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OPTIMAL POLICIES FOR DYNAMIC INVENTORY PROCESSES WITH RANDOM DELIVERY-LAG AND NON-STATIONARY STOCHASTIC DEMANDS

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§1. Introduction.

In this paper we consider an n-periods, $I_1, I_2, \cdots, I_n$, one-commodity dynamic inventory model, where quantities ordered in unit prices, $c_0, c_1, \cdots, c_{k-1}$ and c_k ($c_0>c_1>\cdots>c_{k-1}>c_k$) are delivered, respectively, with $X_0, X_1, \cdots, X_{k-1}$ and X_k-period-lag ($x_i=0, 1, \cdots, r$; $i=0, 1, \cdots, k$), where X_i's are random variable. The cumulative demand in each period is a non-negative random variable whose distribution may change from period to period according to Markov transition law with matrix $P=\|p_{ij}\|$ $(i, j=1, 2, \cdots, m)$ where $p_{ij}\geqq 0$ and $\sum_{j=1}^{m} p_{ij}=1$ for each i. It is assumed that the demand density remains unchanged during one period. In other word, when the demand in a given period is φ_i the demand density in the following period changes to φ_j with probability p_{ij} $(i, j=1, 2, \cdots, m)$. The demand is stochastic but not stationary. At the beginning of a period we decide the amount of the commodity to be ordered in unit prices c_λ for $\lambda=0, 1, \cdots, k$. This amount will depend on the stage number j and the demand density in the period φ_i, and is denoted by $m^i_{\lambda.n-j+1}$, where $i=1, 2, \cdots, m; j=1, 2, \cdots, n; \lambda=0, 1, \cdots, k$. They are all non-negative and some of them are bounded above: $m^i_{k,n-j+1}\geqq 0$, $\beta_\lambda\geqq m^i_{\lambda,n-j+1}\geqq 0$, where $i=1, 2, \cdots, m; j=1, 2, \cdots, n; \lambda=0, 1, \cdots, k-1$. These upper bound depend on the unit price but not stage number. It is assumed that the amount ordered with time lag k is delivered at the end of the period for each k ($k\geqq 1$). Thus the quantity ordered at the beginning of the jth period either delivered immediately or, if delayed, delivered in the unit price c_λ ($\lambda=0, 1, \cdots, k$) at the end of the $j+x_\lambda-1$th period.

In [7], [8], [9] and [11], we have discussed the properties of the optimal policies, the critical numbers and the expected total discounted loss functions in the above mentioned dynamic model under an assumption that the delivery-lag is constant. Papers [7], [9] and [11] are the generalization of corresponding results in [1], [2] and [4]. The model presented here is the generalization of the one discussed in [7], [8] and [9] in the sense that the delivery-lag is random.

We impose the following conditions on the model that hold throughout this paper unless otherwise noted.

(1.1) The interval in ordering is r-period ($r\geqq 1$).

(1.2) There is backlogging of excess demand.

(1.3) The known distribution function of the demand is absolutely continuous with respect to the Lebesgue measure, and the density is denoted by φ_i $(i=1, 2, \cdots, m)$.

(1.4) The holding cost function $h(\eta)$ and the penalty cost function $p(\eta)$ are twice differentiable, convex increasing functions defined on $[0, \infty)$ such that $h(0)=p(0)=0$.

(1.5) There is credit function $v(\eta)$ defined by

$$v(\eta)=\begin{cases} v\eta & \eta\geqq 0 \\ 0 & \eta<0 \end{cases}$$

The reduced penalty cost, that is, the net penalty cost, is defined in the following way. If at the beginning of some period it is known that at the end of this period the amount of size z will be delivered and the demand ξ will occurs, then the net penalty cost for the period is

$$p(\xi-y)-v[\min(z, \xi-y)]$$

where y is the starting stock level of the period.

(1.6) There is a concave, twice differentiable salvage gain function $w(\eta)$ that is increasing for $\eta>0$, and is zero for $\eta\leqq 0$.

(1.7) The ordering cost function $c_j(\eta)$ for unit cost c_j are given by

$$c_j(\eta)=\begin{cases} c_j\eta & 0\leqq\eta\leqq\beta_j \\ 0 & \eta<0 \end{cases} \qquad j=0,\ 1, \cdots, k-1.$$

$$c_k(\eta)=\begin{cases} c_k\eta & \eta\geqq 0 \\ 0 & \eta<0 \end{cases}$$

with $c_0>c_1>\cdots>c_k>0$.

(1.8) The time lag X_i $(i=0, 1, \cdots, k)$ for the commodity of unit cost c_i are non-negative discrete random variable. The joint probability density will be denoted by $q(x_0, x_1, \cdots x_k)$ where $P_r\{X_0=x_0, X_1=x_1, \cdots, X_k=x_k\}=q(x_0, x_1, \cdots, x_k)$ $(x_i=0, 1, \cdots, r; i=0, 1, \cdots, k)$. We assume that

$$q(x_0, x_1, \cdots, x_k)=0 \qquad (x_0, x_1, \cdots, x_k)\notin S$$
$$q(x_0, x_1, \cdots, x_k)\geqq 0 \qquad (x_0, x_1, \cdots, x_k)\in S$$

where

$$S=\{(y_0, y_1, \cdots, y_k)\,|\,y_0\leqq y_1\leqq\cdots, \leqq y_k;\ y_i=0, 1, \cdots r;\ i=0, 1, \cdots, k\}$$

This assumption implies that the commodity of lower unit cost can not deliver faster than the one of higher unit cost.

(1.9) There is a discount factor α, $0<\alpha\leqq 1$.

(1.10) (a) $\alpha\lim_{\eta\to\infty} w'(\eta)=\alpha\lim_{\eta\to\infty}\int_0^\eta w'(\eta-t)\varphi_i(t)dt<c_k$, (b) $w'(0)<v$ $i=1, 2, \cdots, m$.

(1.11) $L(\eta;\varphi_i)-v\int_0^\eta(\eta-t)\varphi_i(t)\,dt$ is convex, where $L(\eta;\varphi_i)$, the expected one-period loss arising from penalty and holding costs, is given by

$$L(\eta;\varphi_i)=\begin{cases}\int_0^{\eta} h(\eta-t)\varphi_i(t)\,dt+\int_{\eta}^{\infty} p(t-\eta)\varphi_i(t)\,dt & \eta>0,\\ \int_0^{\infty} p(t-\eta)\varphi_i(t)\,dt & \eta\leqq 0, i=1,2,\cdots,m.\end{cases}$$

We shall assume that all integrals occuring in this paper exist and are finite, and that interchanges of integration and differentiation where needed are possible.

This impose certain restrictions on the class of demands densities.

At first in the section 2, we state the formulation of the problem and give the optimal policies. In the section 3 and 4, we shall give the optimal policies and discuss the properties of the optimal policies in which the ordering costs are strictly convex-increasing, or composed of a unit costs plus a reorder costs. In the section 5, we shall give remarks concering our model.

§2. Mathematical Formulation.

Let $f_n(x;\varphi_i)$ denote the expected total discounted loss by an optimal policy for the n-period inventory model under the assumption of the ordering policy (1.1), where φ_i is the demand density in first period, and x the initial stock level. From the principle of optimality we obtain for $n\geqq r+1$

$$(2.1)\quad f_n(x;\varphi_i)=\min_{\substack{\beta_\lambda\geqq m_\lambda\geqq 0,\\ \lambda=0,1,\cdots,k-1}}\ \inf_{m_k\geqq 0}\{g_r(x,m_0,m_1,\cdots,m_k;\varphi_i)+f_{n-r,r}(x+m_0+m_1+\cdots+m_k;\varphi_i)\}$$

where $g_r(x,m_0,m_1,\cdots,m_k;\varphi_i)$ is the expected loss incurred in period $1,2,\cdots,r$ if the demand density in first period is φ_i, the initial stock level is x, amounts of orders of size $m_0,m_1,\cdots,m_{k-1}$ and m_k ordered at unit prices, respectivcly, $c_0,c_1,\cdots,c_{k-1}$ and c_k are issued [see (2.8) and (2.10)], and

$$(2.2)\quad f_{n-r,\lambda}(x;\varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^{\infty} f_{n-r,\lambda-1}(x-t;\varphi_j)\varphi_i(t)\,dt$$

$$f_{n-r,0}(x;\varphi_i)=f_{n-r}(x;\varphi_i),\ f_0(x;\varphi_i)=-W_0(x;\varphi_i)=-w(x)$$

$$W_\lambda(x;\varphi_i)=\alpha\sum_{j=1}^{m} p_{ij}\int_0^{x} W_{\lambda-1}(x-t;\varphi_j)\varphi_i(t)\,dt$$

$$\lambda\geqq 1, i=1,2,\cdots,m.$$

It is noticed that $f_{0r}(x;\varphi_i)=-W_r(x;\varphi_i)$ for $x>0$. For $n\leqq r$ we obtain

$$(2.3)\quad f_n(x;\varphi_i)=\min_{\substack{\beta_\lambda\geqq m_\lambda\geqq 0,\\ \lambda=0,1,\cdots,k-1}}\ \inf_{m_k\geqq 0}\{g_n(x,m_0,m_1,\cdots,m_k;\varphi_i)\}$$

where $g_n(x,m_0,m_1,\cdots,m_k;\varphi_i)$ is given by (2.10).

It remains to compute $g_r(x,m_0,m_1,\cdots,m_k;\varphi_i)$. The expected one-period credit cost $V(x,z;\varphi_i)$ is found to be

$$(2.4)\quad V(x,z;\varphi_i)=\begin{cases} v\int_0^{x+z}(t-x)\varphi_i(t)\,dt+vz\int_{x+z}^{\infty}\varphi_i(t)\,dt & x+z>0,\ x\leqq 0,\\ v\int_x^{x+z}(t-x)\varphi_i(t)\,dt+vz\int_{x+z}^{\infty}\varphi_i(t)\,dt & x+z>0,\ x>0,\\ vz & x+z\leqq 0,\end{cases}$$
$$i=1,2,\cdots,m.$$

where x denotes the starting stock level and z the size of order to be delivered at the end of the period. It will be convenient to introduce the function

$$(2.5)\quad V(u;\varphi_i)=\begin{cases} -vu+v\int_0^{u}(u-t)\varphi_i(t)\,dt & u\geqq 0,\\ -vu & u<0,\end{cases}$$
$$i=1,2,\cdots,m.$$

Then $V(x,z;\varphi_i)$ in (2.4) can be written

$$(2.6)\quad V(x,z;\varphi_i)=-V(x+z;\varphi_i)+V(x;\varphi_i).$$

We shall compute $g_r(x,m_0,\cdots,m_k;\varphi_i)$ and $g_n(x,m_0,\cdots,m_k;\varphi_i)$.

Case (a), $n\geqq r+1$. If we are going to order amount $m_0,m_1,\cdots,m_{k-1}$ and m_k at unit prices $c_0,c_1,\cdots,c_{k-1}$ and c_k, respectively, then the ordering cost is given by $\sum_{j=0}^{k}c_jm_j$. When one of permutations $\underline{x}=(x_0,x_1,\cdots,x_k)\in S$ such that $0<x_0<x_1<\cdots<x_k$ and $r>k$ is given, the discounted expected cost over (x_j-x_{j-1}) period from $(x_{j-1}+1)$ th period until x_j th period is given by

$$\sum_{\nu=x_{j-1}}^{x_j-1}L_\nu(x+\sum_{t=0}^{j}m_{t-1};\varphi_i)+V_{x_j-1}(x+\sum_{t=0}^{j}m_t;\varphi_i)-V_{x_j-1}(x+\sum_{t=0}^{j}m_{t-1};\varphi_i)$$

where

$$(2.7)\quad \begin{aligned} &L_\lambda(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^{\infty}L_{\lambda-1}(x-t;\varphi_j)\varphi_i(t)\,dt\\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda\geqq 1\\ &V_\lambda(x;\varphi_i)=\alpha\sum_{j=1}^{m}p_{ij}\int_0^{\infty}V_{\lambda-1}(x-t;\varphi_j)\varphi_i(t)\,dt\\ &L_0(x;\varphi_i)=L(x;\varphi_i),\ V_0(x;\varphi_i)=V(x;\varphi_i),\ V_{-1}(x;\varphi_i)=0\\ &m_{-1}=x_{-1}=0;\end{aligned}$$

hence the sum of the discounted expected cost incurred in period $1,2,\cdots,x_k$ is given by

$$\sum_{j=0}^{k}[\sum_{\nu=x_{j-1}}^{x_j-1}L_\nu(x+\sum_{t=0}^{j}m_{t-1};\varphi_i)+V_{x_j-1}(x+\sum_{t=0}^{j}m_t;\varphi_i)-V_{x_j-1}(x+\sum_{t=0}^{j}m_{t-1};\varphi_i)]$$

On the other hand, the sum of the discounted expected cost incurred in period x_k+1, $\cdots$, r is given by

$$\sum_{\nu=x_k}^{r-1}L_\nu(x+\sum_{t=0}^{k}m_t;\varphi_i)$$

Considering the definition of S and the case $r\leqq k$, we have for $n\geqq r+1$ from above results

$$(2.8)\quad g_r(x, m_0, m_1, \cdots, w_k; \varphi_i) = \sum_{j=0}^{k} c_j m_j + \sum_{\underline{x}\in S} q(\underline{x}) \{ \sum_{j=0}^{k} [\sum_{\nu=x_{j-1}}^{x_j-1} L_\nu (x + \sum_{t=0}^{j} m_{t-1}; \varphi_i)$$

$$+ V_{x_j-1}(x + \sum_{t=0}^{j} m_t; \varphi_i) - V_{x_j-1}(x + \sum_{t=0}^{j} m_{t-1}; \varphi_i)] + \sum_{\nu=x_k}^{r-1} L_\nu (x + \sum_{t=0}^{k} m_t; \varphi_i)\}$$

$$i=1, 2, \cdots, m,$$

where $\sum_{\underline{x}\in S}$ means summation for all permutations such that $\underline{x}=(x_0, x_1, \cdots, x_k)\in S$, and

$$(1.9)\quad \sum_{\nu=j}^{k} L_\nu(\,;\varphi_i)=0 \qquad (k<j) \qquad i=1, 2, \cdots, m.$$

Case (b) $n\leqq r$. Let y_l^* denote the maximum of elements x_i's of a given $(x_0 x_1 \cdots x_k) \in S$ which does not exceed n and has the maximum suffix, where $l=l(x_0, x_1, \cdots, x_k)$ is the maximum suffix. If such a maximum x_i can not be found, let $y_l^*=y_{-1}^*=0$. It is clear from the definition of l that the set S is expressed as the sum of disjoint sets $S_{-1}^{(n)}, S_0^{(n)}, \cdots, S_k^{(n)}$[i.e., $S=\sum_{j=-1}^{k} S_j^{(n)}$], where $S_j^{(n)}=\{\underline{x} \mid l(\underline{x})=j, \underline{x}\in S\}$ $(j=-1, 0, 1, \cdots, k)$ depend on n and j. When one of permutations $(x_0, x_1, \cdots, x_k)\in S$ and n are given, then the integer $l=l(x_0, x_1, \cdots, x_k)$ is uniquely determined and the ordering cost is given by $\sum_{j=0}^{l} c_j m_j$, and the sum of the discounted cost in periods $1, 2, \cdots, y_l^*(=x_l)$ is given by

$$\sum_{j=0}^{l} [\sum_{\nu=x_{j-1}}^{x_j-1} L_\nu (x + \sum_{t=0}^{j} m_{t-1}; \varphi_i) + V_{x_j-1}(x + \sum_{t=0}^{j} m_t; \varphi_i) - V_{x_j-1}(x + \sum_{t=0}^{j} m_{t-1}; \varphi_i)]$$

On the other hand, the sum of the discounted expected costs incurred in periods $y_l^*+1, \cdots, n$ is given by

$$\sum_{\nu=y_l^*}^{n-1} L_\nu (x + \sum_{j=0}^{l} W_j; \varphi_i) - W_n (x + \sum_{j=0}^{l} m_j; \varphi_i)$$

Hence we have for $n\leqq r$

$$(2.10)\quad g_n(x, m_0, m_1, \cdots, m_k; \varphi_i) = \sum_{\underline{x}\in S} q(\underline{x}) \{ \sum_{j=0}^{l(\underline{x})} [c_j m_j + \sum_{\nu=x_{j-1}}^{x_j-1} L_\nu (x + \sum_{t=0}^{j} m_{t-1}; \varphi_i)$$

$$+ V_{x_j-1}(x + \sum_{t=0}^{j} m_t; \varphi_i) - V_{x_j-1}(x + \sum_{t=0}^{j} m_{t-1}; \varphi_i)] + \sum_{\nu=y^*_{l(\underline{x})}}^{n-1} L_\nu (x + \sum_{j=0}^{l(\underline{x})} m_j; \varphi_i)$$

$$- W_n (x + \sum_{j=0}^{l(\underline{x})} m_i; \varphi_i)\}$$

$$i=1, 2, \cdots, m.$$

If we make substitution $x+m_0=u_0, x+m_0+m_1=u_1, \cdots, x+m_0+m_1+\cdots+m_k=u_k$, then (2.8) and (2.10) may be written as

$$(2.11)\quad f_n(x;\varphi_i) = \min_{x\leqq u_0\leqq x+\beta_0} \{[(c_0-c_1)u_0 + H_1(u_0;\varphi_i)] + \min_{u_0\leqq u_1\leqq u_0+\beta_1} \{[(c_1-c_2)u_1$$

$$+ H_2(u;\varphi_i)] + \cdots + \min_{u_{k-2}\leqq u_{k-1}\leqq u_{k-2}+\beta_{k-1}} \{[(c_{k-1}-c_k)u_{k-1} + H_k(u;\varphi_i)]$$

$$+ \inf_{u_{k-1}\leqq u_k} \{c_k u_k + H_{k+1}(u_k;\varphi_i) + f_{n-r,r}(u_k;\varphi_i)\}\cdots\} - c_c x + H_0(x;\varphi_i)$$

$$= \min_{x\leqq u_0\leqq x+\beta_0} [G_{0n}(u_0;\varphi_i)] - c_c x + H_c(x;\varphi_i)$$

$$n\geqq r+1$$

where

$$G_{kn}(u_k;\varphi_i)=c_k u_k+H_{k+1}(u_k;\varphi_i)+f_{n-r,r}(u_k;\varphi_i)$$

$$G_{jn}(u_j;\varphi_i)=(c_j-c_{j+1})u_j+H_{j+1}(u;\varphi_i)+K_{jn}(u_j;\varphi_i)$$

$$K_{k-1,n}(u_{k-1};\varphi_i)=\inf_{u_{k-1}\leq u_k} G_{kn}(u_k;\varphi_i),\ K_{jn}(u_j;\varphi_i)=\min_{u_j\leq u_{j+1}\leq u_j+\beta_{j+1}} G_{j+1\,n}(u_{j+1};\varphi_i)$$

$$j=0,1,\cdots,k-1$$

(2.12)

$$H_j(u_{j-1};\varphi_i)=\sum_{\{\underline{x}\in S;x_{j-1}\neq x_j\}} q(\underline{x})\,[V_{x_{j-1}-1}(u_{j-1};\varphi_i)+\sum_{\nu=x_{j-1}}^{x_j-1} L_\nu(u_{j-1};\varphi_i)-V_{x_j-1}(u_{j-1};\varphi_i)]$$

$$j=0,1,\cdots,k$$

$$H_{k+1}(u_k;\varphi_i)=\sum_{\underline{x}\in S} q(\underline{x})[V_{x_k-1}(u_k;\varphi_i)+\sum_{\nu=x_k}^{r-1} L_\nu(u_k;\varphi_i)]$$

$$i=1,2,\cdots,m$$

and $\sum_{\{\underline{x}\in S,x_{j-1}\neq x_j\}}$ means summation for all permutations such that $\underline{x}=(x_0,x_1,\cdots,x_k)\in S$ and $x_{j-1}\neq x_j$, and

(2.13) $$f_n(x;\varphi_i)=\min_{x\leq u_0\leq x+\beta_0}\{[(a_0c_0-a_1c_1)u_0+H_1^*(u_0;\varphi_i)]+\min_{u_0\leq u_1\leq u_0+\beta_1}\{[(a_1c_1-a_2c_2)u_2$$
$$+H_2^*(u_1;\varphi_i)]+\cdots+\min_{u_k\leq u_{k-1}\leq u_{k-2}+\beta_{k-1}}\{[(a_{k-1}c_{k-1}-a_kc_k)\,u_{k-1}+H_k^*(u_{k-1};\varphi_i)]$$
$$+\inf_{u_{k-1}\leq u_k}\{a_kc_ku_k+H_{k+1}^*(u_k;\varphi_i)\}\cdots\}-a_0c_0x+H_0^*(x;\varphi_i)$$
$$=\min_{x\leq u_0\leq x+\beta_0}\{G_{0n}(u_0;\varphi_i)\}-a_0c_0x+H_0^*(x;\varphi_i)$$

$$n\leq r;i=1,2,\cdots,m.$$

where

$$G_{kn}(u_k;\varphi_i)=a_kc_ku_k+H_{k+1}^*(u_k;\varphi_i)$$

$$G_{jn}(u_j;\varphi_i)=(a_jc_j-a_{j+1}c_{j+1})u_j+H_{j+1}^*(u_j;\varphi_i)+K_{jn}(u_j;\varphi_i)$$

$$K_{k-1,n}(u_{k-1};\varphi_i)=\inf_{u_{k-1}\leq u_k} G_{kn}(u_k;\varphi_i),\ K_{jn}(u_j;\varphi_i)=\min_{u_j\leq u_{j+1}\leq u_j+\beta_{j+1}} G_{j+1,n}(u_{j+1};\varphi_i)$$

$$j=0,1,\cdots,k-1$$

$$a_j=\sum_{h=j}^{k}\sum_{\underline{x}\in S_h^{(n)}} q(\underline{x}) \qquad j=0,1,\cdots,k$$

(2.14)

$$H_j^*(u_{j-1};\varphi_i)=\sum_{\underline{x}\in S_{j-1}^{(n)}} q(\underline{x})\,(V_{x_{j-1}-1}(u_{j-1};\varphi_i)+\sum_{\nu=x_{j-1}}^{n-1} L_\nu(u_{j-1};\varphi_i)-W_n(u_{j-1};\varphi_i))$$
$$+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},x_{j-1}\neq x_j\}} q(\underline{x})\,[V_{x_{j-1}-1}(u_{j-1};\varphi_i)+\sum_{\nu=x_{j-1}}^{x_j-1} L_\nu(u_{j-1};\varphi_i)-V_{x_j-1}(u_{j-1};\varphi_i)]$$

$$j=0,1,\cdots,k$$

$$H_{k+1}^*(u_k;\varphi_i)=\sum_{\underline{x}\in S_k^{(n)}} q(\underline{x})(V_{x_k-1}(u_k;\varphi_i)+\sum_{\nu=x_k}^{n-1} L_\nu(u_k;\varphi_i)-W_n(u_k;\varphi_i))$$

,and $\sum_{\underline{x}\in S_h^{(n)}}$ means summation for all permutations such that $\underline{x}=(x_0,x_1,\cdots,x_k)\in S_h^{(n)}$, and

$\sum_{\{\underline{x}\in S_h^{(n)}, x_{j-1}\neq x_j\}}$ means summation for all permutations such that $\underline{x}=(x_0, x_1, \cdots, x_k)\in S_h^{(n)}$ and $x_{j-1}\neq x_j$.

Remark I. Various cases may be considered in the formulas of (2.12)~(2.14), according to the value of $q(\underline{x})$.

Example I. If $r=k, q(012\cdots k)=1$, then $(012\cdots k)\in S$, $a_0=a_1=\cdots a_n=1, a_j=0$ $(k\geqq j>n)$, $H_0^*(x;\varphi_i)=0$, $H_j^*(u_{j-1};\varphi_i)=L_{j-1}(u_{j-1};\varphi_i)-V_{j-1}(u_{j-1};\varphi_i)+V_{j-2}(u_{j-1};\varphi_i)$ $(j=1, 2, \cdots, n)$, $H_{n+1}^*(u;\varphi_i)=V_{n-1}(u_n;\varphi_i)-W_n(u_n;\varphi_i), H_l^*(u_{l-1};\varphi_i)=0$ $(l>n+1)$; hence (2.13) and (2.14) becomes to the one in [7].

Example II. If $n<r=3, k=1$, then $S_{-1}^{(1)}=\{(22), (23), (33)\}$, $S_0^{(1)}=\{(13), (12), (02), (03)\}$, $S_1^{(1)}=\{(11), (01), (00)\}$; $S_{-1}^{(2)}=\{(33)\}$, $S_0^{(2)}=\{(23), (13), (03)\}$, $S_1^{(2)}=\{(22), (12), (02), (11), (01), (00)\}$, moreover if $\sum_{\underline{x}\in S_{-1}^{(n)}} q(\underline{x})=1$, then $a_0=\sum_{\underline{x}\in S_0^{(n)}} q(\underline{x})+\sum_{\underline{x}\in S_1^{(n)}} q(\underline{x})=0$, $a_1=\sum_{\underline{x}\in S_1^{(n)}} q(x)=0$,

$$H_1^*(u_0;\varphi_i)=\sum_{\underline{x}\in S_0^{(n)}} q(\underline{x})\Big(V_{x_0-1}(u_0;\varphi_i)+\sum_{\nu=x_0}^{n-1} L_\nu(u_0;\varphi_i)-W_n(u_0;\varphi_i)\Big)+\sum_{\{\underline{x}\in S_1^{(n)}, x_0\neq x_1\}} q(\underline{x})\Big(V_{x_0-1}(u_0;\varphi_i)$$
$$+\sum_{\nu=x_0}^{x_1-1} L_\nu(u_0;\varphi_i)-V_{x_1-1}(u_0;\varphi_i)\Big)=0,\quad H_2^*(u_1;\varphi_i)=\sum_{\underline{x}\in S_1^{(n)}} q(\underline{x})\Big(V_{x_1-1}(u_1;\varphi_i)+\sum_{\nu=x_1}^{n-1} L_\nu(u_1;\varphi_i)-$$
$$W_n(u_1;\varphi_i)\Big)=0,\quad f_n(x;\varphi_i)=H_0^*(x;\varphi_i)=\sum_{\underline{x}\in S_{-1}^{(n)}} q(\underline{x})\Big(\sum_{\nu=0}^{n-1} L_\nu(x;\varphi_i)-W_n(x;\varphi_i)\Big)=\sum_{\nu=0}^{n-1} L_\nu(x;\varphi_i)-$$
$$W_n(x;\varphi_i).$$

It follows that the optimal policy is not to order.

Example III. If $n=r=3$, $k=1$ and $q(22)=1$, then $a_0=a_1=1$, $H_0^*(x;\varphi_i)=\sum_{\nu=0}^{1} L_\nu(x;\varphi_i)-V_1(x;\varphi_i)$, $H_1^*(u_0;\varphi_i)=0$, $H_2^*(u_1;\varphi_i)=V_1(u_1;\varphi_i)+L_2(u_1;\varphi_i)-W_3(u_1;\varphi_i)$.

Example IV. If $n\geqq r+1=4$, $k=1$ and $q(22)=1$, then $H_0(x;\varphi_i)=\sum_{\nu=0}^{1} L_\nu(x;\varphi_i)-V_1(x;\varphi_i)$, $H_1(u_0;\varphi_i)=0$, $H_2(u_1;\varphi_i)=V_1(u_1;\varphi_i)+L_2(u_1;\varphi_i)$.

We cite the known results in [2] and [9] that will be necessary for the subsequent analysis.

LEMMA A. (i) *if $g(x)$ is convex function on the real line, then for* $0\leqq m<\infty$

$$h(x)=\min_{x\leqq y\leqq x+m} g(y)$$

is convex, and if $g(x)$ is bounded from below, so is $h(x)$.
(ii) (*a*) *if $g(x)$ is convex and bounded from below, then*

$$k(x)=\inf_{x\leqq y} g(y)$$

is convex and bounded from below. (*b*) *$g(x)$ is convex, and unbounded from below, but non-decreasing, then*

$$k(x)=\inf_{x\leqq y} g(y)=\min_{x\leqq y} g(y)=g(x)$$

COROLLARY A. *Let $\bar{x}$ satisfy $g(\bar{x})\leqq g(x)$ for $-\infty\leqq x\leqq\infty$. Then (i) $y^*(x)$ having the property $h(x)=g(y^*(x))$ is given by*

$$y^*(x)=\begin{cases}\bar{x}+m & x+m<\bar{x}\\ \bar{x} & x<\bar{x}<x+m\\ x & \bar{x}<x,\end{cases}$$

and (ii) $y^*(x)$ *having the property* $k(x)=g(y^*(x))$ *is given by*

$$y^*(x)=\begin{cases}\bar{x} & x\leqq\bar{x},\\ x & x>\bar{x}\end{cases}$$

If $\bar{x}$ is not unique, then $y^*(x)$ is not unique.

LEMMA B. *If conditions of* (1.4), (1.11) *and* (2.31d) *are satisfied, then*

$$L'_{j-1}(x;\varphi_i)-V'_{j-1}(x;\varphi_i)\geqq\alpha^{j-1}(v-\lim_{y\to\infty}p'(y))\ \textit{for all } x$$

$$V'_{j-1}(x;\varphi_i)\geqq-\alpha^{j-1}v \qquad \textit{for all } x$$

$$j\geqq1;\ i=1,2,\cdots,m$$

,and specially when $x\leqq0$

$$L'_{j-1}(x;\varphi_i)-V'_{j-1}(x;\varphi_i)\leqq\alpha^{j-1}(v-p'(0))<0$$

$$V'_{j-1}(x;\varphi_i)=-\alpha^{j-1}v<0$$

$$j\geqq1;\ i=1,2,\cdots,m.$$

THEOREM 2.1. *Under assumptions* (1.1)~(1.11) (i) $G_{jn}(x;\varphi_i)$ *is convex and increasing for* x *large enough*; (ii) *the optimal expected total discounted loss function* $f_n(x;\varphi_i)$ *is convex.*

Proof. We establish the proof by induction on n. There are two possibilities requiring separate treatments $n\leqq r$ and $n\geqq r+1$.

Case I $n\leqq r$, Then from (2.12) we have

$$(2.15)\quad G_{k1}(u;\varphi_i)=\sum_{\{\underline{x}\in S_k^{(1)},x_k=0\}}q(\underline{x})\,[c_k u+L(u;\varphi_i)-W_1(u;\varphi_i)]+\sum_{\{\underline{x}\in S_k^{(1)},x_k\neq0\}}q(\underline{x})\,[c_k u+V(u;\varphi_i)-W_1(u;\varphi_i)]$$

The function $G_{k1}(u;\varphi_i)$ is convex and increasing for u large enough, since $V(u;\varphi_i)-W_1(u;\varphi_i)$ and $L(u;\varphi_i)-W_1(u;\varphi_i)$ are convex, and conditions (1.4), (1.6) and (1.10) hold. By Lemma A, $K_{k-1,1}(u;\varphi_i)$ is convex and increases for u large enough. On the other hand, $H_j^*(u;\varphi_i)$, where $j=0.1,\cdots,k$, is convex since $V(u;\varphi_i)-W_1(u;\varphi_i)$, $L(u;\varphi_i)-V(u;\varphi_i)$ and $L(u;\varphi_i)-W_1(u;\varphi_i)$ are convex by (1.10b) and (1.11). Hence $G_{k-1,1}(u;\varphi_i)$ is convex, since it is the sum of convex functions. From (2.12) we get

$$(2.16)\quad \lim_{u\to\infty}G'_{k-1,1}(u;\varphi_i)=a_{k-1}c_{k-1}-a_kc_k+\lim_{u\to\infty}H_k^{*\prime}(u;\varphi_i)+\lim_{u\to\infty}K'_{k-1,1}(u;\varphi_i)$$

$$=\sum_{\underline{x}\in S_{k-1}^{(1)}}q(\underline{x})\Big[\sum_{v=x_{k-1}}^{0}\lim_{u\to\infty}L'_v(u;\varphi_i)+c_k-\lim_{u\to\infty}W'_1(u;\varphi_i)\Big]$$

$$+\sum_{\underline{x}\in S_k^{(1)}}q(\underline{x})(c_{k-1}-c_k)+\sum_{v=x_{k-1}}^{x_{k-1}}\lim_{u\to\infty}L'_v(u;\varphi_i)+\lim_{u\to\infty}G'_{k1}(u;\varphi_i)$$

$$>0$$

by (1.4), (1.7) and (1.10a). Now suppose that part (i) of Theorem is true for $j=k$, $k-1,\cdots,\ k-\lambda(\lambda\leqq k-1)$. Then we have from (2.12)

$$(2.17)\quad \lim_{u\to\infty}G'_{k-\lambda-1,1}(u;\varphi_i)=a_{k-\lambda-1}c_{k-\lambda-1}-a_{k-\lambda}c_{k-\lambda}+\lim_{u\to\infty}H_{k-\lambda}^{*\prime}(u;\varphi_i)+\lim_{u\to\infty}K'_{k-\lambda-1,1}(u;\varphi_i)$$

$$= \sum_{\underline{x}\in S^{(1)}_{k-\lambda-1}} q(\underline{x})[c_{k-\lambda-1} + \sum_{\nu=x_{k-\lambda-1}}^{0} \lim_{u\to\infty} L'_\nu(u;\varphi_i)] + \sum_{h=k-\lambda}^{k} \sum_{\underline{x}\in S^{(1)}_h} q(\underline{x})[c_{k-\lambda-1} - c_{k-\lambda}$$

$$+ \sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1} \lim_{u\to\infty} L'_\nu(u;\varphi_i)] + \lim_{u\to\infty} G'_{k-\lambda,1}(u;\varphi_i)$$

$$>0 \qquad \text{by (1.4), (1.7) and (1.10a)}$$

The function $G_{k-\lambda-1,1}(u;\varphi_i)$ is convex, since it is the sum of convex functions, and is increasing u large enough. By Lemma A, $\min_{x\leq u\leq x+\beta_0} G_{01}(u;\varphi_i)$ is convex. Hence $f_1(x;\varphi_i)$ is convex.

We assume that the theorem is true for $n-1$, we seek to advance the induction to the integer n. Then, the inductive assumption implies that $V_{n-2}(u;\varphi_i)-W_{n-1}(u;\varphi_i)$, $L_{n-2}(u;\varphi_i)-W_{n-1}(u;\varphi_i)$ and $L_{n-2}(u;\varphi_i)-V_{n-2}(u;\varphi_i)$ are convex, and $c_k u-W_{n-1}(u;\varphi_i)$ is increasing for u large enough. Since

(2.18) $$G_{kn}(u;\varphi_i)=a_kc_ku+H^*_{k+1}(u;\varphi_i)-a_kW_n(u;\varphi_i)$$

$$= \sum_{\underline{x}\in S^{(n)}_k} q(\underline{x})[c_ku+V_{x_k-1}(u;\varphi_i)+\sum_{\nu=x_k}^{n-1} L_\nu(u;\varphi_i)-W_n(u;\varphi_i)]$$

$$= \sum_{\{\underline{x}\in S^{(n)}_k,\, x_k=n\}} q(\underline{x})[c_ku+\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty (V_{n-2}(u-t;\varphi_j)-W_{n-1}(u-t;\varphi_j))\varphi_i(t)\,dt]$$

$$+ \sum_{\{\underline{x}\in S^{(n)}_k,\, x_k\neq n\}} q(\underline{x})[c_ku+V_{x_k-1}(u;\varphi_i)+\sum_{\nu=x_k}^{n-2} L_\nu(u;\varphi_i)$$

$$+\alpha\sum_{j=1}^{m} p_{ij}\int_0^\infty (L_{n-2}(u-t;\varphi_j)-W_{n-1}(u-t;\varphi_j))\varphi_i(t)\,dt]$$

and

(2.19) $$\lim_{u\to\infty} G'_{kn}(u;\varphi_i)= \sum_{\underline{x}\in S^{(n)}_k} q(\underline{x})[c_k-\lim_{u\to\infty} W'_n(u;\varphi_i)+\sum_{\nu=x_k}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)]$$

$$= \sum_{\underline{x}\in S^{(n)}_k} q(\underline{x})[c_k(1-\alpha)+\alpha\sum_{j=1}^{m} p_{ij}\lim_{u\to\infty}\int_0^\infty (c_k-W'_{n-1}(u-t;\varphi_j))\varphi_i(t)\,dt$$

$$+\sum_{\nu=x_k}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)]$$

$$>0$$

,it follows that $G_{kn}(u;\varphi_i)$ is convex and increasing for u large enough. Assume that part (i) of Theorem is true for $j=k, k-1, \cdots, k-\lambda$ $(\lambda\leq k-1)$. Then by the Lemma A, $K_{k-\lambda-1,n}(u;\varphi_i)$ is convex and is increasing for large enough. Then

(2.20) $$\lim_{u\to\infty} G'_{k-\lambda-1,n}(u;\varphi_i)=a_{k-\lambda-1}c_{k-\lambda-1}-a_{k-\lambda}c_{k-\lambda}+\lim_{u\to\infty} H^{*\prime}_{k-\lambda}(u;\varphi_i)+\lim_{u\to\infty} K'_{k-\lambda-1,n}(u;\varphi_i)$$

$$= \sum_{\underline{x}\in S^{(n)}_{k-\lambda-1}} q(\underline{x})[c_{k-\lambda-1}(1-\alpha)+\sum_{\nu=x_{k-\lambda-1}}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i))$$

$$+\alpha\sum_{j=1}^{m} p_{ij}\lim_{u\to\infty}\int_0^\infty (c_{k-\lambda-1}-W'_{n-1}(u-t;\varphi_j))\varphi_i(t)\,dt]$$

$$+\sum_{h=k-\lambda}^{k}\sum_{\underline{x}\in S^{(n)}_h} q(\underline{x})[c_{k-\lambda-1}-c_{k-\lambda}+\sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)]+\lim_{u\to\infty} G'_{k-\lambda,n}(u;\varphi_i)$$

$$>0$$

and

(2.21) $$G_{k-\lambda-1,n}(u;\varphi_i)=\sum_{\underline{x}\in S_{k-\lambda-1}^{(n)}} q(\underline{x})(c_{k-\lambda-1}u+V_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{n-1} L_\nu(u;\varphi_i)-W_n(u;\varphi_i))$$

$$+\sum_{h=k-\lambda}^{k}\sum_{\underline{x}\in S_h^{(n)}} q(\underline{x})(c_{k-\lambda-1}-c_{k-\lambda})u+\sum_{h=k-\lambda}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\ x_{k-\lambda-1}\neq x_{k-\lambda}\}} q(\underline{x})\quad [V_{x_{k-\lambda-1}-1}(u;\varphi_i)$$

$$+\sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1} L_\nu(u;\varphi_i)-V_{x_{k-\lambda}-1}(u;\varphi_i)]+K_{k-\lambda-1,n}(u;\varphi_i)$$

$$=\sum_{\{\underline{x}\in S_{k-\lambda-1}^{(n)},\ x_{k-\lambda-1}=n\}} q(\underline{x})\ [c_{k-\lambda-1}u+\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty(V_{n-2}(u-t;\varphi_j)-W_{n-1}(u-t;\varphi_j))\varphi_i(t)dt]$$

$$+\sum_{\{\underline{x}\in S_{k-\lambda-1}^{(n)},\ x_{k-\lambda-1}\neq n\}} q(\underline{x})\ [c_{k-\lambda-1}u+V_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{n-2} L_\nu(u;\varphi_i)$$

$$+\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty(L_{n-2}(u-t;\varphi_j)-W_{n-1}(u-t;\varphi_j))\varphi_i(t)dt]+\sum_{h=k-\lambda}^{k}\sum_{\underline{x}\in S_h^{(n)}} q(\underline{x})(c_{k-\lambda-1}-c_{k-\lambda})u$$

$$+\sum_{h=k-\lambda}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\ x_{k-\lambda-1}\neq x_{k-\lambda},\ x_{k-\lambda}=n\}} q(\underline{x})\quad [V_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{n-2} L_\nu(u;\varphi_i)+\alpha\sum_{j=1}^{m}p_{ij}\int_0^\infty(L_{n-2}(u-t;\varphi_j)$$

$$-V_{n-2}(u-t;\varphi_j))\varphi_i(t)dt]+\sum_{h=k-\lambda}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\ x_{k-\lambda-1}\neq x_{k-\lambda},\ x_{k-\lambda}\neq n\}} q(\underline{x})\quad [V_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1} L_\nu(u;\varphi_i)$$

$$-V_{x_{k-\lambda}-1}(u;\varphi_i)]+K_{k-\lambda-1,n}(u;\varphi_i)$$

The function $G_{k-\lambda-1,n}(u;\varphi_i)$ is convex, since it is the sum of convex functions, and is increasing for u large enough. By Lemma A, $\lim_{x\leq u\leq x+\beta_0} G_{0n}(u;\varphi_i)$ is convex. Hence $f_n(x;\varphi_i)$ is convex.

Case II $n\geqq r+1$. Since

(2.22) $$\lim_{u\to\infty} G'_{0n}(u;\varphi_i)-a_0c+\lim_{u\to\infty} H_0^{*\prime}(u;\varphi_i)$$

$$=\sum_{j=0}^{k+1}\lim_{u\to\infty} H_j^{*\prime}(u;\varphi_i)$$

$$=\sum_{j=0}^{k}\{\sum_{\underline{x}\in S_{j-1}^{(n)}} q(\underline{x})(\sum_{\nu=x_{j-1}}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to\infty} W'_n(u;\varphi_i))$$

$$+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\ x_{j-1}\neq x_j\}} q(\underline{x})(\sum_{\nu=x_{j-1}}^{x_j-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i))\}+\sum_{\underline{x}\in S_k^{(n)}} q(\underline{x})(\sum_{\nu=x_k}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)$$

$$-\lim_{u\to\infty} W'_n(u;\varphi_i))$$

$$=\sum_{\underline{x}\in S_{-1}^{(n)}} q(\underline{x})(\sum_{\nu=0}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to\infty} W'_n(u;\varphi_i))$$

$$+\sum_{j=0}^{k}\sum_{\underline{x}\in S_j^{(n)}} q(\underline{x})(\sum_{p=0}^{j}\sum_{\nu=x_{p-1}}^{x_p-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)$$

$$+\sum_{\nu=x_j}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i))-\sum_{j=0}^{k}\sum_{\underline{x}\in S_j^{(n)}} q(\underline{x})\lim_{u\to\infty} W'_n(u;\varphi_i)$$

$$=\sum_{j=0}^{k+1}\sum_{\underline{x}\in S_{j-1}^{(n)}} q(\underline{x})(\sum_{\nu=0}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to\infty} W'_n(u;\varphi_i))$$

$$= \sum_{x \in S} q(\underline{x}) \left(\sum_{\nu=0}^{n-1} \lim_{u\to\infty} L'_\nu(u;\varphi_i) - \lim_{u\to\infty} W'_n(u;\varphi_i) \right)$$

$$= \sum_{\nu=0}^{n-1} \mathrm{ltm}_{u\to\infty} L'_\nu(u;\varphi_i) - \lim_{u\to\infty} W'_n(u;\varphi_i)) \qquad n \leqq r,$$

it follows that if $\sum_{\nu=0}^{n-1} \lim_{u\to\infty} L'_\nu(u;\varphi_i) - \lim_{u\to\infty} W'_n(u;\varphi_i) \leqq 0$, then $f_n(x;\varphi_i)$ $(n \leqq r)$ can be convex decreasing, although $G_{0n}(u;\varphi_i)$ $(n \leqq r)$ is increasing for u large enough. Using the fact that $c_k - \lim_{u\to\infty} W'_n(u;\varphi_i) > 0$ for $n \leqq r$ $(i=1,2,\cdots,m)$, we have

$$(2.23) \quad c_k + \lim_{x\to\infty} f'_n(x;\varphi_i) = \lim_{x\to\infty} G'_{0n}(x;\varphi_i) - a_0c_0 + \lim_{x\to\infty} H_1^{*\prime}(x;\varphi_i)$$

$$= \sum_{\nu=0}^{n-1} \lim_{x\to\infty} L'_\nu(x;\varphi_i) + c_k - \lim_{x\to\infty} W'_n(x;\varphi_i) > 0 \textit{ for } n \leqq r$$

,moreover by induction on j, we get

$$(2.24) \quad c_k + \lim_{x\to\infty} f'_{n,j}(x;\varphi_i) > 0 \textit{ for } j \leqq r \ (n \leqq r).$$

Although $f_n(x;\varphi_i)$ $(n \leqq r)$ can be convex decreasing, $G_{k,r+j}(x;\varphi_i)$ is convex and increasing for x large enough, since it is the sum of convex functions and $\lim_{u\to\infty} G'_{k,r+j}(x;\varphi_i)$ $= c_k + \lim_{x\to\infty} H'_{k+1}(x;\varphi_i) + \lim_{x\to\infty} f'_{jr}(x;\varphi_i) > 0$ for $j \leqq r$.

Hence part (i) of the theorem holds for $n = r+1$ and $j = k$. Assume that part (i) of Theorem holds for $j = k, k-1, \cdots, k-\lambda$ $(1 \leqq \lambda \leqq k-1)$, we extend the induction to the integer $k-\lambda-1$. It follows from the inductive assumption

$$(2.25) \quad \lim_{u\to\infty} G'_{k-\lambda-1,r+1}(u;\varphi_i) = c_{k-\lambda-1} - c_{k-\lambda} + \lim_{u\to\infty} H'_{k-\lambda}(u;\varphi_i) + \lim_{u\to\infty} G'_{k-\lambda,r+1}(u;\varphi_i) > 0$$

On the other hand, $(c_{j-1} - c_j)u + H_j(u;\varphi_i)$, where $j = 0,1,\cdots,k$ are convex and increasing for u large enough by assumption. Hence the function $G_{k-\lambda-1,r+1}(u;\varphi_i)$ is convex, since it is the sum of convex functions, and is increasing for u large enough. By the Lemma A, $\lim_{x \leqq u \leqq x+\beta_0} G_{0,r+1}(u;\varphi_i)$ is convex.

Hence $f_n(x;\varphi_i)$ is convex.

Assuming that the theorem holds for $n-1$ $(n-1 \geqq r+1)$, we seek to the advance the induction to the integer n. Then

$$c_k + \lim_{u\to\infty} f'_{n-r}(u;\varphi_i) = \sum_{j=0}^{k} \lim_{u\to\infty} H'_j(u;\varphi_i) + \lim_{u\to\infty} G'_{k,n-r}(u;\varphi_i) > 0.$$

By induction on j, we may deduce that

$$c_k + \lim_{u\to\infty} f'_{n-r,j}(u;\varphi_i) = c_k(1-\alpha) + \alpha \sum_{j=1}^{m} p_{ij} \lim_{u\to\infty} \int_0^\infty (c_k + f'_{n-r,j-1}(u-t;\varphi_j))\varphi_i(t)\,dt > 0 \quad \text{for } j \leqq r$$

, and $f_{n-r,j}(u;\varphi_i)$ is convex. Hence, we have

$$(2.26) \quad \lim_{u\to\infty} G'_{kn}(u;\varphi_i) = c_k + \lim_{u\to\infty} H'_{k+1}(u;\varphi_i) + \lim_{u\to\infty} f'_{n-r,r}(u;\varphi_i) > 0.$$

The function $G_{kn}(u;\varphi_i)$ is convex, since it is the sum of convex functions, and is increasing for u large enough. Now suppose that (i) of the theorem holds for $j = k$, $k-1, \cdots, k-\lambda$ $(1 \leqq \lambda \leqq k-1)$, we extend the induction to the integer $k-\lambda-1$. It follows from the inductive assumption that

(2.27) $\lim_{u\to\infty} G'_{k-\lambda-1,n}(u;\varphi_i)=c_{k-\lambda-1}-c_{k-\lambda}+\lim_{u\to\infty} H'_{k-\lambda}(u;\varphi_i)+\lim_{u\to\infty} G'_{k-\lambda,n}(u;\varphi_i)>0.$

The function $G_{k-\lambda-1,n}(u;\varphi_i)$ is convex, since it is the sum of convex functions, and is increasing for u large enough. By Lemma A, $\min_{x\le u\le x+\beta_0} G_{0n}(u;\varphi_i)$ is convex. Hence $f_n(x;\varphi_i)$ is convex. The proof is complete.

Remark II. *when* $n\geqq r+1$, we have from the simple calculation

$$(2.28)\quad \lim_{u\to\infty} G'_{0n}(u;\varphi_i)-c_0+\min_{u\to\infty} H'_0(u;\varphi_i)$$
$$=\sum_{j=0}^{k+1}\lim_{u\to\infty} H'_j(u;\varphi_i)+\lim_{u\to\infty} f'_{n-r,r}(u;\varphi_i)$$
$$=\sum_{\nu=0}^{r-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)+\lim_{u\to\infty} f'_{n-r,r}(u;\varphi_i).$$

Hence, if $\sum_{\nu=0}^{r-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)+\lim_{u\to\infty} f'_{n-r,r}(u;\varphi_i)\leqq 0$, then $f_n(x;\varphi_i)$ $(n\geqq r+1)$ can be convex decreasing, although $G_{0n}(u;\varphi_i)$ is increasing for u large enough. We note that even in such a situation $G_{k,n+r}(u;\varphi_i)$ is increasing for u large enough. It is clear that if $\lim_{u\to\infty}$ $L'(n;\varphi_i)-\lim_{u\to\infty} W'(u;\varphi_i)>0$, then $\sum_{\nu=0}^{n-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to\infty} W'_n(u;\varphi_i)>0$. Then $f_n(x;\varphi_i)$ is convex and increasing for x large enough.

Remark III. The results of Theorem 2.1 show that infimum in (2.1) and (2.3) can be replaced by minimum.

Theorem 2.2. Let $\bar{x}_{jn}(\varphi_i)$ satisfy $G_{jn}(\bar{x}_{jn}(\varphi_i);\varphi_i)\leqq G_{jn}(x;\varphi_i)$ for $-\infty\leqq\bar{x}_{jn}(\varphi_i)<\infty$. Then the optimal order m^{i*}_{jn} in the first period are given

$$(2.29)\quad m^{i*}_{0n}(x)=\begin{cases}\bar{x}_{0n}(\varphi_i)-x & \bar{x}_{0n}(\varphi_i)-\beta_0<x<\bar{x}_{0n}(\varphi_i),\\ 0 & \bar{x}_{0n}(\varphi_i)<x,\\ \beta_0 & \bar{x}_{0n}(\varphi_i)-\beta_0>x,\end{cases}$$

$$m^{i*}_{1n}(u_0)=\begin{cases}\bar{x}_{1n}(\varphi_i)-u_0 & \bar{x}_{1n}(\varphi_i)-\beta_1<u_0<\bar{x}_{1n}(\varphi_i),\\ 0 & \bar{x}_{1n}(\varphi_i)<u_0,\\ \beta_0 & \bar{x}_{1n}(\varphi_i)-\beta_1>u_0\end{cases}$$

$$\vdots$$

$$m^{i*}_{k-1,n}(u_{k-2})=\begin{cases}\bar{x}_{k-1,n}(\varphi_i)-u_{k-2} & \bar{x}_{k-1,n}(\varphi_i)-\beta_{k-1}<u_{k-2}<\bar{x}_{k-1,n}(\varphi_i)\\ 0 & \bar{x}_{k-1,n}(\varphi_i)<u_{k-2},\\ \beta_{k-1} & \bar{x}_{k-1,n}(\varphi_i)-\beta_{k-1}>u_{k-2},\end{cases}$$

$$m^{i*}_{kn}(u_{k-1})=\begin{cases}\bar{x}_{kn}(\varphi_i)-u_{k-1} & u_{k-1}<\bar{x}_{kn}(\varphi_i),\\ 0 & u_{k-1}>\bar{x}_{kn}(\varphi_i),\end{cases}$$

$i=1,2,\cdots,\quad m,$

where $u_0=x+m^{i*}_{0n}(x)$, $u_1=u_0+m^{i*}_{1n}(u_0)$, $\cdots$, $u_{k-1}=u_{k-2}+m^{i*}_{k-1n}(u_{k-2})$, and β_ν $(\nu=0,1,\cdots,k-1)$ are positive constants, and if there is not finite $\bar{x}_{jn}(\varphi_i)$, then let $\bar{x}_{jn}(\varphi_i)$ denote $-\infty$, when $\bar{x}_{jn}(\varphi_i)$ is a closed interval if there is the smallest root, then let $\bar{x}_{jn}(\varphi_i)$ denote the smallest root, if there is not the smallest root, let $\bar{x}_{jn}(\varphi_i)$ denote $-\infty$.

Additional assumptions guarantee finiteness of $\bar{x}_{jn}(\varphi_i)$. Let us put

$$(2.30)\quad N_j(u;\varphi_i)=\begin{cases} c_{j-1}-c_j+H_j'(u;\varphi_i) & n\geqq r+1, \\ a_{j-1}c_{j-1}-a_jc_j+H_j^{*\prime}(u;\varphi_i) & n\leqq r, \end{cases}$$

$$j=1,2,\cdots,k;\ i=1,2,\cdots,m.$$

A sufficient condition for the validity of (2.31a) is given by Theorem 2.4. If this condition (2.31a) is satisfied, then we obtain the following theorem.

Theorem 2.3. Let conditions (1.1)~(1.11) hold. If

$$(2.31)\quad \begin{aligned} &(a)\ \lim_{u\to\infty} N_j(u;\varphi_i)<0,\ (b)\ \lim_{u\to\infty} L'(u;\varphi_i)-\alpha \lim_{u\to\infty} w'(u)>0, \\ &(c)\ c_0<\alpha^{r-1}v,\ (d)\ v<p'(0),\ (e)\ \varphi_i(t)>0 \textit{ for } t>0, \end{aligned}$$

$$j=1,2,\cdots,k;\ i=1,2,\cdots,m,;r\geqq 1.$$

are satisfied, (i) there exists a unique finite $\bar{x}_{jn}(\varphi_i)$ determined by

$$G_{jn}'(\bar{x}_{jn}(\varphi_i);\varphi_i)=0 \qquad j=0,1,\cdots,k-1;\ i=1,2,\cdots,m,$$

and $\bar{x}_{kn}(\varphi_i)$, where $n\leqq r$, is positive; (ii) $f_n(x;\varphi_i)$ is convex, decreasing for x small enough, increasing for x large enough.

Proof (by induction) Case I. $n\leqq r$. Then we have

$$\begin{aligned} (2.32)\quad G_{k1}'(0;\varphi_i)&=a_kc_k+H_{k+1}^{*\prime}(0;\varphi_i) \\ &=\sum_{\underline{x}\in S_k^{(1)}} q(\underline{x})(c_k+V_{x_k-1}'(0;\varphi_i)+\sum_{\nu=x_k}^{0} L_\nu'(0;\varphi_i)) \\ &=\sum_{\{\underline{x}\in S_{k,}^{(1)} x_k=0\}} q(\underline{x})(c_k+L'(0;\varphi_i))+\sum_{\{\underline{x}\in S_{k,}^{(1)} x_k\neq 0\}} q(\underline{x})(c_k-v) \\ &<\sum_{\{\underline{x}\in S_{k,}^{(1)} x_k=0\}} q(\underline{x})(v-p'(0))+\sum_{\{\underline{x}\in S_{k,}^{(1)} x_k\neq 0\}} q(\underline{x})(c_k-v)<0 \end{aligned}$$

by (2.31c, d), and

$$\lim_{u\to\infty} G_{k1}'(u;\varphi_i)>0 \qquad \text{by the theorem (2.1)}$$

From (1.4), (1.10b) and (2.31d, e), $G_{k1}'(u;\varphi_i)$ is strictly increasing for $u>0$. Hence each equation $G_{k1}'(u_k;\varphi_i)=0$ possesses a unique positive root $\bar{x}_{k1}(\varphi_i)$. Since

$$(2.33)\quad K_{k-1,1}'(u_{k-1};\varphi_i)=\begin{cases} G_{k1}'(u_{k-1};\varphi_i) & u_{k-1}>\bar{x}_{k1}(\varphi_i) \\ 0 & u_{k-1}<\bar{x}_{k1}(\varphi_i) \end{cases}$$

The $\bar{x}_{k1}>0$ implies that $K_{k-1,1}'(u;\varphi_i)=0$ for $u\leqq 0$. Hence from the conditions (2.31a) and (2.14), we have

$$\begin{aligned} (2.34)\quad \lim_{u\to-\infty} G_{k-1,1}'(u;\varphi_i)&=a_{k-1}c_{k-1}-a_kc_k+\lim_{u\to\infty} H_k'^{*}(u;\varphi_i)+\lim_{u\to-\infty} K_{k-1,1}'(u;\varphi_i) \\ &=\sum_{x\in S_{k-1}^{(1)}} q(\underline{x})(c_{k-1}+\lim_{u\to-\infty} V_{x_{k-1}-1}'(u;\varphi_i)+\sum_{\nu=x_{k-1}}^{0}\lim_{u\to\infty} L_\nu'(u;\varphi_i)) \\ &\quad+\sum_{\underline{x}\in S_k^{(1)}} q(\underline{x})(c_{k-1}-c_k)+\sum_{\{\underline{x}\in S_{k,}^{(1)} x_{k-1}\neq x_k\}} q(\underline{x})(\lim_{u\to-\infty} V_{x_{k-1}-1}'(u;\varphi_i)+\sum_{\nu=x_{k-1}}^{x_k-1}\lim_{u\to-\infty} L_\nu'(u;\varphi_i) \\ &\qquad -\lim_{u\to-\infty} V_{x_k-1}'(u;\varphi_i)) \\ &<0 \end{aligned}$$

,and if $p(\eta)$ is linear, then $G'_{k-1,1}(0;\varphi_i)<0$. Therefore by the theorem (2.1), it follows that there exists a finite $\bar{x}_{k-1,1}(\varphi_i)$ such that $G'_{k-1,1}(\bar{x}_{k-1,1}(\varphi_i);\varphi_i)=0$. Uniqueness assertion are consequence of (2.31d, e). If $p(\eta)$ is linear, then $\bar{x}_{k-1,1}(\varphi_i)$ is even positive. Assuming that part (i) of Theorem 2.3. holds for $j=k, k-1, \cdots, k-\lambda$ $(\lambda\leqq k-1)$. Since

$$(2.35)\quad K'_{k-\lambda-1,1}(u_{k-\lambda-1};\varphi_i)=\begin{cases}G'_{k-\lambda,1}(u_{k-\lambda-1}+\beta_{k-\lambda};\varphi_i) & u_{k-\lambda-1}+\beta_{k-\lambda}<\bar{x}_{k-\lambda,1}(\varphi_i)\\ 0 & u_{k-\lambda-1}<\bar{x}_{k-\lambda,1}(\varphi_i)<u_{k-\lambda-1}+\beta_{k-\lambda}\\ G'_{k-\lambda,1}(u_{k-\lambda-1};\varphi_i) & \bar{x}_{k-\lambda,1}(\varphi_i)<u_{k-\lambda-1}\end{cases}$$

,it follows that $\lim_{u\to-\infty} K'_{k-\lambda-1,1}(u;\varphi_i)<0$. Hence from (2.31a) we get

$$(2.36)\quad \lim_{u\to-\infty} G'_{k-\lambda-1,1}(u;\varphi_i)=a_{k-\lambda-1}c_{k-\lambda-1}-a_{k-\lambda}c_{k-\lambda}+\lim_{u\to-\infty} H^{*\prime}_{k-\lambda}(u;\varphi_i)+\lim_{u\to-\infty} K'_{k-\lambda-1,1}(u;\varphi_i)$$

$$<\sum_{\underline{x}\in S^{(1)}_{k-\lambda-1}} q(\underline{x})\,[c_{k-\lambda-1}+\lim_{u\to-\infty} V'_{x_{k-\lambda}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{0}\lim_{u\to-\infty} L'_\nu(u;\varphi_i)]$$

$$+\sum_{h=k-\lambda}^{k}\sum_{\underline{x}\in S^{(1)}_h} q(\underline{x})\,(c_{k-\lambda-1}-c_{k-\lambda})+\sum_{h=k-\lambda}^{k}\sum_{\{\underline{x}\in S^{(1)}_{h},\, x_{k-\lambda-1}\neq x_{k-\lambda}\}} q(\underline{x})\ [\lim_{u\to-\infty} V'_{x_{k-\lambda}-1}(u;\varphi_i)$$

$$+\sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to\infty} V'_{x_{k-\lambda}-1}(u;\varphi_i)]$$

$$<0$$

On the other hand, $\lim_{u\to\infty} G'_{k-\lambda-1,1}(u;\varphi_i)>0$. There exists a finite $\bar{x}_{k-\lambda-1,1}(\varphi_i)$ such that $G'_{k-\lambda-1,1}(\bar{x}_{k-\lambda-1,1}(\varphi_i);\varphi_i)=0$,where $i=1,2,\cdots,m$. Uniqueness follows from (1.4) and (2.31d, e). If $p(\eta)$ is linear, then $\bar{x}_{k-\lambda-1,1}(\varphi_i)$ is even positive. The expression

$$(2.37)\quad f'_1(x;\varphi_i)=\begin{cases}G'_{01}(x+\beta_0;\varphi_i)-a_0c_0+H^{*\prime}_0(x;\varphi_i) & x+\beta_0<\bar{x}_{01}(\varphi_i)\\ -a_0c_0+H^{*\prime}_0(x;\varphi_i) & x<\bar{x}_{01}(\varphi_i)<x+\beta_0\\ G'_{01}(x;\varphi_i)-a_0c_0+H^{*\prime}_0(x;\varphi_i) & \bar{x}_{01}(\varphi_i)<x,\end{cases}$$

used with Lemma B and (2.31b), verifies part (ii). Assume that Theorem holds for the integer $n-1$. Since

$$(2.38)\quad G'_{kn}(0;\varphi_i)=a_kc_k+H^{*\prime}_{k+1}(0;\varphi_i)$$

$$=\sum_{\underline{x}\in S^{(n)}_k} q(\underline{x})\,(c_k+V'_{x_k-1}(0;\varphi_i)+\sum_{\nu=x_k}^{x-1} L'_\nu(0;\varphi_i))$$

$$=\sum_{\{\underline{x}\in S^{(n)}_{k},\, x_k=0\}} q(\underline{x})\,(c_k+\sum_{\nu=0}^{n-1} L'_\nu(0;\varphi_i))+\sum_{\{\underline{x}\in S^{(n)}_{k},\, x_k\neq 0\}} q(\underline{x})\,(c_k-\alpha^{x_k-1}v-\sum_{\nu=x_k}^{x-1} L'_\nu(0;\varphi_i))$$

$$<\sum_{\{\underline{x}\in S^{(n)}_{k},\, x_k=0\}} q(\underline{x})\,(v-\sum_{\nu=0}^{n-1}\alpha^\nu p'(0))+\sum_{\{\underline{x}\in S^{(n)}_{k},\, x_k\neq 0\}} q(\underline{x})\,(c_k-\alpha^{x_k-1}v-\sum_{\nu=x_k}^{n-1}\alpha^\nu p'(0))<0$$

by (2.31c, d), and

$$\lim_{u\to\infty} G'_{kn}(u;\varphi_i)>0$$

,it follows that there exists $\bar{x}_{kn}(\varphi_i)>0$ such that $G'_{kn}(\bar{x}_{kn}(\varphi_i);\varphi_i)=0$. Uniqueness follows from (1.4), (1.10b) and (2.31d, e). Assume that part (i) of theorem hold for $j=k$, k

$-1, \cdots, k-\lambda (\lambda \leqq k-1)$. Then we get from the inductive assumption and (2.31a)

$$(2.39)\quad \lim_{u\to-\infty} G'_{k-\lambda-1,n}(u;\varphi_i)=a_{k-\lambda-1}c_{k-\lambda-1}-a_{k-\lambda}c_{k-\lambda}+\lim_{u\to-\infty} H^{*\prime}_{k-\lambda}(u;\varphi_i)+\lim_{u\to-\infty} K'_{k-\lambda-1,n}(u;\varphi_i)$$

$$<\sum_{\underline{x}\in S^{(n)}_{k-\lambda-1}} q(\underline{x})\,(c_{k-\lambda-1}+\lim_{u\to-\infty} V'_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\nu=x_{k-\lambda-1}}^{n-1}\lim_{u\to-\infty} L'_\nu(u;\varphi_i))$$

$$+\sum_{h=k-\lambda}^{k}\sum_{\underline{x}\in S^{(n)}_h} q(\underline{x})\,(c_{k-\lambda-1}-c_{k-\lambda})+\sum_{h=k-\lambda}^{k}\sum_{\{\underline{x}\in S^{(n)}_h,\,x_{k-\lambda-1}\neq x_{k-\lambda}\}} q(\underline{x})\quad[\lim_{u\to-\infty} V'_{x_{k-\lambda-1}-1}(u;\varphi_i))$$

$$+\sum_{\nu=x_{k-\lambda-1}}^{x_{k-\lambda}-1}\lim_{u\to\infty} L'_\nu(u;\varphi_i)-\lim_{u\to-\infty} V'_{x_{k-\lambda}-1}(u;\varphi_i)]$$

$$<0$$

,it follows by the theorem (2.1), (1.4) and (2.31d, e) that there exists a unique finite $\bar{x}_{k-\lambda-1,n}(\varphi_i)$ such that $G'_{k-\lambda-1,n,}(\bar{x}_{k-\lambda-1,n}(\varphi_i);\varphi_i)=0$. If $p(\eta)$ is linear, then $\bar{x}_{k-\lambda-1,n}(\varphi_i)$ is even positive. Since

$$(2.41)\quad f'_n(x;\varphi_i)=\begin{cases} G'_{0n}(x+\beta_0;\varphi_i)-a_0c_0+H^{*\prime}_0(x;\varphi_i) & x+\beta_0<\bar{x}_{0n}(\varphi_i)\\ -a_0c_0+H^{*\prime}_0(x;\varphi_i) & x<\bar{x}_{0n}(\varphi_i)<x+\beta_0\\ G'_{0n}(x;\varphi_i)-a_0c_0+H^{*\prime}_0(x;\varphi_i) & \bar{x}_{0n}(\varphi_i)<x,\end{cases}$$

it follows from the Lemma B and (2.31b) that part (ii) holds.

Case II. $n\geqq r+1$. Then using the fact that $\lim_{u\to-\infty} f'_1(u;\varphi_i)<0$, we have from (2.31c, d)

$$(2.41)\quad \lim_{u\to-\infty} G'_{k,r+1}(u;\varphi_i)=c_k+\lim_{u\to-\infty} H'_{k+1}(u;\varphi_i)+\lim_{u\to-\infty} f'_{1r}(u;\varphi_i)$$

$$<\sum_{\{\underline{x}\in S,\,x_k=0\}} q(\underline{x})\,(c_k+\sum_{\lambda=0}^{r-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i))+\sum_{\{\underline{x}\in S,\,x_k\neq 0\}} q(\underline{x})\,(c_k-\alpha^{x_k-1}v+\sum_{\lambda=x_k}^{r-1}\lim L'_\lambda(u;\varphi_i))$$

$$<\sum_{\{\underline{x}\in S,\,x_k=0\}} q(\underline{x})\,(v-\sum_{\lambda=0}^{r-1}\alpha^\lambda p'(0))+\sum_{\{\underline{x}\in S,\,x_k\neq 0\}} q(\underline{x})\,(c_k-\alpha^{x_k-1}v-\sum_{\lambda=x_k}^{r-1}\alpha^\lambda p'(0))$$

$$<0$$

If $p(\eta)$ is linear, then $f'_1(0;\varphi_i)<0$, since $H'_0(0;\varphi_i)<0$ and $\bar{x}_{01}(\varphi_i)>0$; hence $G'_{k,r+1}(0;\varphi_i)$ <0. On the other hand, $\lim_{u\to\infty} G'_{k,r+1}(u;\varphi_i)>0$. Hence there exists a finite $\bar{x}_{k,r+1}(\varphi_i)$ such that $G'_{k,r+1}(\bar{x}_{k,r+1}(\varphi_i);\varphi_i)=0$. Uniqueness follows from (2.31d, e). If $p(\eta)$ is linear, $\bar{x}_{k,r+1}(\varphi_i)$ is even positive. Assume that part (i) of the theorem is true for the integer $j=k,\ k-1,\ \cdots,\ k-\lambda(\lambda\leqq k-1)$, we seek to advance the induction to the integer $k-\lambda-1$. Then from the inductive assumption, we have $\lim_{u\to-\infty} K'_{k-\lambda-1,r+1}(u;\varphi_i)<0$. Hence from (2.31a) we get

$$(2.42)\quad \lim_{u\to-\infty} G'_{k-\lambda-1,r+1}(u;\varphi_i)=c_{k-\lambda-1}-c_{k-\lambda}+\lim_{u\to-\infty} H'_{k-\lambda}(u;\varphi_i)+\lim_{u\to-\infty} K'_{k-\lambda-1,r+1}(u;\varphi_i)$$

$$<c_{k-\lambda-1}-c_{k-\lambda}+\sum_{\{\underline{x}\in S,\,x_{k-\lambda-1}\neq x_{k-\lambda}\}} q(x)\,[\lim_{u\to-\infty} V'_{x_{k-\lambda-1}-1}(u;\varphi_i)+\sum_{\lambda=x_{k-\lambda-1}}^{x_{k-\lambda}-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)$$

$$-\lim_{u\to-\infty} V'_{x_{k-\lambda}-1}(u;\varphi_i)]$$

$$<0$$

,and if $p(\eta)$ is linear, then $G'_{k-\lambda-1,r+1}(0;\varphi_i)<0$. On the other hand, $\lim_{u\to\infty} G'_{k-\lambda-1,r+1}(u;\varphi_i)>0$. It follows from above results that $\bar{x}_{k-\lambda-1,r+1}(\varphi_i)$ is finite. Uniqueness is shown as before. If $p(\eta)$ is linear, then $\bar{x}_{k-\lambda-1,r+1}(\varphi_i)$ is even positive. Since

$$(2.43)\quad f'_{r+1}(x;\varphi_i)=\begin{cases} G'_{0,r+1}(x+\beta_0;\varphi_i)-c_0+H'_0(x;\varphi_i) & x+\beta_0<\bar{x}_{0,r+1}(\varphi_i),\\ -c_0+H'_0(x;\varphi_i) & x<\bar{x}_{0,r+1}(\varphi_i)<x+\beta_0,\\ G'_{0,r+1}(x;\varphi_i)-c_0+H'_0(x;\varphi_i) & \bar{x}_{0,r+1}(\varphi_i)<x,\end{cases}$$

it follows from the Lemma B and (2.31b) that part (ii) holds.

We assume that the theorem holds for the integer $n-1(n-1\leqq r+1)$. Then using the fact that $c_k+\lim_{u\to-\infty} H'_{k+1}(u;\varphi_i)<0$ and $\lim_{u\to-\infty} f'_{n-r}(u;\varphi_i)<0$, we find that

$$(2.44)\quad \lim_{u\to-\infty} G'_{kn}(u;\varphi_i)=c_k+\lim_{u\to-\infty} H'_{k+1}(u;\varphi_i)+\lim_{u\to-\infty} f'_{n-r,r}(u;\varphi_i)<0.$$

If $p(\eta)$ is linear, then $f'_{n-r}(0;\varphi_i)<0$, since $H'_0(0;\varphi_i)\leqq 0$ and $\bar{x}_{0,n-r}(\varphi_i)>0$; hence $G'_{kn}(0;\varphi_i)<0$. On the other hand, $\lim_{u\to\infty} G'_{kn}(u;\varphi_i)>0$. Hence there exists a finite $\bar{x}_{kn}(\varphi_i)$ such that $G'_{kn}(\bar{x}_{kn}(\varphi_i);\varphi_i)=0$. Uniqueness follows from (2.31d, e). If $p(\eta)$ is linear, $\bar{x}_{kn}(\varphi_i)$ is even positive. Assume that part (i) of the theorem is true for the integer $j=k, k-1, \cdots, k-\lambda(\lambda\leqq k-1)$, we seek to advance the induction to the integer $k-\lambda-1$. It follows from the inductive assumption and (2.31a)

$$(2.45)\quad \lim_{u\to-\infty} G'_{k-\lambda-1,n}(u;\varphi_i)=c_{k-\lambda-1}-c_{k-\lambda}+\lim_{u\to-\infty} H'_{k-\lambda}(u;\varphi_i)+\lim_{u\to-\infty} K'_{k-\lambda-1,n}(u;\varphi_i)$$
$$<c_{k-\lambda-1}-c_{k-\lambda}+\lim_{u\to-\infty} H'_{k-\lambda}(u;\varphi_i)<0$$

Since $\lim_{u\to\infty} G'_{k-\lambda-1,n}(u;\varphi_i)>0$, and condition (2.31d, e) hold, there exists a unique finite $\bar{x}_{k-\lambda-1,n}(\varphi_i)$ such that $G'_{k-\lambda-1,n}(\bar{x}_{k-\lambda-1,n}(\varphi_i);\varphi_i)=0$. If $p(\eta)$ is linear, then $\bar{x}_{k-\lambda-1,n}(\varphi_i)$ is even positive. Part (ii) is shown as case $n=r+1$. The proof is complete.

Remark IV. $a_kc_k+\lim_{u\to-\infty} H^{*\prime}_{k+1}(u;\varphi_i)<0\ (a_k\neq 0)$ and $c_k+\lim_{u\to-\infty} H'_{k+1}(u;\varphi_i)<0$ are shown from (2.31c, d). Hence the finiteness of $\bar{x}_{kn}(\varphi_i)$ is shown without condition (2.31a).

Let us put

$$(2.46)\quad Q_j(\underline{x})=\begin{cases} \sum_{\{\underline{x}\in S,\ x_{j-1}\neq x_j\}} q(\underline{x}) & n\geqq r+1,\\ \sum_{\{\underline{x}\in S^{(n)}_{j-1},\ x_{j-1}\neq n\}} q(\underline{x}) + \sum_{h=j}^{k} \sum_{\{\underline{x}\in S^{(n)}_h,\ x_{j-1}\neq x_j\}} q(\underline{x}) & ,\ n\leqq r,\end{cases}$$

$$j=1, 2, \cdots, k.$$

THEOREM 2.4. *Let conditions* (1.1)~(1.11) *be valid. If*

$$(2.47)\quad \begin{array}{ll} \text{(a)}\ \max_{1\leqq j\leqq k}(c_{j-1}-c_j)+q[\lim_{u\to-\infty} L'_{r-1}(u;\varphi_i)+\alpha^{r-1}v]<0, & \text{(b)}\ c_0<\alpha^{r-1}v,\\ \text{(c)}\ Q_j(\underline{x})\neq 0, & j=1, 2, \cdots, k;\ i=1, 2, \cdots, m, r\geqq 1.\end{array}$$

,then condition (2.31a) *is satisfied. where* $q=\min\{q(\underline{x})\,|\,q(\underline{x})>0, \underline{x}\in S\}$

Proof. The case when $L'(u;\varphi_i)$ tends to negative infinity as u tends to negative infinity, it is easily seen from (2.12), (2.14) and (2.47c) that the theorem is true. Assume that $L'(u;\varphi_i)$ tends to a negative value as u tends to negative infinity. Then we have

$$(2.48)\quad c_{j-1}-c_j+\lim_{u\to-\infty} H_j'(u;\varphi_i)$$

$$=c_{j-1}-c_j+\sum_{\{\underline{x}\in S,\, x_{j-1}\neq x_j\}} q(\underline{x})\,[\lim_{u\to-\infty} V'_{x_{j-1}-1}(u;\varphi_i)+\sum_{\lambda=x_{j-1}}^{x_j-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)-\lim_{u\to-\infty} V'_{x_{j-1}-1}(u;\varphi_i)]$$

$$=c_{j-1}-c_j+\sum_{\{\underline{x}\in S,\, x_{j-1}\neq x_j,\, x_{j-1}=0\}} q(\underline{x})\,[\sum_{\lambda=0}^{x_j-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)+\alpha^{x_j-1}v]$$

$$+\sum_{\{\underline{x}\ni S,\, x_{j-1}\neq x_j,\, x_{j-1}\neq 0\}} q(\underline{x})\,[-\alpha^{x_{j-1}-1}v+\sum_{\lambda=x_{j-1}}^{x_j-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)+\alpha^{x_j-1}v]$$

$$\leqq c_{j-1}-c_j+\sum_{\{\underline{x}\in S,\, x_{j-1}\neq x_j,\, x_{j-1}=0\}} q(\underline{x})\,[\lim_{u\to-\infty} L'_r(u;\varphi_i)+\alpha^{r-1}v]$$

$$+\sum_{\{\underline{x}\in S,\, x_{j-1}\neq x_j,\, x_{j-1}\neq 0\}} q(\underline{x})\,[\lim_{u\to-\infty} L'_r(u;\varphi_i)+\alpha^{r-1}v]$$

$$\leqq \max_{1\leq j\leq k}(c_{j-1}-c_j)+q[\lim_{u\to-\infty} L'_r(u;\varphi_i)+\alpha^{r-1}v]<0$$

,and

$$(2.47)\quad a_{j-1}c_{j-1}-a_jc_j+\lim_{u\to-\infty} H_j^{*\prime}(u;\varphi_i)$$

$$=\sum_{\underline{x}\in S_{j-1}^{(n)}} q(\underline{x})\,[c_{j-1}+\lim_{u\to-\infty} V'_{x_{j-1}-1}(u;\varphi_i)+\sum_{\lambda=x_{j-1}}^{n-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)]$$

$$+\sum_{h=j}^{k}\sum_{\underline{x}\ni S_h^{(n)}} q(\underline{x})\,(c_{j-1}-c_j)+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\, x_{j-1}\neq x_j\}} q(\underline{x})\,[\lim_{u\to-\infty} V'_{x_{j-1}-1}(u;\varphi_i)$$

$$+\sum_{\lambda=x_{j-1}}^{x_j-1} L'_\lambda(u;\varphi_i)-\lim_{u\to-\infty} V'_{x_j-1}(u;\varphi_i)]$$

$$=\sum_{\{\underline{x}\in S_{j-1}^{(n)},\, x_{j-1}\neq n\}} q(\underline{x})\,[c_{j-1}+\lim_{u\to-\infty} V_{x_{j-1}-1}(u;\varphi_i)+\sum_{\lambda=x_{j-1}}^{n-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)]$$

$$+\sum_{\{\underline{x}\in S_{j-1}^{(n)},\, x_{j-1}=n\}} q(\underline{x})\,[c_{j-1}-\alpha^{n-1}v)+\sum_{h=j}\sum_{x\in S_h^{(n)}} q(\underline{x})\,(c_{j-1}-c_j)$$

$$+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\, x_{j-1}\neq x_j,\, x_{j-1}=0\}} q(\underline{x})\,(\sum_{\lambda=0}^{x_j-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)+\alpha^{x_j-1}v)$$

$$+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\, x_{j-1}\neq x_j,\, x_{j-1}\neq 0\}} q(\underline{x})\,[-\alpha^{x_{j-1}-1}v+\sum_{\lambda=x_{j-1}}^{x_j-1}\lim_{u\to-\infty} L'_\lambda(u;\varphi_i)+\alpha^{x_j-1}v]$$

$$\leqq\sum_{\{\underline{x}\in S_{j-1}^{(n)},\, x_{j-1}\neq n\}} q(\underline{x})\,[\alpha^{r-1}v+\lim_{u\to-\infty} L'_{r-1}(u;\varphi_i)]+\max_{1\leq j\leq k}(c_{j-1}-c_j)$$

$$+\sum_{h=j}^{k}\sum_{\{\underline{x}\in S_h^{(n)},\, x_{j-1}\neq x_j\}} q(\underline{x})\,[\lim_{u\to-\infty} L'_{r-1}(u;\varphi_i)+\alpha^{r-1}v].$$

Three subcase possible. (a) $\sum_{\{\underline{x}\in S_{j-1,}^{(n)} x_{j-1}\neq n\}} q(\underline{x})=0$, $\sum_{h=j} \sum_{\{\underline{x}\in S_{h}^{(n)}, x_{j-1}\neq x_j\}} q(\underline{x})\neq 0$; (b) $\sum_{\{\underline{x}\in S_{j-1,}^{(n)} x_{j-1}\neq n\}} q(\underline{x})\neq 0$, $\sum_{h=j} \sum_{\{\underline{x}\in S_{h}^{(n)}, x_{j-1}\neq x_j\}} q(\underline{x})=0$; (c) $\sum_{\{\underline{x}\in S_{j-1,}^{(n)} x_{j-1}\neq n\}} q(\underline{x})\neq 0$, $\sum_{h=j} \sum_{\{\underline{x}\in S_{h}^{(n)}, x_{j-1}\neq x_j\}} q(\underline{x})\neq 0$.

Then from (2.48) and (2.47a), we have $a_{j-1}c_{j-1}-a_jc_j+\lim_{u\to-\infty} H_j^{*\prime}(u;\varphi_i)<0$.

§3. Optimal Policy for the Model with Set -up Costs.

In this section we analyze the dynamic model for the important practical case in which the ordering costs are given by

$$(1.7)' \quad c_j(\eta)=\begin{cases} c_j\eta+K_j(\eta) & \eta>0,\\ 0 & \eta\leqq 0,\end{cases} \qquad K_j(\eta)=\begin{cases} K_j & \eta>0,\\ 0 & \eta\leqq 0,\end{cases}$$

with $c_0>c_1>\cdots>c_k>0$ and $K_0\geqq K_1\geqq\cdots\geqq K_k>0$. $j=0,1,\cdots,k$.

By the way similar to formulation of §2, we have

$$\begin{aligned}(3.1)\quad f_n(x;\varphi_i)=&\inf_{u_0\geqq x}\{[c_0(u_0-x)+K_0(u_0-x)+H_1(u_1(u_0;\varphi_i)]+\inf_{u_1\geqq u_0}\{[c_1(u_1-u_0)+K_1(u_1-u_0)\\ &+H_2(u_1;\varphi_i)]+\inf_{u_2\geqq u_1}\{[c_2(u_2-u_1)+K_2(u_2-u_1)+H_3(u_2;\varphi_i)]+\cdots\\ &+\inf_{u_k\geqq u_{k-1}}\{[c_k(u_k-u_{k-1})+K_k(u_k-u_{k-1})+H_{k+1}(u_k;\varphi_i)]+f_{n-r,r}(u_k;\varphi_i)\}\cdots\}+H_0(x;\varphi_i)\\ =&\inf_{u_0\geqq x}\{-c_0x+K_0(u_0-x)+G_{0n}(u_0;\varphi_i)\}+H_0(x;\varphi_i) \qquad n\geqq r+1\end{aligned}$$

$$\begin{aligned}(3.2)\quad f_n(x;\varphi_i)=&\inf_{u_0\geqq x}\{a_0[c_0(u_0-x)+K_0(u_0-x)]+H_1^*(u_0;\varphi_i)\inf_{u_1\geqq u_0}\{a_1[c_1(u_1-u_0)+K_1(u_1-u_0)]\\ &+H_2^*(u_1;\varphi_i)+\inf_{u_2\geqq u_1}\{a_2[c_2(u_2-u_1)+K_2(u_2-u_1]+H_3^*(u_2;\varphi_i)+\cdots\\ &+\inf_{u_k\geqq u_{k-1}}\{a_k[c_k(u_k-u_{k-1})+K_k(u_k-u_{k-1})]+H_{k+1}^*(u_k;\varphi_i)\}\cdots\}+H_0^*(x;\varphi_i)\\ =&\inf_{u_0\geqq x}\{a_0[-c_0x+K_0(u_0-x)]+G_{0n}(u_0;\varphi_i)\}+H_0^*(x;\varphi_i)\\ & \qquad n\leqq r\end{aligned}$$

where $f_{n+l}(\cdot;\varphi_i)$, $W_l(\cdot;\varphi_i)$, $H_j(u_{j-1};\varphi_i)$, $H_j^*(u_{j-1};\varphi_i)$ and a_j are given by (2.2), (2.12) and (2.14)

,and

$$G_{kn}(u_k;\varphi_i)=\begin{cases} a_kc_ku_k+H_{k+1}^*(u_k;\varphi_i) & n\leqq r\\ & i=1,2,\cdots,m\\ c_ku_k+H_{k+1}(u_k;\varphi_i)+f_{n-k,k}(u_k;\varphi_i) & n\geqq r+1\end{cases}$$

$$(3.3)\quad G_{jn}(u_j;\varphi_i)=\begin{cases} a_jc_ju_j+H_{j+1}^*(u_j;\varphi_i)+g_{jn}(u_j;\varphi_i) & n\leqq r\\ & j=0,1,\cdots,k-1; i=1,2,\cdots,m\\ c_ju_j+H_{j+1}(u_j;\varphi_i)+g_{jn}(u_j;\varphi_i) & n\geqq r+1\end{cases}$$

$$g_{jn}(u_j;\varphi_i)=\begin{cases} \inf_{u_{j+1}\geqq u_j}\{a_{j+1}[-c_{j+1}u_j+K_{j+1}(u_{j+1}-u_j)]+G_{j+1,n}(u_{j+1};\varphi_i)\} & n\leqq r\\ & j=0,1,\cdots,k-1;\ i=1,2,\cdots,m\\ \inf_{u_{j+1}\geqq u_j}\{-c_{j+1}u_j+K_{j+1}(u_{j+1}-u_j)+G_{j+1,n}(u_{j+1};\varphi_i)\} & n\geqq r+1\end{cases}$$

THEOREM 3.1. *If conditions of Theorem* 2.3 *with* (1.7)′ *in place of* (1.7) *are satisfied except* (2.31d, e), *then* (i) *there exists an unique pair** $(S_{jn}(\varphi_i), s_{jn}(\varphi_i))$ *such that* $S_{jn}(\varphi_i) > s_{jn}(\varphi_i)$, $G_{jn}(S_{jn}(\varphi_i); \varphi_i)$ *is the minimu m value of* $G_{jn}(x;\varphi_i)$, *and* $G_{jn}(s_{jn}(\varphi_i); \varphi_i) = G_{jn}(S_{jn}(\varphi_i); \varphi_i) + K_j$

$$j=0,1,\cdots,k;\ i=1,2,\cdots,m;$$

(ii) $f_n(x;\varphi_i)$ *is* K_0*-convex, decreasing for x small enough, increasing for x large enough.*

THEOREM 3.2. *Under conditions of Theorem* 3.1 *the optimal order* m_{jn}^{i*} *in the first period are of the following form*

$$m_{0n}^{i*}(x) = \begin{cases} S_{0n}(\varphi_i) - x & x < s_{0n}(\varphi_i), \\ 0 & x \geqq s_{0n}(\varphi_i), \end{cases}$$

$$m_{1n}^{i*}(u_0) = \begin{cases} S_{1n}(\varphi_i) - u_0 & u_0 < s_{1n}(\varphi_i), \\ 0 & u_0 \geqq s_{1n}(\varphi_i), \end{cases}$$

$$\vdots$$

$$m_{kn}^{i*}(u_{k-1}) = \begin{cases} S_{kn}(\varphi_i) - u_{k-1} & u_{k-1} < s_{kn}(\varphi_i) \\ 0 & u_{k-1} \geqq s_{kn}(\varphi_i) \end{cases}$$

where $u_0 = x + m_{0n}^{i*}(x),\ u_1 = u_0 + m_{1n}^{i*}(u_0),\ \cdots,\ u_{k-1} = u_{k-2} + m_{k-1,n}^{i*}(u_{k-2})$.

Theorems 2.1 and 2.2 in author's paper [8] stated for the inventory problem with constant delivery-lag are the special case of the above theorems. The proof of Theorems 3.1 and 3.2 is possible to proceed along the same lines as way in Theorems 2.1 and 2.2 of [8], using a part of results in Theorems 2.1 and 2.3 in this paper, and the proof of above theorems is omitted.

§4. Optimal Policy for Convex Ordering Costs.

By the same method as in §2, we obtain the following mathematical formulation for the model with the general ordering costs $c_j(\eta)$ $(0 \leqq \eta)$.

$$(4.1)\quad f_n(x;\varphi_i) = \inf_{u_0 \geqq x} \{[c_0(u_0 - x) + H_1(u_0;\varphi_i)] + \inf_{u_1 \geqq u_0} \{[c_1(u_1 - u_0) + H_2(u_1;\varphi_i)] +$$

$$\cdots\cdots + \inf_{u_k \geqq u_{k-1}} \{[c_k(u_k - u_{k-1}) + H_{k+1}(u_k;\varphi_i)] + f_{n-r,r}(u_k;\varphi_i)]\}\cdots\} + H_0(x;\varphi_i)$$

$$= \inf_{u_0 \geqq x} \{c_0(u_0 - x) + G_{0n}(u_0;\varphi_i)\} + H_0(x;\varphi_i)$$

$$= \inf_{u_0 \geqq x} M_{0n}(u_0, x;\varphi_i) + H_0(x;\varphi_i)$$

$$n \geqq r+1$$

$$(4.2)\quad f_n(x;\varphi_i) = \inf_{u_0 \geqq x} \{[a_0 c_0(u_0 - x) + H_1^*(u_0;\varphi_i)] + \inf_{u_1 \geqq u_0} \{[a_1 c_1(u_1 - u_0) + H_2^*(u_1;\varphi_i)] +$$

$$\cdots + \inf_{u_k \geqq u_{k-1}} \{[a_k c_k(u_k - u_{k-1}) + H_{k+1}^*(u_k;\varphi_i)]\cdots\} + H_0^*(x;\varphi_i)$$

$$= \inf_{u_0 \geqq x} \{c_0 a_0(u_0 - x) + G_{0n}(u_0;\varphi_i)\} + H_0^*(x;\varphi_i)$$

---

*In this section, by the term unique pair $(S_{jn}(\varphi_i), s_{jn}(\varphi_i))$ we mean the pair of the smallest value of $S_{jn}(\varphi_i)$ and $s_{jn}(\varphi_i)$.

$$= \inf_{u_0 \geqq x} M_{0n}(u_0, x; \varphi_i) + H_0^*(x; \varphi_i)$$

$$n \leqq r$$

where $f_{n-r,i}(\cdot; \varphi_i), W_i(\cdot; \varphi_i), H_j(u_{j-1}; \varphi_i), H_j^*(u_{j-1}; \varphi_i)$ and a_j are given by (2.2), (2.12) and (2.14), and

$$G_{kn}(u_k; \varphi_i) = \begin{cases} H_{k+1}^*(u_k; \varphi_i) & n \leqq r \\ & i=1, 2, \cdots, m, \\ H_{k+i}(u_k; \varphi_1) + f_{n-r,r}(u_k; \varphi_i) & n \geqq r+1 \end{cases}$$

$$G_{jn}(u_j; \varphi_i) = \begin{cases} H_{j+1}^*(u_j; \varphi_i) + K_{jn}(u_j; \varphi_i) & n \leqq r \\ & j=0, 1, \cdots, k-1;\ i=1, 2, \cdots, m \\ H_{j+1}(u_j; \varphi_i) + K_{jn}(u_j; \varphi_i) & n \geqq r+1 \end{cases}$$

$$K_{jn}(u_j; \varphi_i) = \inf_{u_{j+1} \geqq u_j} M_{j+1,n}(u_{j+1}, u_j; \varphi_i) \qquad j=0, 1, \cdots, k-1;\ i=1, 2, \cdots, m,$$

$$M_{j+1,n}(u_{j+1}, u_j; \varphi_i) = \begin{cases} a_{j+1}c_{j+1}(u_{j+1}-u_j) + G_{j+1,n}(u_{j+1}; \varphi_i) & n \leqq r \\ & j=-1, 0, 1, \cdots, k-1;\ i=1, 2, \cdots, m, \\ c_{j+1}(u_{j+1}-u_j) + G_{j+1,n}(u_{j+1}; \varphi_i) & n \geqq r+1 \end{cases}$$

Let us put

$$M_{j+1,n}^*(u_{j+1}; \varphi_i) = \begin{cases} a_{j+1}c'_{j+1}(0) + G'_{j+1,n}(u_{j+1}; \varphi_i) & n \leqq r, \\ & j=-1, 0, 1, \cdots, k-1;\ i=1, 2 \cdots, m, \\ c'_{j+1}(0) + G'_{j+1,n}(u_{j+1}; \varphi_i) & n \geqq r+1. \end{cases}$$

We impose the following conditions corresponding to (1,7) and (1.10a).

(1.7)″ There are strict convex and twice differentiable ordering cost functions $c_j(\eta)$ defined on $[0, \infty]$ with $c'(0) > c_1'(0) > \cdots > c_k'(0) > 0$.

(1.10a)″ $\alpha \lim_{\eta\to\infty} w'(\eta) = \alpha \lim_{\eta\to\infty} \int_0^\eta w'(\eta - t)\varphi_i(t)\,dt < c_k'(0)$, and $\varphi_i(t)$ is continuous.

The proof of Theorems 4.1 and 4.2 is possible to proceed along the same line as way in Theorems 2.1 and 2.2 of author's paper [9], using a part of results in Theorems 2.1 and 2.3, and the proof of Theorems 4.1 and 4.2 is omitted.

THEOREM 4.1 *If conditions of Theorem* 2.3 *with* (1.7)″ *and* (1.10a)″ *in place of* (1.7) *and* (1.10a) *are satisfied, then* (i) *there exists a function* $u_{jn}(u_{j-1}; \varphi_i)$ *determined by each equation*

$$\frac{\partial M_{jn}}{\partial u_j}(u_j, u_{j-1}; \varphi_i) = 0 \ \ for\ \ u_{j-1} < \bar{x}_{jn}(\varphi_i),\ j=0, 1, \cdots, k;\ i=1, 2, \cdots, m;$$

where $\bar{x}_{jn}(\varphi_i)$ *is unique finite root of*

$$M_{jn}^*(u_j; \varphi_i) = 0 \qquad j=0, 1, \cdots, k;\ i=1, 2, \cdots, m;$$

(ii) $f_n(x; \varphi_i)$ *is convex, decreasing for x small enough and increasing for x large enough.*

THEOREM 4.2. *Under conditions of Theorem* 4.1 *the optimal order* m_{jn}^{i*} *in the first period is given by*

$$m_{0n}^{i*}(x) = \begin{cases} u_{0n}(x;\varphi_i)-x & x<\bar{x}_{0n}(\varphi_i) \\ 0 & x>\bar{x}_{0n}(\varphi_i) \end{cases}$$

$$m_{1n}^{i*}(u_0) = \begin{cases} u_{1n}(u_0;\varphi_i)-u_0 & u_0<\bar{x}_{1n}(\varphi_i) \\ 0 & u_0>\bar{x}_{1n}(\varphi_i) \end{cases}$$

$$m_{kn}^{i*}(u_{k-1}) = \begin{cases} u_{kn}(u_{k-1};\varphi_i)-u_{k-1} & u_{k-1}<\bar{x}_{kn}(\varphi_i) \\ 0 & u_{k-1}>\bar{x}_{kn}(\varphi_i) \end{cases}$$

$$i=1,2,\cdots,m,$$

where $u_0=x+m_{0n}^{i*}(x)$, $u_1=u_0+m_{1n}^{i*}(u_0)$, $\cdots$, $u_{k-1}=u_{k-2}+m_{k-1,n}^{i*}(u_{k-2})$.

§5. Remarks.

In this section we shall give further remarks concerning our model and the sufficient conditions for (2.31a) for some special cases.

Case I. If $r=k=1$, then $S=\{(01),(01),(11)\}$. The condition (2.31a) is satisfied if $c_0-c_1+q(\lim_{u\to-\infty} L(u;\varphi_i)+v)<0$ and $c_0<v$. Specially when $q(01)=1$, if $c_0-c_1+\lim_{u\to-\infty} L(u;\varphi_i)+v<0$, then (2.31a) holds, and in this case we can prove Theorem 2.3 without the condition (2.31c) and moreover if $\varphi_i=\varphi$, then the model discussed in this paper becomes to the one in [2].

Case II. If $r=k$, $q(01\cdots k)=1$, then the model discussed in this paper becomes to the one in [7], [8] and [9]. In this case (2.31a) is satisfied under less condition than (2.47), that is, if ordering cost is linear, or each ordering cost is composed of a unit cost plus a reorder cost, then $c_0-c_1+\lim_{u\to-\infty} L'(u;\varphi_i)+v<0$ and $c_j<\alpha^{j-1}v$, where $i=1,2,\cdots,m$; $j=1,2,\cdots,k$. Theorem 2.3 and Theorem 3.1 are valid without the condition (2.31c).

Case III. When $r=k$, $q(01\cdots k)=1$ and ordering cost is strict convex and strict increasing, we have discussed in [9]. In this case (2.31a) is satisfied under less condition than (2.47), that is, the conditions in case II with $c_j(0)$ in place of c_j.

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