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ON THE FINITE DIFFERENCE APPROXIMATION TO THE GENERALIZED SOLUTION OF THE FICHERA PROBLEM

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§1. Introduction.

The researches for linear partial differential equations which are not of single type have been made by many authors (cf. Bers [1], Mikhlin [4] and references quoted there). The most original researches for such equations concern elliptic-hyperbolic or mixed equations, which are of partly elliptic and partly hyperbolic types in the domain under consideration. The Tricomi equation and its generalized ones are their examples. The investigations for these equations are made individually and the results are derived by various methods [1], [4].

On his treatment of this mixed equations, Friedrichs [3] constructed his theory through the energy principles regardless of the usual notion of types of partial differential equations. This method is the treatment as symmetric positive linear differential equations (cf. also [6]). This class of equations contains many other elliptic, hyperbolic and parabolic ones with various problem forms (systems of equations, equations of higher order or equations with initial or boundary data imposed).

Another unified theory for linear equations is Fichera's discussion which is related to the second order linear equation of the form

$$a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+b^i\frac{\partial u}{\partial x_i}+cu=f$$

with the condition $a^{ij}\xi_i\xi_j \geqq 0$ [2]. Elliptic, parabolic, elliptic-parabolic and first order equations are included in this class of equations. He proved the maximum principles and the uniqueness of the classical solution, and the existence of the generalized solution. Oleinik [5] established conditions for the uniqueness and the smoothness of the generalized solution.

The present paper reports about the finite difference approximation to the generalized solution of this Fichera problem. The main result is the weak convergenece of the generalized solution of the difference problem to the one of the original problem. For the strongly elliptic equations the similar results are proved by Stummel [7]. Our discussions are not related to all equations concerning to the Fichera problem, but to the ones with some additional restrictions (§5).

§2. Fichera Problem.

Suppose Ω is a bounded domain in R^m with the boundary Σ. Let the functions $a^{ij}(x) \in C^1(\bar{\Omega})$, $b^i(x) \in C^1(\bar{\Omega})$, $c(x) \in C^0(\bar{\Omega})$ $(i, j=1, \cdots, m)$ be given. Let $a^{ij}(x)=a^{ji}(x)$ and

(1) $$a^{ij}(x)\xi_i\xi_j \geqq 0,$$

for all real vectors $\xi=(\xi_1, \cdots, \xi_m)$ and each x in $\bar{\Omega}$. We consider the second order linear equation

(2) $$\frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right)+b^i\frac{\partial u}{\partial x_i}+cu=f \quad \text{in } \Omega.$$

Henceforth we abide by the convention for double indices in the differential geometry. The boundary Σ is divided into three portions in the following way. Let $\tilde{\Sigma}$ be a part of Σ such that

(3) $$a^{ij}(x)n_i(x)n_j(x)=0,$$

where $n_i(x)$ $(i=1, \cdots, m)$ are compornents of inner normal vector on each point x on the boundary. Let us define

(4) $$b(x)=b^i(x)n_i(x)$$

on $\tilde{\Sigma}$ and denote by Σ_1 the part of $\tilde{\Sigma}$ where $b(x)>0$. Let $\Sigma_2=\tilde{\Sigma}\backslash\Sigma_1$ and $\Sigma_3=\Sigma\backslash\tilde{\Sigma}$. For the equation (2), the imposed condition is

(5) $$u=g(x) \quad \text{on} \quad \Sigma_2+\Sigma_3,$$

where $g(x)$ is a given function on $\Sigma_2+\Sigma_3$.

We shall now proceed to the definition of the generalized solution of the Fichera problem. As preliminalies, we shall define some function spaces. Let H_0 be $L_2(\Omega)$ and its inner product and norm be $(u, v)_0$ and $\|u\|_0$, respectively.. The inner product $(u, v)_1$ and the norm $\|u\|_1$ are defined by

(6) $$(u, v)_1=\int_\Omega a^{ij}\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}dx+\int_\Omega uv\,dx+\int_\Gamma buv\,d\sigma$$
$$\|u\|_1=\sqrt{(u, u)_1},$$

where Γ is Σ_1, dx is the volume element in Ω and $d\sigma$ is the surface element on Γ. We also define the inner product and the norm on the boundary :

$$(u, v)_0^\Gamma = \int_\Gamma uv\,d\sigma, \quad \|u\|_0^\Gamma = \sqrt{(u, u)_0^\Gamma}\,.$$

Let the class $\mathring{C}^\infty$ be the set of functions from the intersection of $C^\infty(\Omega)$ and $C^0(\bar{\Omega})$ which vanish on $\Sigma_2 + \Sigma_3$ and the spaces $\mathring{H}_0$ and $\mathring{H}_1$ are defined as the completions of the class $\mathring{C}^\infty$ by the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Moreover H_1 is defined as the completion of the class $C^\infty(\Omega)\cap C^0(\bar{\Omega})$ by the norm $\|\cdot\|_1$. Let the function $g(x)$ which is defined on $\Sigma_2 + \Sigma_3$ be extendable on the whole domain $\bar{\Omega}$ such as the extended function which is again written $g(x)$ belongs to the space H_1.

Definition. Let $f(x) \in H_0$ and $g(x) \in H_1$. The function $u(x) \in H_1$ is called the generalized solution of the Fichera problem (2), (5) if $u(x) - g(x) \in \mathring{H}_1$ and

$$B(u, \varphi) = (f, \varphi)_0 \tag{7}$$

is satisfied for all $\varphi \in \mathring{C}^\infty$, where the bilinear form $B(u, \varphi)$ is defined by

$$\begin{aligned} B(u, \varphi) = &-\int_\Omega a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx - \int_\Omega \frac{\partial}{\partial x_i}(b^i \varphi) u\, dx \\ &+ \int_\Omega cu\varphi\, dx - \int_\Gamma bu\varphi\, d\sigma. \end{aligned} \tag{8}$$

An easy calculation gives that if the function $u(x)$ is a classical solution, it is also the generalized one in the sense of the above definition.

§3. Difference Problem Corresponding to Fichera Problem.

Suppose h is a positive number. Let Φ_h be the set of all points whose coordinates have the form $x_i = nh$, where $i = 1, \cdots, m$ and n is any integer. $\bar{\Omega}_h$ is the set $\bar{\Omega}\cap\Phi_h$ and Ω_h is the set of all points of $\bar{\Omega}_h$ whose neighboring points are all in $\bar{\Omega}_h$, where neighboring points of x mean the points $x \pm he_j$, $e_j = (0, \cdots, \underset{j}{1}, \cdots, 0)$ ($j = 1, \cdots, m$). Let Σ_h be $\bar{\Omega}_h \backslash \Omega_h$. Let the class of functions on Φ_h which vanish on $\Phi_h \backslash \bar{\Omega}_h$ be denoted by $\Pi_h(\bar{\Omega}_h)$ or simply Π_h. Any element of Π_h is written as $u_h(x)$. As for the restriction to $\bar{\Omega}_h$ of a function u defined on $\bar{\Omega}$, we shall write $R_h u \in \Pi_h$ and it is often abbreviated as original u unless there arises any confusion. The translation operators $T^h_{\pm j}$ are mappings of $u_h(x) \in \Pi_h$ to $u_h(x \pm he_j) \in \Pi_h$, ($j = 1, \cdots, m$). The forward and backward difference quotient operators D^h_j and D^h_{-j}, respectively, are defined as

$$\begin{aligned} D^h_j u_h(x) &= \frac{1}{h}(T^h_j u_h(x) - u_h(x)), \\ D^h_{-j} u_h(x) &= \frac{1}{h}(u_h(x) - T^h_{-j} u_h(x)). \end{aligned} \tag{9}$$

For the difference quotient operators there hold the following formulae:

(10) $$D^h_{\pm j}(u_h(x)v_h(x)) = (D^h_{\pm j}u_h(x))(T^h_{\pm j}v_h(x)) + u_h(x)(D^h_{\pm j}v_h(x)),$$

(11) $$h\sum_{l=1}^{k-1}[D^h_{-j}(T^h_j)^l u_h(x)][(T^h_j)^l v_h(x)]$$

$$= -h\sum_{l=1}^{k-1}[(T^h_j)^l u_h(x)][D^h_j(T^h_j)^l v_h(x)] + [(T^h_j)^{k-1}u_h(x)][(T^h_j)^k v_h(x)] - [u_h(x)T^h_j v_h(x)]$$

for any $u_h, v_h \in \Pi_h$ and $1 \leqq j \leqq m$. The formula (11) is often said "summation by parts."

For the simplicity of the sequent discussions the Fichera problem is restricted in the following form. Various modifications are listed in §5. The domain Ω is bounded by a bounded region P_0 on the hyperplane $x_m=0$, a bounded region P_k on $x_m=k>0$ and the surface S connecting ∂P_0 and ∂P_k, where the tangential plane at every point of the surface S is not parallel to the hyperplane $x_m=0$ except at the points on ∂P_0 and ∂P_k. We consider the equation (2) with the conditions $a^{mm}(x)=0$ and $b^m(x)<0$ on $\bar{P}_0+\bar{P}_k$. Then by the assumptions (1), (4) we have $a^{im}(x)=a^{mi}(x)=0$ on $\bar{P}_0+\bar{P}_k$, $b(x)=-b^m(x)>0$ on $\bar{P}_k$ and $b(x)=b^m(x)<0$ on $\bar{P}_0$, and the boundary Σ of Ω consists of three parts $\Sigma_1=\Gamma=P_k$, $\Sigma_2=P_0$, $\Sigma_3=S\cup\partial P_k\cup\partial P_0$. We also assume $a^{ij}(x)\xi_i\xi_j \geqq C\sum_{i=1}^{m-1}\xi_i^2$ for x in $\bar{\Omega}$.

For a positive number h_0, $I(k:h_0)$ denotes the set of positive numbers h of the form k/l, where l is a positive integer such as $0<k/l<h_0$. For $h \in I(k:h_0)$ Γ_h, $\Sigma_{2,h}$ and $\Sigma_{3,h}$ signify $\Gamma\cup\Sigma_h$, $\Sigma_2\cap\Sigma_h$ and $\Sigma_h\backslash(\Gamma_h+\Sigma_{2,h})$, respectively.

Similarly to the continuous case the following inner products and norms can be defined for functions u_h, v_h of Π_h:

(12) $$(u_h, v_h)_{0,h} = h^m\sum_{x\in\Omega_h}u_h(x)v_h(x),$$

(13) $$(u_h, v_h)_{1,h} = h^m\sum_{x\in\Omega_h}a^{ij}(x)(D^h_{-i}u_h(x))(D^h_{-j}v_h(x)) + h^m\sum_{x\in\Omega_h}u_h(x)v_h(x) + h^{m-1}\sum_{x\in\Gamma_h}u_h(x)v_h(x),$$

(14) $$(u_h, v_h)^{\Gamma}_{0,h} = h^{m-1}\sum_{x\in\Gamma_h}u_h(x)v_h(x),$$

(15) $$\|u_h\|_{0,h}=\sqrt{(u_h, u_h)_{0,h}},\quad \|u_h\|_{1,h}=\sqrt{(u_h, u_h)_{1,h}},\quad \|u_h\|^{\Gamma}_{0,h}=\sqrt{(u_h, u_h)^{\Gamma}_{0,h}}$$

and also the bilinear form $B_h(u_h, v_h)$ can be defined as

(16) $$B_h(u_h, v_h) = -h^m\sum_{\Omega_h}a^{ij}(D^h_{-i}u_h)(D^h_{-j}v_h) - \frac{1}{2}h^m\sum_{\Omega_h}u_h[D^h_i(b^i v_h) + D^h_{-i}((T^h_i b^i)v_h)] + h^m\sum_{\Omega_h}cu_h v_h + \frac{1}{2}h^{m-1}\sum_{\Gamma_h}b^m[(T^h_{-m}u_h)v_h + u_h(T^h_{-m}v_h)].$$

Then for any pair $u, v \in \mathring{C}^\infty$ the convergencies

$$(17)\qquad \begin{aligned} &(u, v)_{0,h} \to (u, v)_0, \quad (u, v)_{1,h} \to (u, v)_1, \\ &(u, v)^{\Gamma}_{0,h} \to (u, v)^{\Gamma}_0, \quad B_h(u, v) \to B(u, v) \end{aligned}$$

as $h \to 0$, $h \in I(k:h_0)$, are valid.

For the functions of Π_h, we shall denote the Hilbert spaces defined by the norms $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ by $H_{0,h}$ and $H_{1,h}$, respectively. Let $\mathring{H}_{0,h}$ and $\mathring{H}_{1,h}$ be subspaces of $H_{0,h}$ and $H_{1,h}$, respectively, whose elements vanish on $\Sigma_{2,h}$ and $\Sigma_{3,h}$.

The operator S_h on $H_{0,h}$ into H_0 is defined as follows:

$$(18)\qquad S_h u_h(x) = u_h(x_0)$$

for $x_{0i} - h/2 \leqq x_i < x_{0i} + h/2$, where $i=1, \cdots, m$, $x=(x_1, \cdots, x_m) \in \bar{\Omega}$ and $x_0=(x_{01}, \cdots, x_{0m}) \in \bar{\Omega}_h$. Then we have for $u_h, v_h \in \mathring{H}_{0,h}$

$$(19)\qquad \begin{aligned} (u_h, v_h)_{0,h} &= h^m \sum_{x \in \Omega_h} u_h(x) v_h(x) \\ &= \int_\Omega (S_h u_h(x))(S_h v_h(x))\, dx - \frac{1}{2} h \int_\Gamma (S_h u_h(x))(S_h v_h(x))\, d\sigma. \end{aligned}$$

Definition. Let $f_h \in H_{0,h}$, $g_h \in H_{1,h}$. The function $u_h(x) \in H_{1,h}$ is called the generalized solution of the difference problem corresponding to the Fichera problem, if $u_h - g_h \in \mathring{H}_{1,h}$ and there holds the identity

$$(20)\qquad B_h(u_h, \varphi_h) = (f_h, \varphi_h)_{0,h}$$

for all $\varphi_h \in \mathring{H}_{1,h}$.

Since $B_h(u_h, \varphi_h)$ can be rewritten by using summation by parts in the form

$$(21)\qquad B_h(u_h, \varphi_h) = \left(D^h_j(a^{ij} D^h_{-i} u_h) + \frac{1}{2} b^i D^h_{-i} u_h + \frac{1}{2} (T^h_i b^i) D^h_i u_h + c u_h, \varphi_h \right)_{0,h},$$

the generalized solution u_h of the difference problem satisfies the equation and the boundary condition:

$$(22)\qquad \begin{aligned} &L_h(u_h) = D^h_j(a^{ij} D^h_{-i} u_h) + \frac{1}{2} b^i D^h_{-i} u_h + \frac{1}{2} (T^h_i b^i) D^h_i u_h + c u_h = f_h \quad \text{in } \Omega_h \\ &u_h = g_h \quad \text{on } \Sigma_{2,h} + \Sigma_{3,h} \end{aligned}$$

and conversely the solution of (22) is a generalized solution.

§4. Lemmas and Theorems.

Lemma 1. *There exists a positive constant C_1 such that the inequality*

$$(23)\qquad |B_h(u_h, v_h)| \leqq C_1 \|u_h\|_{1,h} \Big[\sum_{i=1}^{m} \|D_i^h v_h\|_{0,h} + \|v_h\|_{0,h} + \|v_h\|_{0,h}^{\Gamma} + \|T_{-m}^h v_h\|_{0,h}^{\Gamma} \Big]$$

holds for all $h \in I(k:h_0)$ *and for* $u_h \in H_{1,h}$, $v_h \in H_{1,h}$. *If the function* $D_i^h b^i(x) - 2c(x)$ *is bounded away from zero on* $\bar{\Omega}_h$ *for any* $h \in I(k:h_0)$, *there exists a constant* C_2 *such that the inequality*

$$(24)\qquad |B_h(v_h, v_h)| \geqq C_2 \|v_h\|_{1,h}^2$$

holds for all h *in* $I(k:h_0)$ *and for any* $v_h \in \mathring{H}_{1,h}$.

PROOF. Suppose (q^{ij}) is a positive semi-definite matrix, then there holds the inequality

$$(25)\qquad (q^{ij}\xi_i\eta_j)^2 \leqq (q^{ij}\xi_i\xi_j)(q^{ij}\eta_i\eta_j)$$

for all real vectors $(\xi_1, \cdots, \xi_m)$ and $(\eta_1, \cdots, \eta_m)$. Consequently by using the Cauchy inequality and (10) there holds for any pair $u_h, v_h \in H_{1,h}$ with

$$\begin{aligned}
|B_h(u_h, v_h)| &\leqq (a^{ij} D_{-i}^h u_h, D_{-j}^h v_h)_{0,h} \\
&+ \frac{1}{2} |(u_h, (T_i^h b^i) D_i^h v_h + b^i D_{-i}^h v_h)_{0,h}| + |(u_h, (D_i^h b^i) v_h)_{0,h}| \\
&+ |(u_h, c v_h)_{0,h}| + \frac{1}{2} |(T_{-m}^h u_h, b^m v_h)_{0,h}^{\Gamma}| + \frac{1}{2} |(u_h, b^m T_{-m}^h v_h)_{0,h}^{\Gamma}| \\
&\leqq (a^{ij} D_{-i}^h u_h, D_{-j}^h u_h)_{0,h}^{\frac{1}{2}} (a^{ij} D_{-i}^h v_h, D_{-j}^h v_h)_{0,h}^{\frac{1}{2}} \\
&+ \frac{1}{2} \|u_h\|_{0,h} [\|(T_i^h b^i) D_i^h v_h\|_{0,h} + \|b^i D_{-i}^h v_h\|_{0,h}] \\
&+ \|u_h\|_{0,h} [\|(D_i^h b^i) v_h\|_{0,h} + \|c v_h\|_{0,h}] \\
&+ \frac{1}{2} \|T_{-m}^h u_h\|_{0,h}^{\Gamma} \|b^m v_h\|_{0,h}^{\Gamma} + \frac{1}{2} \|u_h\|_{0,h}^{\Gamma} \|b^m T_{-m}^h v_h\|_{0,h}^{\Gamma} \\
&\leqq C_1 \|u_h\|_{1,h} \Big[\sum_{i=1}^{m} \|D_i^h v_h\|_{0,h} + \|v_h\|_{0,h} + \|v_h\|_{0,h}^{\Gamma} + \|T_{-m}^h v_h\|_{0,h}^{\Gamma} \Big].
\end{aligned}$$

Thus we have the first inequality. As to the second assertion we have from (16), (1), (10) and (11)

$$\begin{aligned}
|B_h(v_h, v_h)| &= h^m \sum_{\Omega_h} a^{ij} (D_{-i}^h v_h)(D_{-j}^h v_h) \\
&+ \frac{1}{2} h^m \sum_{\Omega_h} v_h [D_i^h (b^i v_h) + D_{-i}^h ((T_i^h b^i) v_h)] \\
&- h^m \sum_{\Omega_h} c v_h^2 - h^{m-1} \sum_{\Gamma_h} b^m (T_{-m}^h v_h) v_h \\
&= h^m \sum_{\Omega_h} a^{ij} (D_{-i}^h v_h)(D_{-j}^h v_h) \\
&+ \frac{1}{2} h^m \sum_{\Omega_h} [D_i^h b^i - 2c] v_h^2 - \frac{1}{2} h^{m-1} \sum_{\Gamma_h} b^m (T_{-m}^h v_h) v_h.
\end{aligned}$$

Therefore from the positivity of $D_i^h b^i - 2c$ and $-b^m$ there exists a positive constant C_2 such that the inequality (24) holds for all $h \in I(k:h_0)$.

LEMMA 2. *Let $f_h \in H_{0,h}$. If the function $D_i^h b^i(x) - 2c(x)$ is bounded away from zero on $\bar{\Omega}_h$ for any $h \in I(k:h_0)$, then the generalized solution u_h of the difference problem with homogeneous boundary condition satisfies the inequality*

$$\|u_h\|_{1,h} \leqq \frac{1}{C_2} \|f_h\|_{0,h}, \tag{26}$$

where $h \in I(k:h_0)$ and C_2 is a constant as in Lemma 1.

PROOF. Since the solution u_h belongs to $\mathring{H}_{1,h}$, by the definition of the generalized solution and (24) we obtain

$$C_2\|u_h\|_{1,h}^2 \leqq |B_h(u_h, u_h)| = |(f_h, u_h)_{0,h}| \leqq \|f_h\|_{0,h}\|u_h\|_{0,h} \leqq \|f_h\|_{0,h}\|u_h\|_{1,h}.$$

From this the assertion is valid.

THEOREM 1. *Let f_h be in $H_{0,h}$ and let the function $D_i^h b^i(x) - 2c(x)$ be bounded away from zero on $\bar{\Omega}_h$ for $h \in I(k:h_0)$. Then for such h the generalized solution of the difference problem and therefore the solution of the problem (22) exist and are unique.*

PROOF. From Lemma 2 the generalized solution of difference problem is unique. This solution is identical with the solution of (22) by virtue of the remark after Definition in §3. Then the solution of (22) is unique, which implies the existence of the solutions of these problems.

THEOREM 2. *Let $f \in H_0$, $g \in H_1$ and let $\|R_h f\|_{0,h}, \|R_h g\|_{1,h}$ and $\sum_{i=1}^m \|D_i^h R_h g\|_{0,h}$ be bounded independently of h in $I(k:h_0)$. Then the sequence being composed of the generalized solution $u_h \in H_{1,h}$ of the difference problem weakly converges to the one of the original problem as $h \to 0$, $h \in I(k:h_0)$.*

PROOF. Suppose $v_h = u_h - R_h g$ on $\bar{\Omega}_h$, then $v_h \in \mathring{H}_{1,h}$ and

$$B_h(v_h, \varphi_h) = (f, \varphi_h)_{0,h} - B_h(g, \varphi_h) \tag{27}$$

for $\varphi_h \in \mathring{H}_{1,h}$. Therefore by virtue of Lemma 1 we obtain

$$\begin{aligned} C_2\|v_h\|_{1,h}^2 &\leqq |B_h(v_h, v_h)| \leqq |(f, v_h)_{0,h}| + |B_h(g, v_h)| \\ &\leqq \|f\|_{0,h}\|v_h\|_{0,h} + C_1\Big[\sum_{i=1}^m \|D_{-i}^h g\|_{0,h} + \|g\|_{0,h} + \|g\|_{0,h}^{\Gamma} + \|T_{-m}^h g\|_{0,h}^{\Gamma}\Big]\|v_h\|_{1,h} \\ &\leqq C_4\|v_h\|_{1,h} \end{aligned}$$

for all $h \in I(k:h_0)$. Hence $\|v_h\|_{1,h} \leqq C_4/C_2 = C_5$. Then the sequence $\{S_h v_h\}$ is bounded in H_0:

$$\int_\Omega |S_h v_h|^2 dx = \|S_h v_h\|_0^2 = \|v_h\|_{0,h}^2 + \frac{1}{2} h(\|v_h\|_{0,h}^{\Gamma})^2 \leqq C_5^2 + 1 \tag{28}$$

for any $h \in I(k:h_1)$ and some h_1, and there exist a subsequence $\{S_{h_j}v_{h_j}\}$ and a function v in $\mathring{H}_0$ such that for all $\varphi \in \mathring{C}^\infty$

$$\int_\Omega (S_{h_j}v_{h_j}-v)\varphi\, dx \to 0 \tag{29}$$

as $j\to\infty$, $h_j \in I(k:h_1)$. From the assumed relation $a^{ij}\xi_i\xi_j \geqq C\sum_{i=1}^{m-1}\xi_i^2$, we can deduce

$$\|D_i^h v_h\|_{0,h} \leqq C_6\|v_h\|_{1,h} \leqq C_7 \tag{30}$$

for $i=1, \dots, m-1$. Then by using summation by parts the linear functional

$$l_{i,h}(\varphi)=h^m\sum_{\Omega_h} v_h(D_{-i}^h\varphi)=-h^m\sum_{\Omega_h}(D_i^h v_h)\varphi$$

is bounded:

$$|l_{i,h}(\varphi)|=|(D_i^h v_h, \varphi)_{0,h}| \leqq \|D_i^h v_h\|_{0,h}\|\varphi\|_{0,h} \leqq C_7\|\varphi\|_{0,h}.$$

On the other hand by (29)

$$l_{i,h_j}(\varphi)=\int_\Omega (S_{h_j}v_{h_j})(S_{h_j}D_{-i}^{h_j}\varphi)\,dx-\frac{1}{2}h_j\int_\Gamma (S_{h_j}v_{h_j})(S_{h_j}D_{-i}^{h_j}\varphi)\,d\sigma$$

$$\to \int_\Omega v\frac{\partial\varphi}{\partial x_i}\,dx \equiv l_i(\varphi), \tag{31}$$

$$\|\varphi\|_{0,h_j} \to \|\varphi\|_0$$

as $j\to\infty$. Then we obtain the inequality

$$|l_i(\varphi)|=\left|\int_\Omega v\frac{\partial\varphi}{\partial x_i}\,dx\right| \leqq C_7\|\varphi\|_0.$$

Therefore there exists $v_i \in \mathring{H}_0$ such that

$$l_i(\varphi)=\int_\Omega v\frac{\partial\varphi}{\partial x_i}\,dx=-\int_\Omega v_i\varphi\,dx \tag{32}$$

for all $\varphi \in \mathring{C}^\infty$. When the coefficient $a^{mm}(x)$ is not identically equal zero, we have from the boundedness of v_h in the norm $\|\cdot\|_{1,h}$

$$(a^{mm}D_{-m}^h v_h, D_{-m}^h v_h)_{0,h} \leqq 2\sum_{i=1}^{m-1}|(a^{mi}D_{-m}^h v_h, D_{-i}^h v_h)_{0,h}|+C_8.$$

By the positive semi-definiteness of a^{ij} there exists a positive constant C_9 such that

$$|a^{mi}(x)| \leqq C_9 a^{mm}(x), \quad (i=1, \dots, m-1)$$

in $\bar{\Omega}$, then by using the inequality $ab \leqq \varepsilon a^2+(1/4\varepsilon)b^2$ and by (30) we obtain

$$\|\sqrt{a^{mm}}D_{-m}^h v_h\|_{0,h}^2 \leqq \varepsilon\|\sqrt{a^{mm}}D_{-m}^h v_h\|_{0,h}^2+C_{10}.$$

Setting $\varepsilon=1/2$, we have

$$\|\sqrt{a^{mm}}D_{-m}^h v_h\|_{0,h} \leqq C_{11}. \tag{33}$$

Then the linear functional

$$l_{m,h}(\varphi)=h^m\sum_{\Omega_h} v_h D^h_{-m}(a^{mm}\varphi)=-h^m\sum_{\Omega_h} a^{mm}(D^h_m v_h)\varphi$$

is bounded:

$$|l_{m,h}(\varphi)|=|(a^{mm}D^h_m v_h,\ \varphi)_{0,h}|\leqq C_{12}\|\sqrt{a^{mm}}D^h_m v_h\|_{0,h}\|\varphi\|_{0,h}\leqq C_{13}\|\varphi\|_{0,h}.$$

From the relation

$$\begin{aligned} l_{m,h_j}(\varphi)&=\int_\Omega (S_{h_j}v_{h_j})(S_{h_j}D^{h_j}_{-m}(a^{mm}\varphi))\,dx \\ &-\frac{1}{2}h_j\int_\Gamma (S_{h_j}v_{h_j})(S_{h_j}D^{h_j}_{-m}(a^{mm}\varphi))\,d\sigma \qquad (34)\\ &\to\int_\Omega v\frac{\partial(a^{mm}\varphi)}{\partial x_m}\,dx\equiv l_m(\varphi), \end{aligned}$$

we have

$$|l_m(\varphi)|\leqq C_{13}\|\varphi\|_0.$$

Therefore there exists $v_m\in\mathring{H}_0$ such that

$$l_m(\varphi)=\int_\Omega v\frac{\partial(a^{mm}\varphi)}{\partial x_m}\,dx=-\int_\Omega a^{mm}v_m\varphi\,dx \qquad (35)$$

for all $\varphi\in\mathring{C}^\infty$. The relations (31), (32), (34) and (35) mean that the function v has generalized derivatives v_i with respect to the variables x_i $(i=1,\dots,m)$ and

$$\int_\Omega (S_{h_j}D^{h_j}_{-i}v_{h_j})(S_{h_j}\varphi)\,dx\to\int_\Omega v_i\varphi\,dx\ (i=1,\dots,\mathrm{m}-1) \qquad (36)$$

$$\int_\Omega (S_{h_j}a^{mm})(S_{h_j}D^{h_j}_m v_{h_j})(S_{h_j}\varphi)\,dx\to\int_\Omega a^{mm}v_m\varphi\,dx \qquad (37)$$

as $j\to\infty$ for all $\varphi\in\mathring{C}^\infty$. Thus the bilinear form (16)

$$\begin{aligned} B_h(v_h,\varphi)=&-\int_\Omega (S_h a^{ij})(S_h D^h_{-i}v_h)(S_h D^h_{-j}\varphi)\,dx \\ &-\frac{1}{2}\int_\Omega (S_h v_h)(S_h D^h_{-i}(T^h_i b^i)\varphi+S_h D^h_i(b^i\varphi))\,dx+\int_\Omega (S_h c)(S_h v_h)(S_h\varphi)\,dx \\ &+\frac{1}{2}\int_\Gamma [(S_h b^m)(S_h T^h_{-m}v_h)(S_h\varphi)+(S_h b^m)(S_h v_h)(S_h T^h_{-m}\varphi)]\,d\sigma+0(h) \end{aligned}$$

tends to the one (8) as $j\to\infty$, where we set $h=h_j$. And also by easy computation we have

$$B_h(g,\varphi)\to B(g,\varphi)$$

$$(f,\varphi)_{0,h}\to(f,\varphi)_0$$

as $h\to0$. Consequently the function $v\in\mathring{H}_1$ satisfies the equation

$$B(v,\varphi)=(f,\varphi)_0-B(g,\varphi).$$

Thus the function $u=v+g\in H_1$ satisfies the equation (7), which is what we required to prove.

§5. Modifications.

We shall now mention some modifications of the equation and the domain which are restricted in the previous arguments.

i) Case of elliptic equations. If we set $a^{ij}(x)\xi_i\xi_j>0$ for all $x\in\bar{\Omega}$, the domain Ω for the equation (2) has the boundary $\Sigma_3=\Sigma$ and $\Sigma_1=\Sigma_2=\phi$. Hence the restrictions for the domain are removed and the domain may be considered as more general one. The results of this case coincide with the one of Stummel [7]. It is remarkable that the convergence of the sequence of the generalized solution of difference problem in this case is strong one by using the analogous theorem to Rellich's (cf. [8]).

ii) Case of elliptic-parabolic and parabolic equations. The equation in the previous discussions is of elliptic-parabolic or parabolie type. If $b^m(x)>0$ in the previous assumptions, we exchange Σ_1 for Σ_2 and then the discussions are valid. So the same results hold, if $b^m(x)$ has indefinite sign, for example, $u_{xx}+\sin x u_y=f(x)$. For the elliptic equations which degenerate into parabolic type only on the boundary our discussions hold.

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