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A CLOSED SEQUENTIAL PROCEDURE SELECTING THE POPULATION WITH MINIMUM VARIANCE FROM SEVERAL NORMAL POPULATIONS

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§1. Summary.

The problem to select the population in some specified sense from several preassignied populations is very important. A general aspect which generalised a class of closed sequential statistical procedure (CSSP) introduced by Paulson [5] was presented by us [2], where a family of CSSP was applied to the populations whose probability density functions follow one parameter exponential distributions, and where many sequential statistical procedures cited there were classified into several different ways.

A problem which selects the population with minimum variance from several normal populations was discussed by the authors such as Gupta and Sobel [1], Nomachi [2], Paulson [3], [4] and [5] and Truax [6]. However no closed sequential (multiple) statistical procedure for the selection of population having the minimum variance has been used by any of them, except for [2].

Our object in this paper is to present certain family of CSSP for a set of probability 1-α in $0<\alpha<1$ and for a certain configuration of population variances for which the probability by which the best population (with the minimum variance) from several normal populations having unknown means is selected is larger than the preassigned value 1-α in $0<\alpha<1$.

§2. Introduction.

Let us consider a set of (experimental) normal populations $\Pi_i(i=1,2,\cdots,k)$ having mean μ_i and variance σ_i^2 whose values are unknown to us. Let $\{x_{i,j}\}$, $j=1, 2, \cdots$ be the $i-$th sequence of the random samples drawn from the population $\Pi_i(i=1,2,\cdots,k)$.

DEFINITION. For a fixed value of $\rho_0(\rho_0>1)$, let us define a configuration of k population parameters $(\sigma_1, \sigma_2, \cdots, \sigma_k)$ by

(2.1) $$\delta(\sigma_{[1]}, \boldsymbol{\rho}) = \{(\sigma_1, \sigma_2, \cdots, \sigma_k) \mid \sigma_i = \rho_i \sigma_{[1]} \geqq \rho_0 \sigma_{[1]}, \sigma_1 = \sigma_{[1]}, i = 2, 3, \cdots, k\},$$

where $\sigma_{[1]}$ denotes the least one among $(\sigma_1, \sigma_2, \cdots, \sigma_k)$. In this connection, let us define a configuration of k population parameters $(\sigma_1, \sigma_2, \cdots, \sigma_k)$ by

(2.2) $$\delta(\sigma_{[1]}, \boldsymbol{\rho_0}) = \{(\sigma_1, \sigma_2, \cdots, \sigma_k) \mid \sigma_i = \rho_0 \sigma_{[1]}, \sigma_1 = \sigma_{[1]}, i = 2, 3, \cdots, k\}.$$

Then the configuration $\delta(\sigma_{[1]}, \boldsymbol{\rho_0})$ is said a least favorable configuration in a class of configurations $\delta(\sigma_{[1]}, \boldsymbol{\rho})$ whenever $\rho_i \geqq \rho_0 (i=2,3,\cdots,k)$. Let us call the population which has the minimum variance among k populations by the "*best*" population.

Now our object is restated as to give a family of CSSP by which the probability of the correct selecting decision of the best population is larger than the preassigned value $1-\alpha, (0<\alpha<1)$, under a configuration of population parameters $\delta(\sigma_{[1]}, \boldsymbol{\rho})$.

§3. The enunciation of CSSP.

In the first place, let us denote by $\Pi_i^{(l)}$ the population which was not eliminated at the $(l-1)$-th stage of compariaons, and let us put

(3.1) $$\bar{x}_i^{(l)} = \sum_{j=1}^{l} x_{i,j}^{(l)}/l \text{ and } S_{i,(l)}^2 = \sum_{j=1}^{l} (x_{i,j}^{(l)} - \bar{x}_i^{(l)})^2/l,$$

for $i=1, 2, \cdots, k_l$; $l=N_1+1, \cdots, N_2+1$, where N_1 and N_2 will be defined by (3.2) and (3.3), respectively. Let us denote by $\Pi_{[i]}^{(l)}$ the population from which $s_{[i],(l)}^2$ was calculated, where $s_{[i],(l)}^2$ denote the i-th smallest among a set of sample variances $s_{i,(l)}^2 (i=1, 2, \cdots, k_l)$ for each integer l in $N_1 \leqq l \leqq N_2+1$. Then let us present a family of CSSP denoted by S_λ for each value of λ in $1<\lambda<\rho_0$ as follows.

(0) Let us define

(3.2) $$N_1 = [log\{(k-1)/\alpha\}/log\lambda]+2,$$

(3.3) $$N_2 = [log\{(k-1)/\alpha\}/log\{\lambda(1+\rho^2)(\lambda^2+\rho_0^2)^{-1}\}]+2$$

and

(3.4) $$A_\lambda(n; k, \alpha, \rho_0) = \lambda(\lambda - B_n)\{\rho_0^2(\lambda B_n - 1)\}^{-1},$$

where the notation $[\xi]$ denotes the largest integer which is not larger than the value of ξ, and where

(3.5) $$B_n = \{\alpha/(k-1)\}^{1/(n-1)}.$$

(i) In the first stage of comparions, let $\{x_{i,1}^{(1)}, x_{i,2}^{(1)}, \cdots, x_{i,N_1}^{(1)}\}$ be a sample of size N_1 drawn from the population $\Pi_i^{(1)}, (i=1, 2, \cdots, k)$ respectively.

Let us arrange the set of k values $s^2_{1,(N_1)}, s^2_{2,(N_1)}, \cdots, s^2_{k,(N_1)}$ in the ascending order of magnitude, and write them in the following form

(3.6) $$s^2_{[1],(N_1)} \leqq s^2_{[2],(N_1)} \leqq \cdots \leqq s^2_{[k],(N_1)}.$$

If the following relation holds true

(3.7) $$A_\lambda(N_1; k, \alpha, \rho_0) s^2_{[1],(N_1)} \leqq s^2_{[\nu],(N_1)}$$

then we elimiate the population $\Pi^{(1)}_{[\nu]}$, $2 \leqq \nu \leqq k$, at this stage.

Now our statistical procedure proceeds to either one of two alternative ways (i) (a) and (i) (b).

(i) (a) In this case, if only one population $\Pi^{(1)}_{[1]}$ was not elimiated, then we do not draw any more sample and we decide the population $\Pi^{(1)}_{[1]}$ as the best one.

(i) (b) If more than one population were left as the candidates for the best one, then we proceed to the following stage of sampling.

(l^0) $(l = N_1+1, N_1+2, \cdots, N_2)$. Proceeding by induction, at the l-th stage of sampling, let us consider the set of populations denoted by $\Pi^{(l)}_i (i = 1, 2, \cdots, k_l)$ which were not eliminated until the $(l-1)$-th stage of comparisons. Let $x^{(l)}_{i,l}$ be an additional sample of size one drawn from $\Pi^{(l)}_i$, $(i=1, 2, \cdots, k_l)$ respectively, and let us donote by $\{x^{(l-1)}_{i,1}, x^{(l-1)}_{i,2}, \cdots, x^{(l-1)}_{i,l-1}\}$ a pooled sample drawn from $\Pi^{(l-1)}_i$ before and new sample $x^{(l)}_{i,l}$, $(i = 1, 2, \cdots, k_l)$ respectively. Let us arrange the set of k_l values $s^2_{i,(l)} (i = 1, 2, \cdots, k_l)$ in the ascending order of magnitude, and write them in the form $s^2_{[1],(l)} \leqq s^2_{[2],(l)} \leqq \cdots \leqq s^2_{[k_l],(l)}$.

If the following relation holds true

(3.8) $$A_\lambda(l\ ; k, \alpha, \rho_0) s^2_{[1],(l)} \leqq s^2_{[\nu],(l)},$$

then we elimiate the population $\Pi^{(l)}_{[\nu]}$, $2 \leqq \nu \leqq k_l$, at this stage.

In this case, our statistical procedure proceeds to either one of two alternative ways (l^0) (a) and (l^0) (b).

(l^0) (a). If only one population $\Pi^{(l)}_{[1]}$ was not eliminated, then we do not draw any more sample and we decide the population $\Pi^{(l)}_{[1]}$ as the best one.

(l^0) (b). If more than one population were left as the candidates for the best one, then we proceed to the following stage.

We continue these steps of sampling and comparisons in a sequential ways, so far as the best population has not been decided and until the stage number l is smaller than or equal to a preassigned integer N_2. If we attain to the case of (N^0_2) (b), then we proceed to the following (N_2+1)-th stage.

$((N_2+1)^0)$. In this stage, let us consider those population which were not eliminated at the N_2-th stage of comparisons. Let $x^{(N_2+1)}_{i,N_2+1}$ be an additional sample of size one drawn from $\Pi_i^{(N_2+1)}$, $(i=1, 2, \cdots, k_{N_2+1})$ respectively, and let us denote by $\{x_{i,1}^{(N_2+1)}, x_{i,2}^{(N_2+1)}, \cdots, x_{i,N_2+1}^{(N_2+1)}\}$ a pooled sample of samples $\{x_{i,1}^{(N_2)}, x_{i,2}^{(N_2)}, \cdots, x_{i,N_2}^{(N_2)}\}$ drawn from $\Pi_i^{(N_2)}$ before and new sample $x_{i,N_2+1}^{(N_2+1)}$, $i=1,2,\cdots,k_{N_2+1}$ respectively. Let us arrange the set of k_{N_2+1} values $s_{i,(N_2+1)}^2$, $(i=1, 2, \cdots, k_{N_2+1})$ in the ascending order of magnitude, and write them in the form $s_{[1],(N_2+1)}^2 \leqq s_{[2],(N_2+1)}^2 \leqq \cdots \leqq s_{[kN_2+1],(N_2+1)}^2$. Then we decide the population $\Pi_{[1]}^{(N_2+1)}$ as the best one.

§4. Main result.

Let us denote by $P\{\Pi_{[1]} | S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\}$ the probability by which the best population $\Pi_{[1]}$ is eliminated by mean of CSSP S_λ under the configuration of population parameters $\delta(\sigma_{[1]}, \boldsymbol{\rho})$. Then we have the following.

THEOREM. *For a set of constants $\{k, \alpha, \rho_0, \lambda\}$ which were assigned previously in* $k \geqq 2$, $0<\alpha<1$, $\rho_0>1$ and $1<\lambda<\rho_0^2$, we have

$$(4.1) \qquad P\{\Pi_{[1]} | S_\lambda, \delta_{[1]}, \boldsymbol{\rho})\} \leqq \alpha.$$

Before the proof of this theorem, let us enunciate the following preparations.

For a fixed constant ρ_0 in $\rho_0 > 1$, let us consider a test of hypothesis $H_0 : \rho = \rho_0$ against alternative $H_1 : \rho = \rho_0/\lambda$ $(\rho_0^2 > \lambda > 1)$, where $\rho = \sigma_2/\sigma_1$. Let $\{x_{i,1}, x_{i,2}, \cdots, x_{i,n}\}$ $(i=1,2)$ be two independent samples drawn from the normal population $\Pi_i (i=1,2)$, and put that

$$(4.2) \qquad \bar{x}_{i,n} = \sum_{j=1}^{n} x_{i,j}/n, \quad s_{i,n}^2 = \sum_{j=1}^{n} (x_{i,j} - \bar{x}_{i,n})^2/n, \quad \text{and} \quad F_n = s_{2,n}^2/s_{1,n}^2.$$

Now the statistic F_n/ρ_0^2 and F_n/ρ^2 follows F-distribution with a pair of degrees of freedom $(n\text{-}1, n\text{-}1)$ under H_0 and H_1, respectively. Then we have the following lemmas.

LEMMA 1. (Paulson [5]). *We have*

$$(4.3) \qquad P\left\{ \frac{h_n(F_n/\rho_0^2)}{h_n(F_n/(\rho_0/\lambda)^2)} < \beta, \text{ at least one } n, n < \infty \,|\, H_0 \right\} \leqq \beta,$$

where $\beta = \alpha/(k-1)$ *and*

$$(4.4) \qquad h_n(F_n/\rho^2) = \frac{\Gamma(n-1)}{\{\Gamma((n-1)/2)\}^2} \, \frac{F_n^{(n-3)/2}}{\rho^{n-1}(1+F_n/\rho)^{n-1}}, \quad (F_n \geqq 0).$$

LEMMA 2. *For a set of constants $\{k, \alpha, \rho, \lambda\}$ which were assigned previously in* $k \geqq 2$, $0<\alpha<1$, $\rho_0>1$ *and* $1<\lambda<\rho_0^2$, *we have*

$$(4.5) \qquad P\{\Pi_{[1]} | S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\} = \beta,$$

where $\beta = \alpha/(k-1)$.

Proof. From the definition of procedure S_λ, we have

$$(4.6)\quad P\{\Pi_{[1]}\mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}$$
$$= P\{F_n < A_\lambda^{-1}(n\,; k, \alpha, \rho_0),\ \textit{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\},$$

where $F_n = s_{2,n}^2/s_{1,n}^2$ was defined by (4.2). In virtue of (3.2) it easily turns out that the relation $n \geqq N_1$ means the following relation

$$(4.7)\quad \lambda B_n > 1,$$

where $B_n = \{\alpha/(k-1)\}^{1/(n-1)}$. For a set of specified constants $\{k, \alpha, \rho_0. \lambda\}$ which were assigned previously in $k \geqq 2$, $0<\alpha<1$, $\rho_0>1$ and $1<\lambda<\rho_0^2$, noting that $B_n<1$, we can see that $n \leqq N_2$ means $A_\lambda(n\,; k, \alpha, \rho_0) \geqq 1$. Hence after simplification, the relation in (4.3)

$$(4.8)\quad h_n(F_n/\rho_0^2)/h_n(F_n/(\rho_0/\lambda)^2) < \beta$$

can be written in the following equivalent form

$$(4.9)\quad F_n < \rho_0^2(\lambda B_n - 1)\{\lambda(\lambda - B_n)\}^{-1}.$$

Therefore by use of the result of Lemma 1, we have

$$(4.10)\quad P\{\Pi_{[1]}\mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}$$
$$= P\{F_n < \rho_0^2(\lambda B_n - 1)\{\lambda(\lambda - B_n)\}^{-1},\ \text{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}$$
$$= P\{F_n < \rho_0^2(\lambda B_n - 1)\{\lambda(\lambda - B_n)\}^{-1},\ \textit{for some } n < \infty \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}$$
$$= \beta,$$

which is to be proved.

Lemma 3. *Under the condition of Lemma* 2, *we have*

$$(4.11)\quad P\{\Pi_{[1]}\mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\} = P\{\Pi_{[1]}\mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}.$$

Proof. Since that for any $\rho > \rho_0 > 1$, F_n/ρ_0 and F_n/ρ follows F-distribution with a pair of degrees of freedom $(n-1, n-1)$ under H_0 and H_1 (equivalently, under $\delta(\sigma_{[1]}, \boldsymbol{\rho}_0)$ and $\delta(\sigma_{[1]}, \boldsymbol{\rho})$), respectively, we have the following relations

$$(4.12)\quad P\{\Pi_{[1]}\mid S_\lambda, \delta_{[1]}, \boldsymbol{\rho}_0)\}$$
$$= P\{s_{2,n}^2/s_{1,n}^2 < A_\lambda^{-1}(n\,; k, \alpha, \rho_0),\ \textit{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)$$
$$= P\left\{\frac{s_{2,n}^2}{s_{1,n}^2\rho_0^2} < \frac{(\lambda B_n - 1)}{\lambda(\lambda - B_n)},\ \textit{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\right\}$$
$$\geqq P\left\{\frac{s_{2,n}^2}{s_{1,n}^2\rho^2} < \frac{\rho_0^2(\lambda B - 1)}{\rho^2\lambda(\lambda - B_n)},\ \textit{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\right\}$$
$$= P\{F_n < A_\lambda^{-1}(n\,; k, \alpha, \rho_0),\ \textit{for some } n \textit{ in } N_1 \leqq n \leqq N_2 \mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\}$$
$$= P\{\Pi_{[1]}\mid S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\},$$

which is to be proved.

Proof of the thorem. In the lighs of symmetric property of our statiatical procedure S_λ and the results of Lemma 2 and Lemma 3, we have the following relations

$$(4.13)\quad P\{\Pi_{[1]}|S_\lambda, \delta_{[1]}, \boldsymbol{\rho})\}$$
$$=P\{s^2_{1,(n)} > A_\lambda(n\,; k, \alpha, \rho_0)s^2_{\nu,(n)},\ \textit{for some } \nu \textit{ in } 2\leqq\nu\leqq k$$
$$\textit{and for some } n \textit{ in } N_1\leqq n\leqq N_2\,|\,S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\}$$
$$\leqq(k-1)P\{s^2_{1,(n)} > A_\lambda(n; k, \alpha, \rho_0)S^2_{2,n},\ \textit{for some } n \textit{ in } N_1\leqq n\leqq N_2\,|\,S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\}$$
$$=(k-1)P\{\Pi_{[1]}|S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho})\}$$
$$=(k-1)P\{\Pi_{[1]}|S_\lambda, \delta(\sigma_{[1]}, \boldsymbol{\rho}_0)\}$$
$$=\alpha,$$

which is to be proved.

§5. Related problem.

The stage number n which our CSSP terminates eventually is a random variable depending upon total samples drawn before. When we consider about the performance of our procedure, $E\{n\}$, the expectation of n, may be a sort of measures for the performance. In case when $\lambda=\rho_0$, we have

$$\underset{1<\lambda<\rho_0^2}{Min}\ N_2=[log\{(k-1)/\alpha\}/log\{(\rho_0-1/\rho_0)/2\}]+2,$$

however, we do not know what value of λ in $1<\lambda<\rho_0^2$ makes $E\{n\}$ minimum value.

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