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ON COHOMOLOGY GROUPS OF GENERAL LIE TRIPLE SYSTEMS

By

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A general Lie triple system is a tangent algebraic system of the reductive homogeneous space, which was studied by Nomizu in [6], also refer Raševskiĭ [7]. This algebraic system is studied by Sagle in [8, 9, 11, 13, also see 10, 12, 14]. In this paper, it is considered the cohomology groups associated with a weak representation or a representation of general Lie triple system T and will be given an interpretation of the pair of first and second cohomology groups associated with weak representation of T and of the pair of second and third cohomology groups associated with representation of T. Throughout this paper, we assume that the characteristic of the base field Φ is zero and the vector spaces are finite dimensional.

1. Introduction.

A *general Lie triple system* (*general L. t. s.* simply) is an algebraic system T with bilinear composition xy and trilinear composition $[xyz]$ satisfying the following relations [16]:

$$x^2=0, \tag{1.1}$$

$$[xxy]=0, \tag{1.2}$$

$$[xyz]+[yzx]+[zxy]+(xy)z+(yz)x+(zx)y=0, \tag{1.3}$$

$$[(xy)zw]+[(yz)xw]+[(zx)yw]=0, \tag{1.4}$$

$$[xy(zw)]=[xyz]w+z[xyw], \tag{1.5}$$

$$[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]]. \tag{1.6}$$

A linear mapping D of T is a derivation of T if $D(xy)=(Dx)y+x(Dy)$ and $D([xyz])=[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$. Then (1.5) and (1.6) say a linear mapping $D(x, y): z\rightarrow[xyz]$ is a derivation of T, which will be called inner. In the general L. t. s. T, if every ternary product $[xyz]$ vanishes, then above axioms reduce to that of Lie algebras. If all binary products xy vanish, then above axioms reduce to that of Lie triple system (L.t.s. simply), that is, (1.2), (1.6), and

$$(1.3)' \qquad [xyz]+[yzx]+[zxy]=0.$$

A subspace U of general L.t.s. T is called a subsystem if U is closed under both compositions UU and $[UUU]$, this condition is equivalent to that $xy \in U$ and $[xyy] \in U$ for all $x, y \in U$ if the characteristic of Φ is different from 3. A subspace U of T is an ideal if $UT \subseteq U$ and $[UTT] \subseteq U$, in this case it follows $[TTU] \subseteq U$. Hence the ideal is invariant under the inner derivations. Let f be a linear mapping of a general L. t. s. T into a general L. t. s. U, f is called a homomorphism of T into U if $f(xy)=f(x)f(y)$ and $f([xyz])=[f(x)f(y)f(z)]$ (or again equivalently $f(xy)=f(x)f(y)$ and $f([xyy])=[f(x)f(y)f(y)]$ if the characteristic of $\Phi \neq 3$) for all $x, y, z \in T$. The definition of isomorphism is also clear. Let W be a kernel of homomorphism of T onto U, then W is an ideal of T and a quotient system T/W is defined naturally and T/W is isomorphic with U. An ideal W of T is called abelian in T if $WW=(0)$ and $[TWW]=(0)$. If an ideal W is abelian in T then $[WWT]=(0)$.

The following proposition shows that the standard enveloping Lie algebra of an ideal of general L.t.s. T is a subinvariant subalgebra of the standard enveloping Lie algebra of T in the sense of E. Schenkman.

PROPOSITION. *Let U be an ideal of a general L.t.s. T, then the standard enveloping Lie algebra $U+D(U, U)$ of U is an ideal of $U+D(U, T)$ and $U+D(U, T)$ is an ideal of the Lie algebra $T+D(T, T)$.*

2. Examples.

Sagle showed the constructions of remarkable examples of general L. t. s. [11]. In this section we shall concern a two dimensional general L. t. s. over $\boldsymbol{C}$.

(1) Let M be a Malcev algebra with product xy. Then M becomes a general L.t.s.[1)] relative to xy and $[xyz]=x(yz)-y(xz)+(xy)z$ [17].

(2) There are five different types of two demensional general L. t. s.

THEOREM. *Any two dimensional general L.t.s. T over the complex field $\boldsymbol{C}$ can be reduced to one of the following:*

$$\text{(i)}\begin{cases} ab=0 \\ [abb]=0, \\ [baa]=0 \end{cases} \quad \text{(ii)}\begin{cases} ab=0 \\ [abb]=a, \\ [baa]=0 \end{cases} \quad \text{(iii)}\begin{cases} ab=0 \\ [abb]=a, \\ [baa]=b \end{cases} \quad \text{(iv)}\begin{cases} ab=a \\ [abb]=0, \\ [baa]=0 \end{cases} \quad \text{(v)}\begin{cases} ab=a \\ [abb]=\alpha a, \quad \alpha\neq 0. \\ [baa]=0 \end{cases}$$

1) In [5], Loos proved that any Malcev algebra has a structure of L. t. s. relative to the composition $[xyz]=x(yz)-y(xz)+2(xy)z$.

Proof. Let a, b be a basis of 2-dimensional general L. t. s. T over $\boldsymbol{C}$. If we put $[abb]=\alpha a+\beta b$, $[baa]=\delta a+\gamma b$, $ab=\lambda a+\mu b$, $\alpha, \beta, \gamma, \delta, \lambda, \mu \in \boldsymbol{C}$, then we have the relations (*) $\beta=\delta$ and (**) $\lambda\beta=\alpha\mu$, $\lambda\gamma=\beta\mu$. Indeed, from (1.6) we obtain $[[xyy]xy]+[[yxx]yy]=0$ and $[[xyy]xx]+[[yxx]yx]=0$ for all $x, y \in T$, hence for the basis a, b of T $(\beta-\delta)[abb]=0$ and $(\beta-\delta)[baa]=0$, from which we obtain (*). Next, $[ab(ab)]=[aba]b+a[abb]=b(\beta a+\gamma b)+a(\alpha a+\beta b)=0$ by (*), on the other hand $[ab(ab)]=(\alpha\mu-\lambda\beta)a+(\beta\mu-\lambda\gamma)b$, hence we have (**).

If $\lambda=\mu=0$, then T is a 2-dimensional L.t.s. with respect to $[xyz]$, therefore T can be reduced to one of (i), (ii), (iii) from [15]. In the case, one of λ, μ is not zero, we may assume $\lambda\neq 0$. By the basis transformation $a'=\lambda a+\mu b$, $b'=(1/\lambda)b$, we obtain $a'b'=a'$, $[a'b'b']=\alpha' a'$, $[b'a'a']=0$ from (**). It is easy to show the existence of general L. t. s. of the types (i), ⋯, (v).

3. Weak representations.

Definition. Let ρ be a linear mapping of a general L. t. s. T into the algebra $E(V)$ of linear endomorphisms of a vector space V and D and θ be the bilinear mappings of T into $E(V)$. (ρ, D, θ), or (ρ, θ) simply, is called a *weak representation* of T into V if

$$D(x, y)+\theta(x, y)-\theta(y, x)=[\rho(x), \rho(y)]-\rho(xy), \tag{3.1}$$

$$[D(x, y), \rho(z)]=\rho([xyz]), \tag{3.2}$$

$$[D(x, y), \theta(z, w)]=\theta([xyz], w)+\theta(z, [xyw]). \tag{3.3}$$

The weak representation space V is called a *weak GT-module*.

Let $(\rho, D, \theta; V)$ be a weak representation of general L. t. s. T such that every composition $[xyz]$ vanishes. If $D=\theta=0$, then $(\rho; V)$ is an usual representation of Lie algebra T relative to the product xy, that is, ρ is a Lie algebra homomorphism of T into $\mathfrak{gl}(V)$. If ρ is a representation of Lie algebra L with product xy, then by putting $D(x, y)=\rho(xy)$, $\theta(x, y)=\rho(y)\rho(x)$ $(\rho, D, \theta; V)$ is a weak representation of general L. t. s. associated with L. If $(\rho; V)$ is a weak representation of Malcev algebra M [18], then putting $D(x, y)=[\rho(x), \rho(y)]+\rho(xy)$, $\theta(x, y)=\rho(x)\rho(y)+\rho(y)\rho(x)-\rho(xy)$, $(\rho, D, \theta; V)$ is a weak representation of general L.t.s. associated with M. Let $(\rho, D, \theta; V)$ be a weak representation of general L. t. s. T such that every product xy vanishes. If $\rho=0$, then (3.1), (3.2), and (3.3) reduce to the definition of weak representation of L. t. s. relative to the product $[xyz]$ [19].

In a general L.t.s. T, put $D(x, y): z\to[xyz]$, $\theta(x, y): z\to[zxy]$, $\rho(x): y\to xy$, then (ρ, D, θ) is a weak representation of T into itself, we call this representation to be regular.

We have

$$[D(x, y), D(z, w)] = D([x y z], w) + D(z, [x y w]). \tag{3.4}$$

Hence, if (ρ, D, θ) is a weak representation of general L.t.s. into a vector space V, then a linear space spanned by $\sum_i D(x_i, y_i)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

For the general L.t.s. T, put $T^{(0)} = T$, $T^{(i)} = [T T^{(i-1)} T^{(i-1)}]$, $i = 1, 2, \dots$ If $(\rho, D, \theta; V)$ is a weak representation of T and $D(T, T)^{(k)}$ is a kth derived subalgebra of Lie algebra $D(T, T)$ generated by all $\sum_i D(x_i, y_i)$, then, following Lister [4] by using (3.4), $T^{(i+1)} \subseteq T^{(i)}$ and the induction on k, we have the following

PROPOSITION. *Let $(\rho, D, \theta; V)$ be a weak representation of general L.t.s. T and $D(T, T)^{(k)}$ be a kth derived subalgebra of Lie algebra $D(T, T)$, then it holds*

$$D(T, T)^{(2k)} \subseteq \sum_{i=0}^{k} D(T^{(i)}, T^{(2k-i)}),$$

$$D(T, T)^{(2k+1)} \subseteq \sum_{i=0}^{k} D(T^{(i)}, T^{(2k+1-i)}).$$

Hence, for T suth that $T^{(n)} = (0)$ for some integer n, the Lie algebra $D(T, T)$ is solvable, and in this case, under the assumption the base field of V is algebraically closed, there is a one dimensional D-invariant subspace of V.

4. Cohomology groups associated with a weak representation.

Let (ρ, D, θ) be a weak representation of a general L.t.s. T into a vector space V. We shall define the cohomology groups associated with (ρ, D, θ). A $(2p+1)$-linear mapping f of T into V is called a $(2p+1)$-V-cochain if

$$f(x_1, \dots, x_{2i-1}, x_{2i}, \dots, x_{2p}, x_{2p+1}) = 0$$

for $x_{2i-1} = x_{2i}$, $i = 1, 2, \dots, p$. Similarly, a $(2p+2)$-V-cochain is a $(2p+2)$-linear mapping f of T into V satisfying

$$f(x_1, \dots, x_{2i-1}, x_{2i}, \dots, x_{2p+1}, x_{2p+2}) = 0$$

if $x_{2i-1} = x_{2i}$, $i = 1, 2, \dots, p$. Denote $C^n(T, V)$, $n = 0, 1, 2, \dots$, a vector space over Φ spanned by n-V-cochains, where we identify $C^0(T, V)$ with V.

We shall introduce a notion of coboundary operation for the product space of cochain groups. For a pair (f, f) of the same element f in $C^0(T, V)$ define

$$(\delta_{\mathrm{I}} f)(x) = \rho(x) f, \tag{4.1}$$

$$(\delta_{\mathrm{II}} f)(x, y) = \theta(x, y) f. \tag{4.2}$$

For each $(f, g) \in C^{2p-1}(T, V) \times C^{2p}(T, V)$ we define a mapping δ: $\delta(f, g) = (\delta_{\mathrm{I}} f, \delta_{\mathrm{II}} g)$

of $C^{2p-1}(T, V)\times C^{2p}(T, V)$ into $C^{2p+1}(T, V)\times C^{2p+2}(T, V)$ by the following formulas:

$$(\delta_{\mathrm{I}} f)(x_1, x_2, \cdots, x_{2p+1})$$

$$\begin{aligned}
&=(-1)^p\rho(x_{2p+1})[g(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1})-g(x_1, \cdots, x_{2p}) \\
&\quad +\rho(x_{2p-1})f(x_1, \cdots, x_{2p-2}, x_{2p})-\rho(x_{2p})f(x_1, \cdots, x_{2p-1}) \\
&\qquad -f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p})] \qquad (4.3)\\
&\quad +\sum_{k=1}^{p}(-1)^{k+1}D(x_{2k-1}, x_{2k})f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+1}) \\
&\quad +\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+1}(-1)^k f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2p+1}),
\end{aligned}$$

$$(\delta_{\mathrm{II}} g)(x_1, x_2, \cdots, x_{2p+2})$$

$$\begin{aligned}
&=(-1)^p\theta(x_{2p+1}, x_{2p+2})[g(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1})-g(x_1, \cdots, x_{2p}) \\
&\quad +\rho(x_{2p-1})f(x_1, \cdots, x_{2p-2}, x_{2p})-\rho(x_{2p})f(x_1, \cdots, x_{2p-1}) \\
&\qquad -f(x_1, \cdots, x_{2p-2}, x_{2p-1}x_{2p})] \qquad (4.4)\\
&\quad +\sum_{k=1}^{p}(-1)^{k+1}D(x_{2k-1}, x_{2k})g(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+2}) \\
&\quad +\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^k g(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2p+2}),
\end{aligned}$$

where the sign $\wedge$ over a letter indicates that this letter is to be omitted.

For example, if $(f, g)\in C^1(T, V)\times C^2(T, V)$

$$\begin{aligned}
(\delta_{\mathrm{I}} f)(x, y, z)&=-\rho(z)[g(y, x)-g(x, y)+\rho(x)f(y)-\rho(y)f(x)-f(xy)] \\
&\quad +D(x, y)f(z)-f([xyz]),
\end{aligned}$$

$$\begin{aligned}
(\delta_{\mathrm{II}} g)(x, y, z, w)&=-\theta(z, w)[g(y, x)-g(x, y)+\rho(x)f(y)-\rho(y)f(x)-f(xy)] \\
&\quad +D(x, y)g(z, w)-g([xyz], w)-g(z, [xyw]).
\end{aligned}$$

We shall prove for any (f, g) $\delta\delta(f, g)=0$ or

$$\delta_{\mathrm{I}}\delta_{\mathrm{I}} f=0, \tag{4.5}$$

$$\delta_{\mathrm{II}}\delta_{\mathrm{II}} g=0. \tag{4.6}$$

For $f\in C^0(T, V)$, (4.5) follows from (3.1), (3.2) and (4.6) follows from (3.1), (3.3). To prove the general case we define the following operations.

Let $(f, g)\in C^{2p-1}(T, V)\times C^{2p}(T, V)$, $p=2, 3, \cdots$, then $\kappa(x, y)$ is a linear mapping of $C^{2p-1}(T, V)\times C^{2p}(T, V)$ into itself defined by

$$\begin{aligned}
(\kappa(x, y)f)(x_1, \cdots, x_{2p-1})&=D(x, y)f(x_1, \cdots, x_{2p-1}) \\
&\quad -\sum_{j=1}^{2p-1}f(x_1, \cdots, [xyx_j], \cdots, x_{2p-1}).
\end{aligned} \tag{4.7}$$

$\iota(x, y)$ is a linear mapping of $C^{2p-1}(T, V)\times C^{2p}(T, V)$ into $C^{2p-3}(T, V)\times C^{2p-2}(T, V)$ defined by

$$(\iota(x, y)f)(x_1, \cdots, x_{2p-3})=f(x, y, x_1, \cdots, x_{2p-3}). \tag{4.8}$$

The same definitions are applied also for g. By direct calculations we have

$$\iota(x, y)\delta+\delta\iota(x, y)=\kappa(x, y), \tag{4.9}$$

$$[\kappa(x, y), \iota(z, w)]=\iota([xyz], w)+\iota(z, [xyw]). \tag{4.10}$$

By the induction on p we obtain

$$[\kappa(x, y), \kappa(z, w)]=\kappa([xyz], w)+\kappa(z, [xyw]), \tag{4.11}$$

$$\kappa(x, y)\delta=\delta\kappa(x, y). \tag{4.12}$$

Assume $\delta_{\mathrm{I}}\delta_{\mathrm{I}}f=\delta_{\mathrm{II}}\delta_{\mathrm{II}}g=0$ for all $(f, g)\in C^{2p-1}(T, V)\times C^{2p}(T, V)$, then for $(f, g)\in C^{2p+1}(T, V)\times C^{2p+2}(T, V)$ $\iota(x, y)\delta_{\mathrm{I}}\delta_{\mathrm{I}}f=(\kappa(x, y)-\delta_{\mathrm{I}}\iota(x, y))\delta_{\mathrm{I}}f=\delta_{\mathrm{I}}\delta_{\mathrm{I}}\iota(x, y)f=0$, hence $\delta_{\mathrm{I}}\delta_{\mathrm{I}}f=0$ for all f. Similarly $\delta_{\mathrm{II}}\delta_{\mathrm{II}}g=0$ for all g. Thus we have proved (4.5) and (4.6).

Let $Z^{2p-1}(T, V)$ be the set of $f\in C^{2p-1}(T, V)$ such that $\delta_{\mathrm{I}}f=0$, then the $(2p-1)$-th cohomology group $H^{2p-1}(T, V)$ is the factor group $Z^{2p-1}(T, V)/\delta_{\mathrm{I}}C^{2p-3}(T, V)$. Similarly the $2p$th cohomology group $H^{2p}(T, V)$ is defined as $Z^{2p}(T,V)/\delta_{\mathrm{II}}C^{2p-2}(T,V)$ and we have the product of cohomology groups $H^{2p-1}(T, V)\times H^{2p}(T, V)$, $p=1,2,\cdots$. We define $H^0(T, V)$ as the set of f in $C^0(T, V)$ such that $\delta_{\mathrm{I}}f=0$ and $\delta_{\mathrm{II}}f=0$. Then from (4.1) and (4.2)

$H^0(T, V)$ is a subspace of V spanned by the invariant elements under the weak representation of T.

5. Extensions of weak GT-modules.

The purpose of this section is to give an interpretation of the pair of first and second cohomology groups following the method of [3].

DEFINITION. Let $(\rho_1, D_1, \theta_1 ; V)$ and $(\rho_2, D_2, \theta_2 ; W)$ be weak GT-modules. An *extension* of $(\rho_1, D_1, \theta_1 ; V)$ by $(\rho_2, D_2, \theta_2 ; W)$ is a weak GT-module $(\rho^*, D^*, \theta^* ; V^*)$ and two GT-homomorphisms $\iota : W\to V^*$ and $\pi : V^*\to V$ such that

$$0\longrightarrow W\xrightarrow{\ \iota\ }V^*\xrightarrow{\ \pi\ }V\longrightarrow 0$$

is an exact sequence.

Two extensions: $0\longrightarrow W\xrightarrow{\iota}V^*\xrightarrow{\pi}V\longrightarrow 0$ and $0\longrightarrow W\xrightarrow{\iota'}V^{*\prime}\xrightarrow{\pi'}V\longrightarrow 0$ of V by W are *equivalent* if the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{\iota} & V^* & \xrightarrow{\pi} & V & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow \kappa & & \downarrow 1 & & \\ 0 & \longrightarrow & W & \xrightarrow{\iota'} & V^{*\prime} & \xrightarrow{\pi'} & V & \longrightarrow & 0 \end{array}$$

is commutative.

Now, suppose that $(\rho^*, D^*, \theta^*; V^*)$ is an extension of $(\rho_1, D_1, \theta_1; V)$ by $(\rho_2, D_2, \theta_2; W)$. Let l be a linear mapping of V into V^* such that $\pi l = 1$ on V. For $x, y \in T$, $f(x)$ and $g(x, y)$ are the elements of $\mathrm{Hom}(V, V^*)$ defined as follows:

$$f(x)(v) = \rho^*(x)\, l(v) - l\rho_1(x)(v),$$

$$g(x, y)(v) = \theta^*(x, y)\, l(v) - l\theta_1(x, y)(v) \qquad v \in V.$$

Then $f(x)$, $g(x, y) \in \mathrm{Hom}(V, W)$ since $\pi f(x)(v) = \pi g(x, y)(v) = 0$. The vector space $\mathrm{Hom}(V, W)$ becomes a weak GT-module by defining (ρ, D, θ) as $(\rho(x)f)(v) = \rho_2(x)f(v) - f(\rho_1(x)(v))$, $(\theta(x, y)f)(v) = \theta_2(x, y)f(v) - f(\theta_1(x, y)(v))$, and $(D(x, y)f)(v) = D_2(x, y)f(v) - f(D_1(x, y)(v))$, $v \in V$, $f \in \mathrm{Hom}(V, W)$. Then $(f, g) \in C^1(T, \mathrm{Hom}(V, W)) \times C^2(T, \mathrm{Hom}(V, W))$. Furthermore it follows $(f, g) \in Z^1(T, \mathrm{Hom}(V, W)) \times Z^2(T, \mathrm{Hom}(V, W))$. In fact, by using (3.1) and (3.2) $(\delta_{\mathrm{I}} f)(x, y, z)(v) = -\rho(z)[g(y, x) - g(x, y) + \rho(x)f(y) - \rho(y)f(x) - f(xy)](v) + D(x, y)f(z)(v) - f([xyz])(v) = \{[D^*(x, y), \rho^*(z)] - \rho^*([xyz])\}\, l(v) - l\,\{[D_1(x, y), \rho_1(z)] - \rho_1([xyz])\}(v) = 0$. Hence $f \in Z^1(T, \mathrm{Hom}(V, W))$. Similarly $g \in Z^2(T, \mathrm{Hom}(V, W))$ follows from (3.1) and (3.3).

Let l_1 and l_2 be two linear mappings of V into V^* such that $\pi l_1 = \pi l_2 = 1$ on V. If we put $f_i(x)(v) = \rho^*(x) l_i(v) - l_i\rho_1(x)(v)$, $g_i(x, y)(v) = \theta^*(x, y) l_i(v) - l_i\theta_1(x, y)(v)$, $v \in V$, $i = 1, 2$, then $f_1(x) - f_2(x) = \rho(x) l = (\delta_{\mathrm{I}} l)(x)$ and $g_1(x, y) - g_2(x, y) = \theta(x, y) l = (\delta_{\mathrm{II}} l)(x, y)$, where $l = l_1 - l_2$. Therefore to each extension of V by W corresponds uniquely an element of $H^1(T, \mathrm{Hom}(V, W)) \times H^2(T, \mathrm{Hom}(V, W))$. It is clear that two equivalent extensions of V by W determine the same element of $H^1(T, \mathrm{Hom}(V, W)) \times H^2(T, \mathrm{Hom}(V, W))$.

Conversely, given $(f, g) \in Z^1(T, \mathrm{Hom}(V, W)) \times Z^2(T, \mathrm{Hom}(V, W))$ and put $V^* = V \oplus W$ (vector space direct sum). The linear mappings $\rho^*(x)$, $\theta^*(x, y)$, $D^*(x, y)$ of V^* are defined as follows:

$$\rho^*(x)(v, w) = (\rho_1(x)(v), f(x)(v) + \rho_2(x)(w)),$$

$$\theta^*(x, y)(v, w) = (\theta_1(x, y)(v), g(x, y)(v) + \theta_2(x, y)(w)),$$

$$D^*(x, y) = \theta^*(y, x) - \theta^*(x, y) + [\rho^*(x), \rho^*(y)] - \rho^*(xy),$$

$(v, w) \in V^*$, then by a strightforward calculation it follows that $(\rho^*, D^*, \theta^*; V^*)$ is a weak representation of T. If we put $\iota(w) = (0, w)$ and $\pi(v, w) = v$, then π

is a GT-homomorphism and $0 \longrightarrow W \xrightarrow{\iota} V^* \xrightarrow{\pi} V \longrightarrow 0$ is an exact sequence. Define a linear mapping l of V into V^* by $l(v)=(v, 0)$, then $\pi l=1$ on V. $(\rho^*(x)l - l\rho_1(x))(v) = (0, f(x)(v))$ implies $f(x) = \rho^*(x)l - l\rho_1(x)$ and similarly $g(x, y) = \theta^*(x, y)l - l\theta_1(x, y)$, hence l defines a given 1-cocycle f and 2-cocycle g. Thus we obtain an extension V^* of V by W.

Suppose that $0 \longrightarrow W \xrightarrow{\iota} V^* \xrightarrow{\pi} V \longrightarrow 0$ and $0 \longrightarrow W \xrightarrow{\iota'} V^{*\prime} \xrightarrow{\pi'} V \longrightarrow 0$ are two extensions of V by W correspond to the same element of $H^1(T, \mathrm{Hom}(V, W)) \times H^2(T, \mathrm{Hom}(V, W))$. Then we have relations $f(x) = f'(x) + (\delta_{\mathrm{I}} h)(x)$ and $g(x, y) = g'(x, y) + (\delta_{\mathrm{II}} h)(x, y)$, $h \in \mathrm{Hom}(V, W)$, hence $\rho^*(x)l - l\rho_1(x) = \rho^{*\prime}(x)l' - l'\rho_1(x) + \rho_2(x)h - h\rho_1(x)$ and $\theta^*(x, y)l - l\theta_1(x, y) = \theta^{*\prime}(x, y)l' - l'\theta_1(x, y) + \theta_2(x, y)h - h\theta_1(x, y)$. So that, we obtain

$$(*)\quad \begin{cases} l'\rho_1(x) - l\rho_1(x) + h\rho_1(x) = \rho^{*\prime}(x)l' - \rho^*(x)l + \rho_2(x)h, \\ l'\theta_1(x, y) - l\theta_1(x, y) + h\theta_1(x,y) = \theta^{*\prime}(x, y)l' - \theta^*(x, y)l + \theta_2(x, y)h. \end{cases}$$

A linear mapping κ of V^* into $V^{*\prime}$ is defined as

$$\kappa(v^*) = l'\pi(v^*) + v^* - l\pi(v^*) + h\pi(v^*) \qquad v^* \in V^*,$$

then it is shown that $\pi'\kappa = \pi$ and κ is a GT-homomorphism of V^* into $V^{*\prime}$ by making use of the relation $(*)$, therefore two extensions $0 \longrightarrow W \xrightarrow{\iota} V^* \xrightarrow{\pi} V \longrightarrow 0$ and $0 \longrightarrow W \xrightarrow{\iota'} V^{*\prime} \xrightarrow{\pi'} V \longrightarrow 0$ are equivalent.

Thus we have the following theorem.

THEOREM. *Let V and W be the weak GT-modules of general L. t. s. T. Then, there is a one-to-one correspondence between the equivalence classes of extensions of V by W and the elements of* $H^1(T, \mathrm{Hom}(V, W)) \times H^2(T, \mathrm{Hom}(V, W))$.

6. Cohomology groups associated with a representation.

First we recall a definition of representation[1)] of general L. t. s. [19].

DEFINITION. Let ρ be a linear mapping of a general L. t. s. T into the algebra $E(V)$ of linear endomorphisms of a vector space V and D and θ be the bilinear mappings of T into $E(V)$. $(\rho, D, \theta\,; V)$ is called a *representation* of T if ρ, D and θ satisfy the following relations:

$$(6.1)\qquad D(x, y) + \theta(x, y) - \theta(y, x) = [\rho(x), \rho(y)] - \rho(xy),$$

$$(6.2)\qquad \theta(x, yz) - \rho(y)\theta(x, z) + \rho(z)\theta(x, y) = 0,$$

$$(6.3)\qquad \theta(xy, z) - \theta(x, z)\rho(y) + \theta(y, z)\rho(x) = 0,$$

1) This definition of representation is an application to general L. t. s. of the definition by Eilenberg in [2].

(6.4) $\theta(z, w)\theta(x, y)-\theta(y, w)\theta(x, z)-\theta(x, [yzw])+D(y, z)\theta(x, w)=0,$

(6.5) $[D(x, y), \rho(z)]=\rho([xyz]),$

(6.6) $[D(x, y), \theta(z, w)]=\theta([xyz], w)+\theta(z, [xyw]).$

According (6.1) we shall sometimes denote by (ρ, θ) the representation (ρ, D, θ) simply. From (6.1), (6.2), (6.3), and (6.5) we have

(6.7) $D(xy, z)+D(yz, x)+D(zx, y)=0.$

The regular mapping (ρ, D, θ) of T is a representation of T into itself and an ideal of T is a subspace of T invariant under this representation. If ρ is a representation of Malcev algebra M, by putting $\theta(x, y)=\rho(x)\rho(y)+\rho(y)\rho(x)-\rho(xy)$, (ρ, θ) becomes a representation of a general L. t. s. associated with M [19].

Let $(\rho, D, \theta\,;V)$ be a representation of general L. t. s. T. Let (f, g) be a pair of $2p$- and $(2p+1)$-linear mappings of T into V such that

$$f(x_1, \cdots, x_{2i-1}, x_{2i}, \cdots, x_{2p})=0$$

and

$$g(x_1, \cdots, x_{2i-1}, x_{2i}, \cdots, x_{2p+1})=0$$

if $x_{2i-1}=x_{2i}$, $i=1, 2, \cdots, p$. We denote by $C^n(T, V)$, $n\geqq 1$, a vector space spanned by such linear mappings. For each element (f, g) of the product space $C^{2p}(T, V)\times C^{2p+1}(T, V)$ a coboundary operator $\delta:(f, g)\to(\delta_{\mathrm{I}}f, \delta_{\mathrm{II}}g)$ is a mapping of $C^{2p}(T, V)\times C^{2p+1}(T, V)$ into $C^{2p+2}(T, V)\times C^{2p+3}(T, V)$ defined by the following formulas:

$$\begin{aligned}
&(\delta_{\mathrm{I}}f)(x_1, x_2, \cdots, x_{2p+2})\\
&=(-1)^p[\rho(x_{2p+1})g(x_1, \cdots, x_{2p}, x_{2p+2})-\rho(x_{2p+2})g(x_1, \cdots, x_{2p+1})\\
&\qquad -g(x_1, \cdots, x_{2p}, (x_{2p+1}x_{2p+2}))]\\
&\quad+\sum_{k=1}^{p}(-1)^{k+1}D(x_{2k-1}, x_{2k})f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+2})\\
&\quad+\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^k f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2p+2}),
\end{aligned}$$

$$\begin{aligned}
&(\delta_{\mathrm{II}}g)(x_1, x_2, \cdots, x_{2p+3})\\
&=(-1)^p[\theta(x_{2p+2}, x_{2p+3})g(x_1, \cdots, x_{2p+1})-\theta(x_{2p+1}, x_{2p+3})g(x_1, \cdots, x_{2p}, x_{2p+2})]\\
&\quad+\sum_{k=1}^{p+1}(-1)^{k+1}D(x_{2k-1}, x_{2k})g(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+3})\\
&\quad+\sum_{k=1}^{p+1}\sum_{j=2k+1}^{2p+3}(-1)^k g(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2p+3})
\end{aligned}$$

for $(f, g)\in C^{2p}(T, V)\times C^{2p+1}(T, V)$, $p=1, 2, 3, \cdots$.

In the case $n=1$, we shall only consider a subspace spanned by the diagonal elements $(f, f) \in C^1(T, V) \times C^1(T, V)$ and $\delta(f, f) = (\delta_{\mathrm{I}} f, \delta_{\mathrm{II}} f)$ is an element of $C^2(T, V) \times C^3(T, V)$ defined by

$$(\delta_{\mathrm{I}} f)(x, y) = \rho(x) f(y) - \rho(y) f(x) - f(xy),$$

$$(\delta_{\mathrm{II}} f)(x, y, z) = \theta(y, z) f(x) - \theta(x, z) f(y) + D(x, y) f(z) - f([xyz]).$$

Further, for each $(f, g) \in C^2(T, V) \times C^3(T, V)$ another coboundary operation $\delta^* = (\delta_{\mathrm{I}}^*, \delta_{\mathrm{II}}^*)$ of $C^2(T, V) \times C^3(T, V)$ into $C^3(T, V) \times C^4(T, V)$ is defined by

$$\begin{aligned}(\delta_{\mathrm{I}}^* f)(x, y, z) = & -\rho(x) f(y, z) - \rho(y) f(z, x) - \rho(z) f(x, y) \\ & + f(xy, z) + f(yz, x) + f(zx, y) \\ & + g(x, y, z) + g(y, z, x) + g(z, x, y).\end{aligned}$$

$$\begin{aligned}(\delta_{\mathrm{II}}^* g)(x, y, z, w) = & \ \theta(x, w) f(y, z) + \theta(y, w) f(z, x) + \theta(z, w) f(x, y) \\ & + g(xy, z, w) + g(yz, x, w) + g(zx, y, w).\end{aligned}$$

For each $f \in C^1(T, V)$ a direct calculation shows that $\delta_{\mathrm{I}} \delta_{\mathrm{I}} f = \delta_{\mathrm{I}}^* \delta_{\mathrm{I}} f = 0$ and $\delta_{\mathrm{II}} \delta_{\mathrm{II}} f = \delta_{\mathrm{II}}^* \delta_{\mathrm{II}} f = 0$. In generally, by the same way as §4, for each $(f, g) \in C^{2p}(T, V) \times C^{2p+1}(T, V)$ it is shown that $\delta_{\mathrm{I}} \delta_{\mathrm{I}} f = 0$ and $\delta_{\mathrm{II}} \delta_{\mathrm{II}} g = 0$, or $\delta\delta(f, g) = 0$. For the case $p \geqq 2$, $Z^{2p}(T, V) \times Z^{2p+1}(T, V)$ is a subspace of $C^{2p}(T, V) \times C^{2p+1}(T, V)$ spanned by (f, g) such that $\delta(f, g) = 0$. The cohomology group $H^{2p}(T, V) \times H^{2p+1}(T, V)$ of T associated with representation (ρ, D, θ) is defined as the factor space $(Z^{2p}(T, V) \times Z^{2p+1}(T, V)) / (B^{2p}(T, V) \times B^{2p+1}(T, V))$, where $B^{2p}(T, V) \times B^{2p+1}(T, V) = \delta(C^{2p-2}(T, V) \times C^{2p-1}(T, V))$. $H^1(T, V) = \{f \in C^1(T, V) ; \delta_{\mathrm{I}} f = 0, \delta_{\mathrm{II}} f = 0\}$ by definition. In the case $p=1$, let $Z^2(T, V)$ be a subspace of $C^2(T, V)$ spanned by f such that $\delta_{\mathrm{I}} f = \delta_{\mathrm{I}}^* f = 0$ and $Z^3(T, V)$ be a subspace of $C^3(T, V)$ spanned by g such that $\delta_{\mathrm{II}} g = \delta_{\mathrm{II}}^* g = 0$, then $H^2(T, V) \times H^3(T, V)$ is defined as the factor space $(Z^2(T, V) \times Z^3(T, V)) / (B^2(T, V) \times B^3(T, V))$, where $B^2(T, V) \times B^3(T, V) = \{\delta(f, f) ; f \in C^1(T, V)\}$. Let f be a linear mapping of T into a representation space V, f is called a derivation of T into V if $f(xy) = \rho(x) f(y) - \rho(y) f(x)$ and $f([xyz]) = \theta(y, z) f(x) - \theta(x, z) f(y) + D(x, y) f(z)$. If (ρ, θ) is the regular representation of T, then f is a derivation of T. Then, by the definition of $H^1(T, V)$

$H^1(T, V)$ is the vector space spanned by derivations of T into V.

7. Extensions of general L. t. s.

In this section it will be given an interpretation of $H^2(T, V) \times H^3(T, V)$ in the relation with an extension of general L. t. s. following the method of [1].

DEFINITION. Let T, T^*, U be the general L.t.s. over Φ. An *extension* of T by U is an exact sequence $0 \longrightarrow U \xrightarrow{\iota} T^* \xrightarrow{\pi} T \longrightarrow 0$ of general L.t.s. Two extensions $0 \longrightarrow U \xrightarrow{\iota} T^* \xrightarrow{\pi} T \longrightarrow 0$ and $0 \longrightarrow U \xrightarrow{\iota'} T^{*\prime} \xrightarrow{\pi'} T \longrightarrow 0$ are *equivalent* if there is a homomorphism κ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\iota} & T^* & \xrightarrow{\pi} & T & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow \kappa & & \downarrow 1 & & \\ 0 & \longrightarrow & U & \xrightarrow{\iota'} & T^{*\prime} & \xrightarrow{\pi'} & T & \longrightarrow & 0 \end{array}$$

is commutative.

We shall consider the case $\iota(U)$ is abelian in T^*. Let T^* be an extension of T with an abelian kernel $\iota(U)$. Identifying U with its image $\iota(U)$ by the injection mapping ι, we have for $a=x+p$, $b=y+q$ in T^*, p, q in U, $\rho(a)(v)=(x+p)(v)=xv$, $\theta(a, b)(v)=[v, x+p, y+q]=[vxy]$, $v \in U$. Let l be a linear mapping of T into T^* such that $\pi l=1$ on T. For $x, y \in T$ define linear mappings of U into itself by $\rho(x)(v)=l(x)(v)$, $\theta(x, y)(v)=[vl(x)l(y)]$, and $D(x, y)(v)=[l(x)l(y)v]$, then $\rho(x)$, $\theta(x, y)$, and $D(x, y)$ do not depend on the selection of l. Since π is a homomorphism of T^* onto T, $l(x)l(y)-l(xy)$ and $[l(x)l(y)l(z)]-l([xyz])$ are in the kernel U, by using the assumption U is abelian in T^*, it is shown that (ρ, D, θ) is a representation of T with U as the representation space. Put

$$f(x, y)=l(x)l(y)-l(xy),$$

$$g(x, y, z)=[l(x)l(y)l(z)]-l([xyz]),$$

then $(f, g) \in C^2(T, U) \times C^3(T, U)$. We shall further show that $(f, g) \in Z^2(T, U) \times Z^3(T, U)$. For this purpose we see some relations followed from the axioms of general L.t.s. for T^*. From (1.3): $[l(x)l(y)l(z)]+[l(y)l(z)l(x)]+[l(z)l(x)l(y)] + (l(x)l(y))l(z)+(l(y)l(z))l(x)+(l(z)l(x))l(y)=0$ we have $(\delta_{\mathrm{I}}^* f)(x, y, z)=0$. Similarly, from (1.4), (1.5), (1.6) we have $(\delta_{\mathrm{II}}^* g)(x, y, z, w)=0$, $(\delta_{\mathrm{I}} f)(x, y, z, w)=0$, $(\delta_{\mathrm{II}} g)(x, y, z, v, w)=0$ respectively. Therefore $(f, g) \in Z^2(T, U)\times Z^3(T, U)$. If l' is another linear mapping of T into T^* such that $\pi l'(x)=x$ for each $x \in T$, then $h(x)=l'(x)-l(x)$ is in U, h is a 1-U-cochain. Let (f', g') be the pair of factor sets corresponding to l', that is $f'(x, y)=l'(x)l'(y)-l'(xy)$ and $g'(x, y, z) =[l'(x)l'(y)l'(z)]-l'([xyz])$, then it follows $f'=f+\delta_{\mathrm{I}} h$ and $g'=g+\delta_{\mathrm{II}} h$, therefore the cohomology class of f and the cohomology class of g do not depend on the choice of section l, hence the extension of T by abelian U determines uniquely an element of $H^2(T, U) \times H^3(T, U)$. Two equivalent extensions define the same element of $H^2(T, U) \times H^3(T, U)$.

Conversely, let (ρ, D, θ) be a representation of a general L.t.s. T into a vector space U and $(f, g) \in Z^2(T, U) \times Z^3(T, U)$. Put $T^*=T \oplus U$ (vector space direct sum) and define two compositions in T^* by

$$(x, u)(y, v) = (xy, \rho(x)v - \rho(y)u + f(x, y)),$$

$$[(x, u)(y, v)(z, w)] = ([xyz], \theta(y, z)u - \theta(x, z)v + D(x, y)w + g(x, y, z)),$$

then we have the relations (1.1), (1.2), ..., (1.6) with respect to these compositions and T^* becomes a general L.t.s. We have an exact sequence $0 \longrightarrow U \xrightarrow{\iota} T^* \xrightarrow{\pi} T \longrightarrow 0$, where $\iota(u) = (0, u)$ and $\pi(x, u) = x$. If we define a linear mapping l of T into T^* by $l(x) = (x, 0)$, then $\pi l = 1$ on T. Since $l(x)l(y) - l(xy) = (0, f(x, y))$ and $[l(x)l(y)l(z)] - l([xyz]) = (0, g(x, y, z))$, (f, g) is one of the pairs of cocycles defined by the extension. Therefore, to each $(f, g) \in Z^2(T, U) \times Z^3(T, U)$ corresponds an extension of T by U.

Thus, we have the following theorem.

THEOREM. *To each equivalent class of extensions T^* of a general L.t.s. T by abelian ideal U in T^* corresponds an element of $H^2(T, U) \times H^3(T, U)$. Let (ρ, D, θ) be a representation of a general L.t.s. T into a vector space U. If (f, g) is a pair of cocycles belonging to the element of $H^2(T, U) \times H^3(T, U)$, then there is an extension T^* of T by U such that U is abelian in T^*.*

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