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ON HOMOGENEOUS COMPLEX MANIFOLDS

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1. In his paper on the complex manifolds, [1] A. BLANCHARD remarked that a holomorphic mapping of the complex projective space is non-degenerate or reduced to a constant mapping. R. REMMERT and K. STEIN treated this fact in more general circumstances as a part of their studies on proper holomorphic mappings, [6]. Subsequently, a topological approach was made by R. REMMERT and T. VAN de VEN, [7]. The results there are mostly topological.

Let X be a complex space, G a group of automorphisms, i.e., biholomorphic transformations of X, and G_α, $\alpha \in X$, the isotropy subgroup of G; $G_\alpha=\{g \in G: g(\alpha)=\alpha\}$. Then, a result of REMMERT and STEIN reads as follows:

Let X be a compact connected normal complex space and satisfy the condition:

(A) *For any different two points x', x'' of X there exists no analytic subset of X other than X itself which contains both x' and $\{g(x''): g \in G_{x'}\}$.*

Then any holomorphic mapping $\tau: X \to Y$ of X to another complex space Y is non-degenerate, and therefore is a covering mapping, or reduced to a constant mapping.

One of the modified rather geometric variants of condition (A) will be given as follows:

(B) The isotropy subgroup G_α operates transitively on $X-\{\alpha\}$.

Under the condition (B) the space X is necessarily a complex manifold, for the condition (B) obviously implies the transitivity of G, i.e., X is homogeneous. For the sake of simplicity we shall call *B-manifold* the complex manifold satisfying the condition (B). The complex projective space P^n and the complex number space C^n are the wellknown examples of B-manifold. The purpose of this note is to make some remarks on the B-manifolds.

One of our results is

THEOREM I. *Let X be a connected compact B-manifold. If the field $K(X)$ of all meromorphic functions on X is different from the complex number field C, then X is projective-algebraic.*

2. We denote by $O(X)$ the ring of holomorphic functions on X.

Definition 1. *A complex space X is called holomorphically separable, if for any different two points x', $x'' \in X$ there exists a holomorphic function $h(x) \in O(X)$ which* *separates these points:* $h(x') \neq h(x'')$. *The space X is called meromorphically separable, if for any different two points* $x', x'' \in X$ *there exists a meromorphic function* $m(x) \in K(X)$ *which is holomorphic at* x' *and* x'', *and separates these points:* $m(x') \neq m(x'')$.

By a wellknown result due to R. REMMERT, [3], the field $K(X)$ of the meromorphic functions on a connected compact irreducible complex space X is algebraic, i.e., $K(X)$ is a simple algebraic extension of a transcendental extension of the complex number field C; the transcendence degree $t(X)$ is finite and not larger than the topological dimension of X. $t(X)$ is also called the algebraic dimension of X.

It is easily verified that:

Proposition 1. *The algebraic dimension of a compact meromorphically separable complex space is equal to its topological dimension.*

Proposition 2. *Let X be a connected B-manifold. Then, the group G of automorphisms of X operates transitively on* $X \times X - \Delta$, Δ *the diagonal subset of* $X \times X$, *in such a way that for any different two points* (p, q), $(p', q') \in X \times X - \Delta$ *there exists an automorphism* $g \in G$ *such that* $g(p, q) \equiv (g(p), g(q)) = (p', q')$. *The converse is also true.*

Proof. There may occur essentially following three cases: (i) $p = p'$, (ii) $p \neq p'$ and $p \neq q'$, (iii) $p = q'$ and $p' = q$.

The case (i). By the definition of the B-manifold the isotropy subgroup G_p operates transitively on $X - \{p\}$, and therefore there exists an automorphism $g \in G$ such that $g(q) = q'$.

The case (ii). Again, by definition there exist $g \in G_p$ such that $g(q) = q'$ and $g' \in G_{q'}$ such that $g'(p) = p'$. Then, $g' \cdot g(p, q) \equiv (g' \cdot g(p), g' \cdot g(q)) = (p', q')$ in $X \times X$.

The case (iii). Take any $g \in G$ such that $g(p) = q$, this is possible, for X is homogeneous, and put $g(q) = q'$. Then $q' \neq p$. We take a $g' \in G_q$ such that $g(q') = p$. The automorphism $g' \cdot g$ satisfies $g' \cdot g(p) = q$ and $g' \cdot g(q) = p$. The proof of the latter half is trivial.

As an immediate consequence of Prop. 2 we have:

Proposition 3, *Let X be a B-manifold. If there exists at least one non-constant meromorphic function on X, then X is meromorphically separable.*

Proof. Let x', x'' be any different two points of X. By assumption there exists a non-constant meromorphic function $m(x)$ on X. It is possible to choose different two points y', $y'' \in X$ such that the function $m(x)$ is holomorphic at y', y'' and separates these points: $m(y') \neq m(y'')$. By Prop. 2 there exists an automorphism $g \in G$ such that $g(x') = y'$ and $g(x'') = y'$. Then, the meromorphic function $m(g(x))$ separates x' and x'': $m(g(x')) = m(y') \neq m(y'') = m(g(x''))$. Thus, X is meromorphically separable.

Combining Prop. 1 and Prop. 3 we obtain:

Proposition 4. *Let X be a connected compact B-manifold and* $K(X) \neq C$. *Then, the algebraic dimension of X is equal to its topological dimension.*

Now, the proof of Th. I is a direct consequence of the following theorem due to H. GRAUERT and R. REMMERT, [2], for a B-manifold is necessarily homogeneous:

Theorem. *A connected compact homogeneous complex manifold, the algebraic dimen-*

sion and the topological dimension of which coincide, is projective-algebraic.

3. In this section we shall consider some non-compact B-manifolds.

Definition 2. *Let X be a complex space.*

a) *X is called k-complete, if for any point $x \in X$ there exist a neighborhood $U(x)$ of $x$and a finite number of functions $f_1, f_2, \cdots, f_k$; $f_i \in O(X)$ which define a holomorphic mapping $\tau : X \to C^k$ such that $\tau^{-1}(y) \cap U(x)$ is discrete in $U(x)$ for any $y \in C^k$,*

b) *X is called holomorphically convex, if for any compact subset $K \subset X$, its envelope $\hat{K} = \{x \in X : |f(x)| \leqq \sup_{x \in K} |f(x)|$ for any $f \in O(X)\}$ is compact.*

c) *X is called holomorphically complete, if X is k-complete and holomorphically convex.*

We adopted the definition c) following the fundamental theorem of H. GRAUERT. Before stating our theorem we need the follwing theorem due to R. REMMERT, [5]:

Theorem. *If X is a holomorphically convex complex manifold, then there exists a proper kernel of X. The kernel space X^* is uniquely determined and analytically isomorphic to the complex quotient space X/S, where S is the holomorphic separation, in particular, the kernel space is holomorphically complete.*

For the detailed account of this theorem we refer to [5]. From this theorem it follows immediately:

Proposition 5. *A holomorphically separable holomorphically convex complex manifold is holomorphically complete.*

Now, we assert:

Theorem II. *A non-compact holomorphically convex B-manifold X is holomorphically complete.*

Proof. Since X is non-compact and holomorphically convex, there exists at least one non-constant holomorphic function $h(x) \in O(X)$, for, if $O(X) = C$, for any compact subset $K \subset X$ its envelope $\hat{K}$ coincides with the whole space X. Let $x', x'' \in X$ be any different two points. Since $h(x)$ is non-constant, there exist different two points y', $y'' \in X$ such that $h(y') \neq h(y'')$. By Prop. 2 there exists an automorphism $g \in G$ such that $g(x') = y'$ and $g(x'') = y''$. Then, the function $h(g(x))$ separates x', x'', hence X is holomorphically separable. Then, Prop. 5 applies.

4. We know little about the structures of general B-manifolds. The following is a trivial consequence of the result of R. REMMERT and K. STEIN, cf. [6]:

Proposition 6. *Any compact B-manifold can not be a product space, in general, a fibre bundle with positive-dimensional fibre.*

Also by Prop. 2 we have:

Proposition 7. *Any B-manifold is connected.*

Therefore the connectedness condition in Th. I is unnecessary.

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