Kumamoto J. Sci., Ser. A, Vol. 8, No. 3, pp. 107–114 (1968)

ON WEAK REPRESENTATIONS OF LIE TRIPLE SYSTEMS

By

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(Received May 31, 1968)

A Lie triple system is a tangent algebra of the symmetric space. In [4], Lister defined a representation of Lie triple system T as a triple homomorphism of T into a Lie triple system of linear transformations. In [5], we defined a representation according the method of Eilenberg. In this paper, we shall introduce a more generalized notion of representation, which will be called a weak representation. This corresponds to the weak representation of Malcev algebra in [6].

The cohomology theory of Lie triple systems was studied in [2] and [5]. In §2, we consider the cohomology groups associated with a weak representation, following the method of Chevalley and Eilenberg [1]. In §3, it is shown that there exists a one-to-one correspondence between the equivalence classes of extensions of a weak representation module V by a weak representation module W and the elements of two dimensional cohomology group $H^2(T, \mathfrak{L}(V, W))$, according the method of Hochschild in [3]. This result corresponds to the fact that the one dimensional cohomology groups $H^1(L, V)$ associated with a representation (ρ, V) of Lie algebra L is explained in relation with extensions of representation modules.

Prof. W. G. Lister obtained the interpretation of the second cohomology group as module extension classes independently, in which he called our weak T-module a pseudo T-module.

Throughout this paper, we assume that a characteristic of the base field is zero and the vector spaces are finite dimensional.

1. Weak representations. A *Lie triple system* (simply *L. t. s.*) is a vector space T over a field Φ with a trilinear composition $[xyz]$ satisfying the following relations:

(1.1) $\qquad [xxy]=0,$

(1.2) $\qquad [xyz]+[yzx]+[zxy]=0,$

(1.3) $[xy[zwv]]=[[xyz]wv]+[z[xyw]v]+[zw[xyv]]$.

A derivation of a L.t.s. T is a linear mapping D of T such that $D([xyz])=[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$ for all x, y, z in T. From (1.3) it follows that $\sum_i D(x_i, y_i)$: $z\to \sum_i [x_iy_iz]$ is a derivation, which is called an inner derivation.

Let ρ be a linear mapping of a L. t. s. T into an associative algebra $\mathfrak{E}(V)$ of linear transformations of a vector space V. If ρ is a L.t.s. homomorphism of T into $\mathfrak{E}(V)$, i.e. $\rho([xyz])=[[\rho(x), \rho(y)]\rho(z)]$ for all $x, y, z\in T$, then ρ is called a representation of T into V [4], here we shall call a representation of this type a special representation.

DEFINITION. A *weak representation* of a L. t. s. T into a vector space V is a pair (D, θ) of bilinear mappings of T into $\mathfrak{E}(V)$ satisfying

(1.4) $D(x, y)=\theta(y, x)-\theta(x, y)$

and

(1.5) $[D(x, y), \theta(z, w)]=\theta([xyz], w)+\theta(z, [xyw])$

for all x, y, z, w in T. If a weak representation satisfies the following relation, then it is called a representation.

(1.6) $\theta(z, w)\theta(x, y)-\theta(y, w)\theta(x, z)-\theta(x, [yzw])+D(y, z)\theta(x, w)=0$.

From (1.4) we may say a (weak) representation (D, θ) a (weak) representation θ simply. We call a (weak) representation space V a (weak) T-module. A special representation ρ of a L.t.s. T induces a representation θ of T by putting $\theta(x, y)=\rho(y)\rho(x)$. For x, y in T, let $D(x, y)$ and $\theta(x, y)$ be linear mappings $z\to[xyz]$ and $z\to[zxy]$ of T into itself respectively, then (D, θ) is a representation of T with T as a representation space, which will be called a regular representation of T, and $D(x, y)$ becomes an inner derivation of T. In an algebraic system with a trilinear composition $[xyz]$, a pair (D, θ) of regular mappings satisfying (1.4) and (1.5) characterizes a Lie triple system.

Let θ_1, θ_2 be weak representations of T with representation spaces V, W respectively. Let $\mathfrak{L}(V, W)$ be a vector space spanned by a linear mappings of V into W. For $x, y\in T$ define a linear mapping $\theta_3(x, y)$ of $\mathfrak{L}(V, W)$ into itself by

(1.7) $\theta_3(x, y)f=\theta_2(x, y)f-f\theta_1(x, y)$ $\quad f\in\mathfrak{L}(V, W)$,

(1.8) $D_3(x, y)=\theta_3(y, x)-\theta_3(x, y)$,

then (D_3, θ_3) is a weak representation of T with representation space $\mathfrak{L}(V, W)$, and $D_3(x, y)f=D_2(x, y)f-fD_1(x, y)$.

From (1.4) and (1.5), for the weak representation (D, θ) we have

(1.9) $$[D(x,y), D(z,w)] = D([xyz], w) + D(z, [xyw]).$$

Therefore, the vector space $D(T,T)$ spanned by $\sum_i D(x_i, y_i)$ forms a subalgebra of $\mathfrak{gl}(V)$.

We recall that the derived system $T^{(i)}$ of L. t. s. T is defined by $T^{(1)}=[TTT]$ and $T^{(k)}=[TT^{(k-1)}T^{(k-1)}]$, $k=2, 3, \cdots$, and T is called solvable (in T) if there exists a positive integer n such that $T^{(n)}=(0)$ [4]. Then a result by Lister [4] is slightly generalized to a weak representation.

PROPOSITION. *Let (D, θ) be a weak representation of a L. t. s. T and let $D(T,T)$ be the Lie algebra generated by all $\sum_i D(x_i, y_i)$, $x_i, y_i \in T$. Denote $T^{(k)}$ and $D(T,T)^{(k)}$ the derived subsystem of order k of T and the derived subalgebra of order k of $D(T,T)$ respectively. Then*

(1.10) $$D(T,T)^{(2k)} \subseteqq \sum_{i=0}^{k} D(T^{(i)}, T^{(2k-i)}),$$ $$D(T,T)^{(2k+1)} \subseteqq \sum_{i=0}^{k} D(T^{(i)}, T^{(2k+1-i)}).$$

Hence if T is a solvable L.t.s., then $D(T,T)$ is a solvable Lie algebra. If (D, θ) is a weak representation of a solvable L.t.s. T into a vector space over an algebraically closed field, then there is a one dimensional D-invariant subspace of V.

We give an example of a weak representation which is not a representation. Let T be a 3-dimensional L. t. s. over Φ with basis X_1, X_2, X_3, in which a multiplication is defined by

$$[X_1X_2X_1]=-2X_1,\quad [X_1X_2X_2]=2X_2,\quad [X_1X_2X_3]=X_3,\quad [X_1X_3X_1]=0,$$
$$[X_1X_3X_2]=X_3,\quad [X_1X_3X_3]=0,\quad [X_2X_3X_1]=0,\quad [X_2X_3X_2]=0,$$
$$[X_2X_3X_3]=0,\quad [X_iX_jX_k]=-[X_jX_iX_k].$$

Then a vector subspace V spanned by X_1, X_2 is a subsystem of T and a vector subspace W spanned by X_3 is an ideal of T.

Let θ_1 and θ_2 be weak representations of V into V and V into W, respectively, induced by regular mappings of T, and $\mathfrak{L}(V, W)$ be a vector space spanned by linear mappings of V into W. By the definitions (1.7) and (1.8) $(D_3, \theta_3; \mathfrak{L}(V, W))$ is a weak representation of L. t. s. V. Define f in $\mathfrak{L}(V, W)$ by $f(\lambda X_1 + \mu X_2) = (\lambda + \mu) X_3$, then for $x=y=X_1$, $z=w=X_2$, $v=X_2$ $(\theta_3(z, w)\theta_3(x, y) - \theta_3(y, w)\theta_3(x, z) - \theta_3(x, [yzw]) + D_3(y, z)\theta_3(x, w)) f(v) = -4X_3$, hence θ_3 is not a representation.

2. Cohomology groups associated with a weak representation.

Let (D, θ) be a weak representation of a L. t. s. T into a vector space V and let f be a $2p$-linear mapping of $T\times\cdots\times T$ ($2p$ times) into V satisfying

$$f(x_1, x_2, \cdots, x_{2i-1}, x_{2i}, \cdots, x_{2p-1}, x_{2p})=0$$

for $x_{2i-1}=x_{2i}$, $i=1, 2, \cdots, p-1$. Then f is called a $2p$-V-cochain. $C^{2p}(T, V)$ denotes a vector space spanned by $2p$-V-cochains, where we define $C^0(T, V)=V$.

A coboundary operator δ is a linear mapping of $C^{2p}(T, V)$ into $C^{2p+2}(T, V)$ defined as follows:

$$(\delta f)(x, y)=\theta(x, y)f \qquad \text{for } f\in C^0(T, V),$$

$$\begin{aligned}(2.1)\qquad &(\delta f)(x_1, x_2, \cdots, x_{2p+2})\\ &=(-1)^{p+1}\theta(x_{2p+1}, x_{2p+2})[f(x_1, x_2, \cdots, x_{2p})-f(x_1, \cdots, x_{2p-2}, x_{2p}, x_{2p-1})]\\ &\quad+\sum_{k=1}^{p}(-1)^{k+1}D(x_{2k-1}, x_{2k})f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2p+2})\\ &\quad+\sum_{k=1}^{p}\sum_{j=2k+1}^{2p+2}(-1)^{k}f(x_1, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}\, x_{2k}\, x_j], \cdots, x_{2p+2})\\ &\qquad\qquad \text{for } f\in C^{2p}(T, V), p=1, 2, \cdots,\end{aligned}$$

where the sign $\wedge$ over a letter indicates that this letter is to be omitted.

If $f\in C^0(T, V)$, then $(\delta\delta f)(x_1, x_2, x_3, x_4)$

$$\begin{aligned}&=\theta(x_3, x_4)[(\delta f)(x_1, x_2)-(\delta f)(x_2, x_1)]+D(x_1, x_2)(\delta f)(x_3, x_4)-(\delta f)([x_1x_2x_3], x_4)\\ &\qquad-(\delta f)(x_3, [x_1x_2x_4])\\ &=\{\theta(x_3, x_4)(\theta(x_1, x_2)-\theta(x_2, x_1))+D(x_1, x_2)\theta(x_3, x_4)-\theta([x_1x_2x_3], x_4)\\ &\qquad-\theta(x_3, [x_1x_2x_4])\}f\\ &=\{[D(x_1, x_2), \theta(x_3, x_4)]-\theta([x_1x_2x_3], x_4)-\theta(x_3, [x_1x_2x_4])\}f\end{aligned}$$

$=0$, by (1.5). Similarly $\delta\delta f=0$ for every $f\in C^2(T, V)$.

For $x, y\in T$, $\kappa(x, y)$ is a linear mapping of $C^{2p}(T, V)$ into itself defined by

$$\begin{aligned}(2.2)\qquad &\kappa(x, y)f=D(x, y)f \qquad \text{for } f\in C^0(T, V),\\ &(\kappa(x, y)f)(x_1, x_2, \cdots, x_{2p})=D(x, y)f(x_1, \cdots, x_{2p})-\sum_{j=1}^{2p}f(x_1, \cdots, [xyx_j], \cdots, x_{2p})\\ &\qquad\qquad \text{for } f\in C^{2p}(T, V), p=1, 2, 3, \cdots\end{aligned}$$

$\iota(x, y)$ is a linear mapping of $C^{2p}(T, V)$ into $C^{2p-2}(T, V)$ defined by

$$\begin{aligned}(2.3)\qquad &\iota(x, y)f=0 \qquad \text{for } f\in C^0(T, V),\\ &(\iota(x, y)f)(x_1, x_2, \cdots, x_{2p-2})=f(x, y, x_1, x_2, \cdots, x_{2p-2})\\ &\qquad\qquad \text{for } f\in C^{2p}(T, V), p=1, 2, 3, \cdots\end{aligned}$$

Then, by direct calculations we have

(2.4) $\iota(x, y)\delta f+\delta\iota(x, y)f=\kappa(x, y)f$ for $f\in C^{2p}(T, V), p=2, 3, 4, \cdots$

(2.5) $[\kappa(x, y), \iota(z, w)]f=\iota([xyz], w)f+\iota(z, [xyw])f$ for $f\in C^{2p}(T, V), p=2, 3, 4, \cdots$

Next, we have

(2.6) $[\kappa(x, y), \kappa(z, w)]f=\kappa([xyz], w)f+\kappa(z, [xyw])f$ for $f\in C^{2p}(T, V), p=0. 1, 2, \cdots$

For $p=0, 1$ this is proved by a direct calculation, so that we assume (2.6) holds for every $f\in C^{2p}(T, V)$ and let $f\in C^{2p+2}(T, V)$.
Then for arbitrary u, v in T,

$$\begin{aligned}&\iota(u, v)([\kappa(x, y), \kappa(z, w)]f-\kappa([xyz], w)f-\kappa(z, [xyw])f)\\&=([\kappa(x, y), \kappa(z, w)]-\kappa([xyz], w)-\kappa(z, [xyw]))\iota(u, v)f+\iota([zw[xyu]], v)f\\&\quad+\iota([[xyz]wu], v)f+\iota([z[xyw]u], v)f\\&\quad-\iota([xy[zwu]], v)f+\iota(u, [[xyz]wv])f+\iota(u, [z[xyw]v])f\\&\quad+\iota(u, [zw[xyv]])f-\iota(u, [xy[zwv]])f=0\end{aligned}$$

by the induction assumption and (1.3). Hence (2.6) follows. Similarly by induction on p we have

(2.7) $\kappa(x, y)\delta f=\delta\kappa(x, y)f$ for $f\in C^{2p}(T, V), p=0, 1, 2, \cdots$

Using these relations and by induction on p we have

(2.8) $\delta\delta f=0$ for $f\in C^{2p}(T, V), p=0, 1, 2, \cdots$

Let $Z^{2p}(T, V)$ be a subspace spanned by elements $f\in C^{2p}(T, V)$ such that $\delta f=0$, i.e., f is $2p$–V–cocycle, and let $B^{2p}(T, V)$ a subspace spanned by elements of $C^{2p}(T, V)$ of the form δf, $f\in C^{2p-2}(T, V)$. By (2.8) $B^{2p}(T, V)$ is a subspace of $Z^{2p}(T, V)$. The $2p$–th cohomology group $H^{2p}(T, V)$ of T relative to the weak representation θ is the factor space $Z^{2p}(T, V)/B^{2p}(T, V)$, where we define $B^0(T, V)=0$.

$H^0(T, V)$ is the subspace of V spanned by the invariant elements under the weak representation of T.

In particular, if θ is a regular representation, then $H^0(T, V)$ is an ideal of T. In the next section, we consider the meanning of $H^2(T, V)$.

3. Extensions of weak T–modules.

DEFINITION. Let (θ_1, V) and (θ_2, W) be weak T–modules. (θ_3, V^*) is called an *extension* of V by W if V^* is a weak T–module with W as submodule, and there is a T–homomorphism π of V^* onto V such that $\mathrm{Ker}(\pi)=W$.

Let (θ_3, V^*) is an extension of (θ_1, V) by (θ_2, W) and let l be a linear mapping of V into V^* such that $\pi l=1$ on V. For $x, y \in T$, define an element $g(x,y)$ in $\mathfrak{L}(V, V^*)$ by

$$g(x,y)(v)=\theta_3(x,y)l(v)-l\theta_1(x,y)(v) \qquad v\in V.$$

Then $g(x,y)\in\mathfrak{L}(V,W)$ since $\pi g(x,y)(v)=0$.

For $x,y\in T$ and $f\in\mathfrak{L}(V,W)$, if we define a linear mapping $\theta(x,y)$ of $\mathfrak{L}(V,W)$ by

$$\theta(x,y)f(v)=\theta_3(x,y)f(v)-f\theta_1(x,y)(v) \qquad v\in V,$$

$$D(x,y)=\theta(y,x)-\theta(x,y),$$

(D,θ) is a weak representation of T with representation space $\mathfrak{L}(V,W)$ and $g\in C^2(T,\mathfrak{L}(V,W))$. Furthermore, $g\in Z^2(T,\mathfrak{L}(V,W))$, because

$$\begin{aligned}
&(\delta g)(x_1,x_2,x_3,x_4)(v)\\
&=\theta(x_3,x_4)(g(x_1,x_2)-g(x_2,x_1))(v)+D(x_1,x_2)g(x_3,x_4)(v)-g([x_1x_2x_3],x_4)(v)\\
&\quad -g(x_3,[x_1x_2x_4])(v)\\
&=\theta_3(x_3,x_4)(g(x_1,x_2)-g(x_2,x_1))(v)-(g(x_1,x_2)-g(x_2,x_1))\theta_1(x_3,x_4)(v)\\
&\quad +D_3(x_1,x_2)g(x_3,x_4)(v)-g(x_3,x_4)D_1(x_1,x_2)(v)-g([x_1x_2x_3],x_4)(v)\\
&\quad -g(x_3,[x_1x_2x_4])(v)\\
&=-\theta_3(x_3,x_4)D_3(x_1,x_2)l(v)+D_3(x_1,x_2)\theta_3(x_3,x_4)l(v)-\theta_3([x_1x_2x_3],x_4)l(v)\\
&\quad -\theta_3(x_3,[x_1x_2x_4])l(v)-lD_1(x_1,x_2)\theta_1(x_3,x_4)(v)+l\theta_1(x_3,x_4)D_1(x_1,x_2)(v)\\
&\quad +l\theta_1([x_1x_2x_3],x_4)(v)+l\theta_1(x_3,[x_1x_2x_4])(v)\\
&=0 \quad \text{by (1.5)}.
\end{aligned}$$

Let l_1 and l_2 be two linear mappings of V into V^* such that $\pi l_1=\pi l_2=1$. If we put

$$g_1(x,y)=\theta_3(x,y)l_1-l_1\theta_1(x,y),$$

$$g_2(x,y)=\theta_3(x,y)l_2-l_2\theta_1(x,y),$$

then $(g_1(x,y)-g_2(x,y))(v)=\theta_3(x,y)l(v)-l\theta_1(x,y)(v)$, $v\in V$, where $l=l_1-l_2$. Since $\pi l(v)=0$, $l\in\mathfrak{L}(V,W)$, hence $(g_1(x,y)-g_2(x,y))(v)=\theta(x,y)l(v)=(\delta l)(x,y)\cdot(v)$, i.e., $g_1-g_2=\delta l$. Hence, an extension of V by W determines uniquely an element of $H^2(T,\mathfrak{L}(V,W))$.

Conversely, given an element $g\in Z^2(T,\mathfrak{L}(V,W))$. Denote V^* a vector space direct sum $V\oplus W$, and for $x,y\in T$ put

$$\theta_3(x,y)(v,w)=(\theta_1(x,y)(v),\ g(x,y)(v)+\theta_2(x,y)(w)),$$

$$D_3(x,y)=\theta_3(y,x)-\theta_3(x,y), \qquad (v,w)\in V\oplus W.$$

Then

$$([D_3(x,y),\theta_3(z,u)]-\theta_3([xyz],u)-\theta_3(z,[xyu]))(v,w)$$

$$= \left(0, \begin{array}{l} \theta_2(z,u)(g(x,y)-g(y,x))(v) \\ -(g(x,y)-g(y,x))\theta_1(z,u)(v) \\ +D_2(x,y)g(z,u)(v)-g(z,u)D_1(x,y)(v) \\ -g([xyz],u)(v)-g(z,[xyu])(v) \end{array} \right)$$

$$= \left(0, \begin{array}{l} \theta(z,u)(g(x,y)-g(y,x))(v)+D(x,y)g(z,u)(v) \\ -g([xyz],u)(v)-g(z,[xyu])(v) \end{array} \right)$$

$$= (0, \quad (\delta g)(x,y,z,u)(v))$$

$$=(0,0).$$

Hence V^* is a weak T-module. Put $\iota(w) = (0,w)$, $\pi(v,w) = v$, then π is a T-homomorphism of V^* onto V with kernel W. We identify W with its image $\iota(W)$ in V^*, and if we put $l(v)=(v,0)$ for $v \in V$, $\pi l = 1$. $(\theta_3(x,y)l-l\theta_1(x,y))(v)=(0,g(x,y)(v))$, hence $g(x,y)=\theta_3(x,y)l-l\theta_1(x,y)$, so that l defines a given 2-cocycle g. V^* is an extension of V by W.

Now, suppose (θ_3, V^*) and $(\theta_3', V^{*\prime})$ be two extensions of V by W with same element of $H^2(T,\mathfrak{L}(V,W))$, i.e. $g(x,y)=g'(x,y)+(\delta\rho)(x,y)$, $\rho \in \mathfrak{L}(V,W)$, and let π and π' be T-homomorphisms of V^* onto V and $V^{*\prime}$ onto V, respectively.

From the assumption we have

$$\theta_3(x,y)l-l\theta_1(x,y)=\theta_3'(x,y)l'-l'\theta_1(x,y)+\theta_3'(x,y)\rho-\rho\theta_1(x,y),$$

hence

$$(3.1) \qquad l'\theta_1(x,y)-l\theta_1(x,y)+\rho\theta_1(x,y)=\theta_3'(x,y)l'-\theta_3(x,y)l+\theta_3(x,y)\rho.$$

Define a linear mapping σ of V^* into $V^{*\prime}$ by

$$\sigma(v^*)=l'\pi(v^*)+(v^*-l\pi(v^*))+\rho\pi(v^*) \qquad v^* \in V^*.$$

Then $\pi'\sigma=\pi$. By using (3.1) we have

$$\begin{aligned} \theta_3'(x,y)\sigma(v^*) &= \theta_3'(x,y)\{l'\pi(v^*)+(v^*-l\pi(v^*))+\rho\pi(v^*)\} \\ &= l'\theta_1(x,y)\pi(v^*)+\theta_3(x,y)(v^*)-l\theta_1(x,y)\pi(v^*)+\rho\theta_1(x,y)\pi(v^*) \\ &= l'\pi\theta_3(x,y)(v^*)+\theta_3(x,y)(v^*)-l\pi\theta_3(x,y)(v^*)+\rho\pi\theta_3(x,y)(v^*) \\ &= \sigma(\theta_3(x,y)(v^*)), \end{aligned}$$

hence $\theta_3'(x,y)\sigma=\sigma\theta_3(x,y)$, i.e., σ is a T-homomorphism of V^* into $V^{*\prime}$. If $\sigma(v^*)=0$ then $\pi(v^*)=\pi'\sigma(v^*)=0$, $v^* \in \mathrm{Ker}(\pi)=W$, $\mathrm{Ker}(\sigma)\subset W$, but $\sigma=1$ on W, hence σ is an isomorphism of V^* into $V^{*\prime}$. Moreover σ is surjective, since for $v^{*\prime} \in V^{*\prime}$, if we put $v^*=l\pi'(v^{*\prime})+v^{*\prime}-l'\pi'(v^{*\prime})-\rho\pi'(v^{*\prime})$, then $v^* \in V^*$, and $\sigma(v^*)=v^{*\prime}$. Therefore, there exists a T-isomorphism σ of V^* onto $V^{*\prime}$ such that $\sigma=1$ on W and $\pi'\sigma=\pi$, i.e., two extensions V^* and $V^{*\prime}$ are equivalent.

Two equivalent extensions of V by W determine the same element of $H^2(T,\mathfrak{L}(V,W))$.

Summaring above results we have the following theorem.

THEOREM. *There is a one-to-one correspondence between the equivalence classes of extensions of V by W and the elements of $H^2(T, \mathfrak{L}(V, W))$.*

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