n-u_{n-1})+K_n(u_n-u_{n-1})+V_{n-1}(u_n;\varphi_i)\\ &-W_n(u_n;\varphi_i)\}\cdots\}\\ =&\min_{u_0\geqq x}\{-c_0x+K_0(u_0-x)+G_{0n}(u_0;\varphi_i)\} \qquad n<k\end{aligned}$$

where $G_{jn}(u_j;\varphi_i)$ in (2.3) are given by

$$G_{nn}(u_n;\varphi_i)=c_n u_n+V_{n-1}(u_n;\varphi_i)-W_n(u_n;\varphi_i)$$

(2. 4) $$G_{jn}(u_j;\varphi_i)=c_j u_j+L_j(u_j;\varphi_i)-V_j(u_j;\varphi_i)+V_{j-1}(u_j;\varphi_i)+g_{jn}(u_j;\varphi_i)$$

$$n<k;\ j=0,1,\cdots,n-1$$

$$g_{jn}(u_j;\varphi_i)=\min_{u_{j+1}\geqq u_j}\{-c_{j+1}u_j+K_{j+1}(u_{j+1}-u_j)+G_{j+1,n}(u_{j+1};\varphi_i)\}$$

$$g_{-1}(u_{-1};\varphi_i)=f_n(x;\varphi_i)\qquad n<k;\ j=-1,0,1,\cdots,n-1.$$

We cite the known results in [10] that will be needed in the analysis that follows.

Definition. Let $K\geqq 0$, and let $f(x)$ be a differentiable function. We say that $f(x)$ is K-convex if

(2. 5) $$K+f(x+a)-f(x)-af'(x)\geqq 0\qquad \text{for all } a>0 \text{ and all } x.$$

If differentiability is not assumed, the appropriate definition of K-convexity is

(2. 6) $$K+f(x+a)-f(x)-a\left[\frac{f(x)-f(x-b)}{b}\right]\geqq 0$$

$$\text{for all } a>0,\ \text{all } b>0,\ \text{and all } x.$$

It is readily verified that K-convexity has the following properties [10, p. 199]:

(i) O-convex is equivalent to ordinary convexity.

(ii) If $f(x)$ is K-convex, then $f(x+h)$ is K-convex for all h.

(iii) If f and g are K-convex and M-convex, respectively, then $\alpha f+\beta g$ is $(\alpha K+\beta M)$-convex when α and β are positive. This property may be extend to denumerable sum and integrals whenever the interchange of limits is permissible.

THEOREM 2.1. *If conditions of* (1.1)~(1.11) *are satisfied, then*

(i) *there exists a unique pair** $(S_{jn}(\varphi_i), s_{jn}(\varphi))$ *such that* $S_{jn}(\varphi_i)>s_{jn}(\varphi_i)$, $G_{jn}(S_{jn}(\varphi_i);\varphi_i)$ *is the minimum value of* $G_{jn}(x;\varphi_i)$, *and* $G_{jn}(s_{jn}(\varphi_i);\varphi_i)=G_{jn}(S_{jn}(\varphi_i);\varphi_i)+K_j$

$$j=0,1,\cdots,n \text{ for } n\leqq k;\ j=0,1,\cdots,k \text{ for } n>k,\ i=1,2,\cdots,m.$$

(ii) $f_n(x,\varphi_i)$ *is* K_0*-convex, decreasing for* x *small enough, increasing for* x *large enough.*

Proof (by induction). There are two possibilities requiring separate treatment $n\leqq k$ and $n\geqq k+1$.

* If $S_{jn}(\varphi_i)$ and $s_{jn}(\varphi_i)$ are not a unique, we choose the smallest such value. By the term "unique pair $(S_{jn}(\varphi_i), s_{jn}(\varphi_i))$" we mean the pair of the smallest value of $S_{jn}(\varphi_i)$ and $s_{jn}(\varphi_i)$. Similar remarks will be apply whenever we speak about the unique pair.

Case (a) $n \leqq k$. Suppose first that $n=1$. Then we have

$$\lim_{u\to\infty} G'_{11}(u;\varphi_i)=c_1-\alpha \lim_{u\to\infty} w'(u)>0 \quad \text{by (1.8a) and (2.4)} \tag{2.7}$$

and

$$G'_{11}(t;\varphi_i)=c_1-v<0 \qquad \text{for } t\leqq 0 \qquad \text{by (1.8a) and (2.4)} \tag{2.8}$$

Hence, each equation $G'_{11}(u;\varphi_i)=0$ possesses a unique root or single closed interval of zeros. Let $S_{11}(\varphi_i)>0$ denote the smallest root. For definiteness, henceforth, whenever we speak of the root of such an equation we shall mean the smallest root. Since $G_{11}(u;\varphi_i)$ is strictly decreasing for $u<S_{11}(\varphi_i)$, and $K_1>0$, there exists a unique finite $s_{11}(\varphi_i)$ such that $s_{11}(\varphi_i)<S_{11}(\varphi_i)$ and $G_{11}(s_{11}(\varphi_i);\varphi_i)=G_{11}(S_{11}(\varphi_i);\varphi_i)+K_1$. Based on this result we obtain from (2.4)

$$g_{01}(u_0;\varphi_i)=\begin{cases}-c_1u_0+K_1+G_{11}(S_{11}(\varphi_i);\varphi_i) & u_0<s_{11}(\varphi_i)\\ -c_1u_0+G_{11}(u_0;\varphi_i) & u_0\geqq s_{11}(\varphi_i)\end{cases} \tag{2.9}$$

From Scarf [10], it follows that $g_{01}(u;\varphi_i)$ is K_1-convex. We have from (2.4), (1.9 a, b), (2.7), and (2.9)

$$\lim_{u\to\infty} G'_{01}(u;\varphi_i)=c_0-c_1+\lim_{u\to\infty} L'(u;\varphi_i)+\lim_{u\to\infty} G'_{11}(u;\varphi_i)>0 \tag{2.10}$$

and

$$\lim_{u\to-\infty} G'_{01}(u;\varphi_i)=c_0-c_1+v+\lim_{u\to-\infty} L'(u;\varphi_i)<0 \tag{2.11}$$

Since $c_0u+L(u;\varphi_i)-V(u;\varphi_i)$ is convex, and $g_{01}(u;\varphi_i)$ is K_1-convex, $G_{01}(u,\varphi_i)$ is K_1 convex by the properties (i), (ii), and (iii) above and therefore K_0-convex. Hence there exists a unique pair $(S_{01}(\varphi_i), s_{01}(\varphi_i))$ such that $S_{01}(\varphi_i)>s_{01}(\varphi_i)$, $G_{01}(S_{01}(\varphi_i);\varphi_i)$ is the minimum value of $G_{01}(u_1;\varphi_i)$, and $G_{01}(s_{01}(\varphi_i);\varphi_i)=G_{01}(S_{01}(\varphi_i);\varphi_i)+K_0$. Hence we have from above result and (2.4)

$$f_1(x;\varphi_i)=\begin{cases}-c_0x+K_0+G_{01}(S_{01}(\varphi_i);\varphi_i) & x<s_{01}(\varphi_i),\\ -c_0x+G_{01}(x;\varphi_i) & x\geqq s_{01}(\varphi_i),\end{cases} \tag{2.12}$$

$$f'_1(x;\varphi_i)=\begin{cases}-c_0 & x<s_{01}(\varphi_i),\\ -c_0+G'_{01}(x;\varphi_i) & x>s_{01}(\varphi_i).\end{cases} \tag{2.13}$$

It is easily seen that $f_1(x;\varphi_i)$ is K_0-convex by the same method as Scarf [10], th another part of (ii) is immediate from (2.13). Assuming that the theorem 2. hold the integer $n-1$ and that

$$V'_l(u;\varphi_i)=-\alpha^l v \qquad \text{for all } u\leqq 0.$$

(2.14)
$$\lim_{u\to\infty}(L'_l(u;\varphi_i)-V'_l(u;\varphi_i)>\alpha^{l+1}\lim_{u\to\infty}w'(u)$$
$$\lim_{u\to-\infty}(L'_l(u;\varphi_i)-V'_l(u;\varphi_i))<0$$
$$\lim_{u\to\infty}W'_{l+1}(u;\varphi_i)=\alpha^{l+1}\lim_{u\to\infty}w'(u)$$
$$l=0,1,\dots,n-2;\ i=1,2,\dots,m.$$

then we have

(2.15)
$$\lim_{u\to\infty}G'_{nn}(u;\varphi_i)=c_n+\alpha\sum_{j=1}^{m}p_{ij}\lim_{u\to\infty}\int_0^\infty(V'_{n-2}(u-t;\varphi_j)-W'_{n-1}(u-t;\varphi_j))\varphi_i(t)dt$$
$$=c_n-\alpha^n\lim_{u\to\infty}w'(u)>0 \qquad \text{by (1.8a)}$$

and

(2.16)
$$G'_{nn}(u;\varphi_i)=c_n+\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty(V'_{n-2}(u-t;\varphi_i)-W'_{n-1}(u-t;\varphi_j))\varphi_i(t)dt$$
$$=c_n-\alpha^{n-1}v<0 \qquad \text{for } u\leqq 0 \qquad \text{by (1.8a)}$$

Hence, there exists a unique positive $S_{nn}(\varphi_i)>0$ such that $G'_{nn}(S_{nn}(\varphi_i);\varphi_i)=0$. The function $G_{nn}(u;\varphi_i)$ is strictly decreasing for $u<S_{nn}(\varphi_i)$ and $K_n>0$. Hence for each i there exists a unique finite $s_{nn}(\varphi_i)$ such that $s_{nn}(\varphi_i)<S_{nn}(\varphi_i)$, and $G_{nn}(s_{nn}(\varphi_i);\varphi_i)=G_{nn}(S_{nn}(\varphi_i);\varphi_i)+K_n$. Based on this result we obtain from (2.4)

(2.17)
$$g_{n-1,n}(u_{n-1};\varphi_i)=\begin{cases}-c_nu_{n-1}+K_n+G_{nn}(S_{nn}(\varphi_i);\varphi_i) & u_{n-1}<s_{nn}(\varphi_i)\\ -c_nu_{n-1}+G_{nn}(u_{n-1};\varphi_i) & u_{n-1}\geqq s_{nn}(\varphi_i)\end{cases}$$

From Scarf [10], it follows that $g_{n-1,n}(u;\varphi_i)$ is K_n-convex. Assume that part (i) of the theorem is true for the integer r $(0<r\leqq n)$, and that $\lim_{u\to\infty}G'_{rn}(u;\varphi_i)>0$ for r $(0<r\leqq n)$, and that $g_{r-1,n}(u;\varphi_i)$ is K_r-convex, and that

(2.18)
$$g_{r-1,n}(u_{r-1};\varphi_i)=\begin{cases}-c_ru_{r-1}+K_r+G_{rn}(S_{rn}(\varphi_i);\varphi_i) & u_{r-1}<s_{rn}(\varphi_i)\\ -c_ru_{r-1}+G_{rn}(u_{r-1};\varphi_i) & u_{r-1}\geqq s_{rn}(\varphi_i)\end{cases}$$

Then we get from (2.4)

(2.19)
$$\lim_{u\to\infty}G'_{r-1,n}(u;\varphi_i)=(c_{r-1}-c_r)+\alpha\sum_{j=1}^{m}p_{ij}\lim_{u\to\infty}\int_0^\infty(L'_{r-2}(u-t;\varphi_j)-V'_{r-2}(u-t;\varphi_j)$$
$$+V'_{r-3}(u-t;\varphi_j))\varphi_i(t)dt+\lim_{u\to\infty}G'_{rn}(u;\varphi_i)$$
$$\geqq(c_{r-1}-c_r)+\alpha^r\lim_{u\to\infty}w'(u)+\lim_{u\to\infty}G'_{rn}(u;\varphi_i)>0$$

and

$$\text{(2.20)}\qquad \lim_{u\to-\infty} G'_{r-1,n}(u;\varphi_i)=c_{r-1}-c_r+\lim_{u\to-\infty}(L'_{r-1}(u;\varphi_i)-V'_{r-1}(u;\varphi_i)+V'_{r-2}(u;\varphi_i))$$
$$<(c_{r-1}-\alpha^{r-2}v)-c_r<0$$

Since $g_{r-1,n}(u;\varphi_i)$ is K_r-convex and $(c_{r-1}u+L_{r-1}(u;\varphi_i)-V_{r-1}(u;\varphi_i)+V_{r-2}(u;\varphi_i))$ is convex, it is easily seen that $G_{r-1,n}(u;\varphi_i)$ is K_r-convex, therefore $G_{r-1,n}(u;\varphi_i)$ is K_{r-1}-convex. Hence there exists a unique pair $(S_{r-1,n}(\varphi_i), s_{r-1,n}(\varphi_i))$ such that $S_{r-1,n}(\varphi_i)>s_{r-1,n}(\varphi_i)$, $G_{r-1,n}(S_{r-1,n}(\varphi_i);\varphi_i)$ is the minimum value of $G_{r-1,n}(u;\varphi_i)$, and $G_{r-1,n}(s_{r-1,n}(\varphi_i);\varphi_i)=G_{r-1,n}(S_{r-1,n}(\varphi_i);\varphi_i)+K_{r-1}$. Hence we get from (2.4)

$$\text{(2.21)}\qquad g_{r-2,n}(u_{r-2};\varphi_i)=\begin{cases}-c_{r-1}u_{r-2}+K_{r-1}+G_{r-1,n}(S_{r-1,n}(\varphi_i);\varphi_i) & u_{r-2}<s_{r-1,n}(\varphi_i)\\ -c_{r-1}u_{r-2}+G_{r-1,n}(u_{r-2};\varphi_i) & u_{r-2}\geqq s_{r-1,n}(\varphi_i)\end{cases}$$

$$\text{(2.22)}\qquad g'_{r-2,n}(u_{r-2};\varphi_i)=\begin{cases}-c_{r-1} & u_{r-2}<s_{r-1,n}(\varphi_i)\\ -c_{r-1}+G'_{r-1,n}(u_{r-2};\varphi_i) & u_{r-2}>s_{r-1,n}(\varphi_i)\end{cases}$$

Since $G_{r-1,n}(u;\varphi_i)$ is K_{r-1}-convex, it follows that $g_{r-2,n}(u;\varphi_i)$ is K_{r-1}-convex by the same method as Scarf [10]. The another part of (ii) is immedite from (2.22). Using results in the case $n\leqq k$, we can prove inductively the case $n>k$ by the similar method to that above, except that $-W_n(x;\varphi_i)$ is replaced by $f_{n-k,k}(x;\varphi_i)$ and, therefore, we will omit it. Moreover we obtain from the argument in Theorem 2.1 the following theorems.

THEOREM 2.2. *If conditions of Theorem 2.1 are satisfied, then the optimal order m^{i*}_{jn} ($j=0,1,\dots,k$ for $n>k$; $j=0,1,\dots,n$ for $n\leqq k$) in the first period are of the following form.*

(i) $n\leqq k$

$$m^{i*}_{0n}(x)=\begin{cases}S_{0n}(\varphi_i)-x & x<s_{0n}(\varphi_i)\\ 0 & x\geqq s_{0n}(\varphi_i)\end{cases}$$

$$m^{i*}_{1n}(u_1)=\begin{cases}S_{1n}(\varphi_i)-u_0 & u_0<s_{1n}(\varphi_i)\\ 0 & u_0\geqq s_{1n}(\varphi_i)\end{cases}\qquad i=1,2,\dots,m.$$

$$\vdots$$

$$m^{i*}_{nn}(u_{n-1})=\begin{cases}S_{nn}(\varphi_i)-u_{n-1} & u_{n-1}<s_{nn}(\varphi_i)\\ 0 & u_{n-1}\geqq s_{nn}(\varphi_i)\end{cases}$$

(ii) $n>k$

$$m^{i*}_{0n}(x)=\begin{cases}S_{0n}(\varphi_i)-x & x<s_{0n}(\varphi_i)\\ 0 & x\geqq s_{0n}(\varphi_i)\end{cases}$$

$$m_{1n}^{i*}(u_0)=\begin{cases} S_{1n}(\varphi_i)-u_0 & u_0<s_{1n}(\varphi_i) \\ 0 & u_0\geqq s_{1n}(\varphi_i) \end{cases} \qquad i=1, 2, \cdots, m$$

$$\vdots$$

$$m_{kn}^{i*}(u_{k-1})=\begin{cases} S_{kn}(\varphi_i)-u_{k-1} & u_{k-1}<s_{kn}(\varphi_i) \\ 0 & u_{k-1}\geqq s_{kn}(\varphi_i) \end{cases}$$

whese

$$u_0=x+m_{0n}^{i*}(x),\ u_1=u_0+m_{1n}^{i*}(u_0)=x+m_{0n}^{i*}(x)+m_{1n}^{i*}(x+m_{0n}^{i*}(x)),$$

$$\cdots\cdots u_j=u_{j-1}+m_{jn}^{i*}(u_{j-1})$$

$$j=2, 3, \cdots, n-1 \text{ for } n\leqq k;\ j=2, \cdots, k-1 \text{ for } n>k.$$

THEOREM 2.3. *Let conditions of Theorem* 3.1 *hold except* (1.8b) *and* (1.9). *If there exists a unique pair* $(S_{jn}(\varphi_i), s_{jn}(\varphi_i))$ *such that the properties* (i) *of the theorem* 2.1 *hold, then the optimal ordering policy in the first period is of the* (S, s) *type, i.e. the form of Theorem* 2.2.

REMARK. *The* K_0*-convexity of* $f_n(x, \varphi_i)$ *may be proved under less strigent conditions, i.e., ones of* (1.1)～(1.11) *except* (1.8b) *and* (1.9).

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References

[1] BARANKIN, E.W. A Delivery-Lag Inventory Model with Emergency Provision (The Single-Period Case). Naval Res. Logist. Quart. 1961, **8** (3), 285-311.

[2] DANIEL, K.H., A Delivery-Lag Inventory Model with Emergency, Chapter 2 in SCARF, H.E., D.M. GILFORD, and M.W. SHELLY (eds), Multistage Inventory Models and Techniques, Standford, Calif; Stanford Univ. Press, 1963.

[3] KARLIN, S. Dynamic Inventory Policy with Varying Stochastic Demands. Management Sci., 1960, **6** (3), 231-258.

[4] FUKUDA, Y. Optimal Policy for the Inventory Problem with Negotiable Leadtime. Management Sci., 1964, **10** (4), 690-708.

[5] ARROW, K.J., S. KARLIN, and H. SCARF. Studies in the Mathematical Theory of Inventory and Production. Standford, Calif; Stanford Univ. Press, 1958.

[6] IGLEHART, D. and S. KARLIN. Optimal Policy for Dynamic Inventory Process with Non-stationary Stochastic Demands. Studies in Applied Probability and Management Science. Stanford, Calif: Standford Univ. Press, 1962.

[7] KODAMA, M. A Type of Statistical Inventory Problems with Emergency and Delivery-Lag, Memoirs of the Faculty of General Education, Kumamoto University. Sersies of Natural Sciences, No. 1 (March, 1966).

[8] ———. A Delivery-Lag Inventory Control Process with Emergency and Non-Stationary Stochastic Demands, Bulletin of Mathematical Statistics, Research Association of Statistical Sciences, Vol. 12, No. 1-2.

[9] ———. A Delivery-Lag Inventory Control Process with Emergency and Non-Stationary Stochastic Demands II. Kumamoto J. Sci. A. Vol. 7, No. 3, 1966.

[10] SCARF, H., The Optimality of (S, s) Policies in the Dynamic Inventory Problem, Chapter 13 in ARROW, K. J., S. KARLIN, and P. SUPPES (eds), Mathematical Methods in the Social Sciences, 1959, Stanford, California: Stanford University Press, 1960,