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ON DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE EIGHT

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Introduction :

B. Huppert, in his paper [2], studied on primitive, solvable permutation groups of degree $p+1$ or of degree p, where p is a prime number. However the determination of the doubly transitive groups of a given degree is open to the question. In this paper we shall prove the following theorem :

THEOREM *Let G be a doubly transitive permutation group of degree 8. Then G is isomorphic to one of the following groups;*

(i) *Frobenius group of order 56 with an elementary abelian group of order 8 as a Frobenius kernel, $PSL(2,7)$, $\Gamma(2^3)$ or $PGL(2,7)$,*

(ii) *Holomorph of N; $Hol(N)$, where N is an elementary abelian group of order 8,*

(iii) *the symmetric group S_8 of degree 8 or the alternating group A_8 of degree 8.*

These groups can be, of course, represented as doubly transitive groups of degree 8.

Now, we use the following notation. For a finite set Ω we denote the symmetric group on Ω by S^{Ω}. The points of Ω are denoted by $\alpha, \beta, \cdots$ and the elements of a group are denoted by $\sigma, \tau, \ldots$. When a permutation group G on Ω is either symmetric or alternating, G is called an AS-group. For a permutation group G on Ω, the stabilizer of α in Ω is denoted by G_α. For a subset Δ of Ω we denote the intersection of all G_αs for α in Δ by G_Δ.

§1. Lemmas on transitive groups of prime degree

We shall show the several lemmas.

LEMMA 1. *Let G be a transitive group on Ω consisting of p elements, where p is a prime number. Then G is solvable if and only if $G_{\alpha,\beta}=1$ for every pair α, β of Ω with $\alpha\neq\beta$.*

Proof: If G is solvable, then a minimal normal subgroup N of G is a regular subgroup of order p. Therefore the stabilizer G_α of α in Ω can be regarded as a subgroup of the group of automorphisms of N, and so G_α is semi-regular on $\Omega-\{\alpha\}$, i.e. $G_{\alpha,\beta}=1$ for any β in $\Omega-\{\alpha\}$.

Conversely for every pair α, β in $\Omega(\alpha\neq\beta)$ we assume $G_{\alpha,\beta}=1$. Since the order of G is divisible by p and is not larger than $p(p-1)$, a Sylow p-subgroup P of G is a cyclic normal subgroup of order p. Hence P is a self-centralizing subgroup of G. Therefore the factor group G/P is isomorphic to a subgroup of the group of automorphisms of P which is a cyclic group of order $p-1$. Thus G is solvable.

The next lemma which is due to Bochert is useful to the determination of the order of G.

LEMMA 2. *Let G be a primitive group on the set Ω of n elements, If G is not an AS-group of degree n, then the following inequality holds;*

$$[S_n : G] \geq \left[\frac{n+1}{2}\right]!$$

where $[m]$ denotes the maximal integer that does not exceed m.

Proof: Let Δ be a subset consisting of m elements of Ω, and let S^Δ be the subgroup of S^Ω the permutations of which permute the elements of Δ arbitrarily but fix the set $\Gamma=\Omega-\Delta$ elementwise. Now we choose the set Δ satitfying the following two conditions:

(i) $S^\Delta\cap G=1$,

(ii) m is as large an integer as possible.

(In this case we have $1\leqq m$.)

Then $\left[\frac{n+1}{2}\right]\leqq m$ holds. If $[n+1/2]\leqq m$ does not hold, we have $m<n/2$. Since Γ consists of n-m elemnts, by the choice of Δ there is an element $\sigma(\neq 1)$ in $S^\Gamma\cap G$ which moves some element α of Γ.

Of course σ is containted in G_Δ. Likewise, we have an element $\tau(\neq 1)$ in $S^{\Delta\cup\{\alpha\}}\cap G$ because of the maximality of m. Then it is easily verified that α is the only element of Ω that is moved by σ and τ. Therefore G contains a 3-cycle $[\sigma,\tau]$, the commutator of σ and τ, which is contrary to the assumption that G is not an AS-group of degree n. Hence $[n+1/2]\leqq$m holds. By $S^\Delta\cap$G$=1$, none of cosets of S^Ω by G contain two elements of S^Δ. Thus we have

$$[S^\Omega : G]\geqq m\,!\geqq[n+1/2]\,!$$

Now it is well-known that a non-abelian simple group of order pqr^m, where p, q, r, are distinct prime numbers, is isomorphic to the alternating group A_5 of degree 5 or to $PSL(2, 7)$.

LEMMA 3. *Let p, q and r be distinct prime numbers and G be a transitive group of degree p, of order pqr^m, where m is a positive integer. If G does not contain a non-abelian simple group, then G is solvable.*

Proof: We prove this lemma by the induction on the number of prime factors dividing the order of G. If $m=1$, G is clearly solvable. Suppose $m>1$. By the assumption, G is not a simple group, and so G has a proper normal subgroup N such that $|N|=pq^a r^b$ with $a=0$ or 1, $0\leqq b\leqq m$ and $(a, b)\neq(1,m)$. Then by Burnside's theorem the factor group G/N is solvable, while N is a transitive group of degree p and the number of prime factors of $|N|$ is less than that of $|G|$. Hence by the inductive hypothesis N is solvable. Since both G/N and N are solvable, G is also solvable.

The next lemma is well-known as the lemma on transitive extension of a simple group.

LEMMA 4. *Let G be a primitive permutation group containing no regular normal subgroup. If the stabilizer G_α of α is simple, then G is also simple.*

Proof: Let N be a non-trivial normal subgroup of G. Since N is not regular, N has a non-trivial intersection to G_α. Since G_α is simple and $G_\alpha\cap N$ is normal in G_α, N contains G_α. By the primitivity of G, G_α is a maximal subgroup of G and N is a transitive group. Therefore $N=G$. Hence G is simple.

§2. The proof of theorem

By putting $p=7$ in lemmas of §1, we shall prove our theorem. Let G be a doubly transitive group of degree 8. Since the stabilizer G_α of α in Ω is a transitive group of degree 7, by lemma 1 G_α is solvable if and only if G is a Zassenhaus group. In this case $|G|=8\cdot7\cdot a$, $a=1$, 3 or 6. (i) $a=1$; it is clear that G is a Frobenius group with an elementary abelian group of order 8 as a Frobenius kernel. (ii) $a=3$; since $|G|=2^3\cdot3\cdot7$, G is simple or solvable. If G is simple, G is isomorphic to $PSL(2, 7)$. If G is solvable, then by B. Huppert [3] G is a subgroup of the group $\Gamma(2^3)$ of all semilinear transformations over the Galois Field $GF(2^3)$. However $|\Gamma(2^3)|=2^3\cdot(2^3-1)\cdot3$ and $|G|=2^3\cdot3\cdot7$. Therefore G is isomorphic to $\Gamma(2^3)$. (iii) $a=6$; since in this case G is exactly 3-fold transitive, by Zassenhaus [6] we have that $G\cong PGL(2, 7)$.

Next, we assume that G_α is not solvable. Then by Burnside's theorem G_α is a doubly transitive group. Therefore $|G_\alpha|$ is divisible by $7\cdot6$. On the other hand, if G is not an AS-group, then by lemma 2 we have the following inequalities:

$$[S_8 : G]\geqq[8+1/2]!=4!$$
$$\text{i.e. } |G|\leqq 8\cdot7\cdot6\cdot5.$$

Hence we can put $|G_\alpha|=7\cdot 6\cdot a$, $a=1,2,3,4$ or 5.
Since G_α is not solvable, by lemma 3 $a\neq 1,2,3$. If $a=5$, by $|G_\alpha|=7\cdot 6\cdot 5$ G_α contains a proper normal subgroup N of order divisible by 7. In this case G/N and N are solvable, and so G_α is solvable, which is contrary to the nonsolvability of G_α. Hence $a=4$ i.e. $|G_\alpha|=7\cdot 6\cdot 4$. Again by using lemma 3 it follows that G_α is simple i.e. $G_\alpha \cong PSL(2,7)$. Since $|G|=8\cdot 7\cdot 6\cdot 4$, G is not simple. Therefore by lemma 4, G must contain an elementary abelian regular normal subgroup N of order 8. By the following relation;

$$G_\alpha \cong G/N \leq Aut(N) \cong PSL(3,2) \cong PSL(2,7),$$

we have that G is isomorphic to the Holomorph of N.
Hence the proof is completed.

Moreover, since we can prove that every primitive group of degree 8 is doubly transitive, in the assumption of our theorem a doubly transitive group G is to be replaced by a primitive group G (see [5]).

References

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