Kumamoto J. Sci. (Math.)
9 (1972), 1-9

# ON BOUNDED SOLUTIONS FOR ELLIPTIC EQUATIONS WITH COEFFICIENTS SINGULAR AT THE BOUNDARY IN AN UNBOUNDED DOMAIN

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(Received November 25, 1971)

## 1. Introduction.

In [1], M. Schechter has established in an original way unique solvability for the Dirichlet problem of the second order linear elliptic partial differential equation defined in a bounded domain

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + a(x)u = f(x), \tag{1.1}$$

where $a_{ij}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ are permitted to become infinity at a portion of the boundary. In this paper we shall deal with bounded solutions of the Dirichlet problem for the equation (1.1) defined in an unbounded domain, where $a_{ij}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ are permitted to possess the singularities of the hypotheses similar to those in [1].

In the Euclidean $n$-space of variables $x=(x_1, x_2, \cdots, x_n)$ we shall now consider an unbounded domain $\Omega$ with a suitably smooth boundary $\dot{\Omega}$ (cf. Section 2) such that (i) $\Omega$ lies in the half-space $E=\{x: x_n \geqq 0\}$, (ii) $\dot{\Omega} \cap \dot{E}$ is a nonempty, bounded region of $(n-1)$-dimension, and (iii) for an arbitrarily small $r>0$ and a sufficiently large $R>0$, the set $\Omega \cap \{x: x_n \leqq r\} \cap \{x: |x| \geqq R\}$ is empty. Furthermore, for convenience's sake we shall assume that the points of $\dot{\Omega}$ of singularities of the coefficients $a_{ij}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ in (1.1) are contained in the hyperplane $x_n=0$.

Then we consider the Dirichlet problem for the equation (1.1) with the boundary condition

$$u=\varphi \qquad \text{on } \dot{\Omega}, \tag{1.2}$$

where $\varphi$ is a continuous function prescribed on $\dot{\Omega}$.

Here we also consider the functions $p(t)$ and $q(t)$ satisfying the following conditions:

(1.3) $q(t) \geqq 0$ and $p(t)$ are continuous in $0<t<\infty$ and bounded at infinity.

(1.4) $$\exp\{\int_0^\infty p(s)ds\} \text{ and } \int_0^\infty [\exp\{-\int_z^\infty p(s)ds\}q(z)]\,dz \quad \text{exist,}$$

and for an arbitrary $t_0>0$, the function

(1.5) $$h(t)\equiv\mu\int_0^t \exp\{P(z)\}dz+\int_0^t [\exp\{P(z)\}\int_z^{t_0}\exp\{-P(w)\}q(w)dw]\,dz$$

is defined in $0\leqq t\leqq t_0$, where $P(t)\equiv\int_t^{t_0} p(s)ds$ and $\mu$ is a positive constant.

By virtue of these functions $p(t)$ and $q(t)$, we shall assume the following conditions on (1.1):

(1.6) $$|a_{ij}(x)|,\ |a_i(x)|,\ |a(x)|,\ |f(x)|\leqq q(x_n) \text{ in } \Omega,\ a_n(x)\leqq p(x_n) \text{ in } \Omega,$$

where we have normalized the equation (1.1) by assuming $a_{nn}(x)=1$. Under the above conditions we shall consider the existence and uniqueness of bounded solution of the problem (1.1)–(1.2).

The author wishes to express his hearty thanks to Professors M. Inaba and N. Ikebe for their kind advices and constant encouragements.

## 2. Statement of the theorems.

In order to state our theorems, we should provide the followings.

ASSUMPTIONS (A). In addition to the preceding assumptions (1.6), we shall assume:

(2.1) the functions $a_{ij}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ are $\alpha$-Hölder continuous in any compact subset of $\bar{\Omega}=\Omega\cup\dot{\Omega}$ which does not meet the hyperplane $x_n=0$;

(2.2) $a_{ij}(x)=a_{ji}(x)$, and the operator $L$ is uniformly elliptic in $\Omega$, that is, there exists a positive constant $\lambda$ such that

$$\lambda\sum_{i=1}^n \xi_i^2 \leqq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j$$

for any $x\in\Omega$ and for any real vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$;

(2.3) $a(x)\leqq -c^2<0$, $c$ being a constant;

(2.4) the boundary $\dot{\Omega}$ belongs to class $C^{2+\alpha}$, where $0<\alpha<1$.

ANTI-BARRIER AT INFINITY. In order to consider the treatment of bounded solutions defined in an unbounded domain, we have to introduce the concept of anti-barrier at infinity for the operator $L$. For each number $R>0$, we shall denote by $\Sigma_R$ the subdomain $\Omega\cap\{x:|x|>R\}$ of $\Omega$.

DEFINITION: A function $V(x)$ defined in $\Sigma_R$ will be called an anti-barrier at infinity for the operator $L$, if (i) $V(x)$ is positive and tends to $+\infty$ as $|x|\to\infty$, and (ii) $V(x)$ is of class $C^2(\Sigma_R)$ and satisfies $LV(x)\leqq 0$ in $\Sigma_R$ (cf. [2]).

For example, since $p(t)$ and $q(t)$ are bounded at infinity, the function $V(x)=\prod_{i=1}^{n}\cosh kx_i$ defined in $\Sigma_R$ is an anti-barrier at infinity for the operator $L$, where $k>0$ is a suitable small number.

We shall now state our theorems.

THEOREM 1. *Let the assumptions* (1.6) *and* (A) *be fulfilled. Then there exists a bounded solution of the problem* (1.1)–(1.2) *which belongs to class* $C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$.

THEOREM 2. *Let the assumptions* (1.6) *and* (A) *be fulfilled. Then there exists at most one solution of the problem* (1.1)–(1.2) *which is bounded and belongs to class* $C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$.

## 3. The proof of Theorem 1.

In proving Theorem 1, we shall use the results of Schauder [3] and Schechter [1], which we restate as lemmas of the form most suitable for our purpose.

Let $\mathfrak{D}$ be a bounded domain. Then we define the norm of $u\in C^m(\mathfrak{D})$ by

$$\|u\|_m^{\mathfrak{D}}=\sum_{|k|\leq m}\max_{x\in\mathfrak{D}}|D^k u(x)|,$$

where $D^k=\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdot\frac{\partial^{k_2}}{\partial x_2^{k_2}}\cdots\cdots\frac{\partial^{k_n}}{\partial x_n^{k_n}}$ and $|k|=k_1+k_2+\cdots+k_n$. We define the norm of $u\in C^{m+\alpha}(\mathfrak{D})$ $(0<\alpha<1)$ by

$$\|u\|_{m+\alpha}^{\mathfrak{D}}=\|u\|_m^{\mathfrak{D}}+\max_{|k|=m}\max_{x,y\in\mathfrak{D}}|D^k u(x)-D^k u(y)|/|x-y|^\alpha$$

where $|x-y|$ denotes the distsnce of the points $x$ and $y$.

LEMMA 1 (*Schauder*). *Let the assumptions* (A) *be fulfilled, and assume that* $\|a_{ij}(x)\|_\alpha^{\mathfrak{D}}$, $\|a_i(x)\|_\alpha^{\mathfrak{D}}$, $\|a(x)\|_\alpha^{\mathfrak{D}}$ and $\|f(x)\|_\alpha^{\mathfrak{D}}$ *are bounded by a positive constant* $k_0$. *If $u$ is a function of class $C^{2+\alpha}(\mathfrak{D})$ satisfying the equation*

$$Lu=f \quad in \quad \mathfrak{D},$$

*we have for any subdomain $M$ such that $\bar{M}\subset\mathfrak{D}$ the following interior estimate*

(3.1) $$\|u\|_{2+\alpha}^{M} \leqq K(\|f\|_{\alpha}^{\mathfrak{D}} + \|u\|_{0}^{\mathfrak{D}}),$$

*where* $K$ *is a constant depending only on* $\alpha$, $\lambda$, $k_0$, $M$ *and* $\mathfrak{D}$.

*Furthermore, in case* $\dot{\mathfrak{D}} \in C^{2+\alpha}$ *we consider the problem*;

$$Lu=f \quad \text{in} \quad \mathfrak{D},$$

$$u=\varphi \quad \text{on} \quad \dot{\mathfrak{D}}$$

*where* $\varphi$ *is a prescribed function belonging to class* $C^{2+\alpha}$ *on* $\dot{\mathfrak{D}}$. *If* $M'$ *is any subdomain of* $\mathfrak{D}$ *such that* $M' \cap \dot{\mathfrak{D}}$ *is not void, then we have the following boundary estimate*

(3.2) $$\|u\|_{2+\alpha}^{M'} \leqq K'(\|f\|_{\alpha}^{\mathfrak{D}} + \|u\|_{0}^{\mathfrak{D}} + \|\varphi\|_{2+\alpha}^{\dot{\mathfrak{D}}}),$$

*where* $K'$ *is a constant depending only on* $\alpha$, $\lambda$, $k_0$, $M'$ *and* $\mathfrak{D}$.

We denote by $G_r$ the half-space $x_n \geqq r > 0$. Then we consider a bounded domain $D$ such that $(\Omega - G_r) \subset D \subset \Omega$, and hereafter we assume so.

LEMMA 2 (*Schechter*). *Under the assumptions* (1.6) *and* (A), *consider the Dirichlet problem* (*cf. problem* (1.1)-(1.2))

(3.3) $$Lu=f \quad \text{in} \quad D,$$
$$u=\varphi \quad \text{on} \quad \dot{D}.$$

*Then there exists a solution* $u$ *of class* $C^{2+\alpha}(D) \cap C(\bar{D})$.

Moreover, we consider an interval $[0, t_0]$ such that the strip $0 \leqq x_n \leqq t_0$ contains the domain $D$, and recall the function $h(t)$ defined by (1.4) in the interval $[0, t_0]$. Then we have

LEMMA 3 (*Schechter*). *Let* $u$ *be a solution of the problem* (3.3). *Then the following estimate holds*

(3.4) $$|u(x)| \leqq h(x_n) \quad \text{in} \quad \bar{D}.$$

Now we are ready to prove Theorem 1 by using the above Lemmas.

PROOF OF THEOREM 1. Without loss of generality, we may consider that it is sufficient to prove Theorem 1 in the case $\varphi=0$, that is, the problem:

(3.5) $$Lu=f \quad \text{in} \quad \Omega,$$

(3.6) $$u=0 \quad \text{on} \quad \dot{\Omega}.$$

We first take a domain $\Omega_r$ such that $(G_r \cap \Omega) \subset \Omega_r \subset (G_{r/2} \cap \Omega)$ and

$\dot{\Omega}_r \in C^{2+\alpha}$, where $r>0$ is an arbitrarily small number. Then we consider a sequence of domains $\{D_m\}$ such that, for every integer $m \geqq 1$, $D_m$ is bounded, $(\Omega-\Omega_r) \subset D_m \subset \Omega$, $D_m \subset D_{m+1}$, $D_m$ tends to $\Omega$ as $m \to \infty$, and $\dot{D}_m \in C^{2+\alpha}$. By Lemma 2 we can solve the following problem in each $D_m$:

(3.7) $$Lu_m = f \quad \text{in} \quad D_m,$$

(3.8) $$u_m = 0 \quad \text{on} \quad \dot{D}_m.$$

Therefore we obtain a sequence of functions $u_m(x)$ of class $C^{2+\alpha}(D_m) \cap (\bar{D}_m)$ yielded by the above problem (3.7)-(3.8). Then we can select a convergent subsequence $\{u_{m_\nu}\}$ of $\{u_m\}$ whose limit function satisfies the problem (3.5)-(3.6) and moreover is bounded in $\Omega$. Indeed, let $M$ and $B$ be any subdomains of $\Omega$ such that $\bar{B} \subset M$, $\bar{M}$ is compact and contained in $\Omega$. Then there exists a number $N$ such that $D_m \supset M$ for $m \geqq N$. By the assumptions (A) and the properties of $u_m$ given by Lemma 2, we can apply the Schauder's interior estimate (3.1) in Lemma 1 to each functions $u_m$ for $m \geqq N$:

(3.1)′ $$\|u_m\|_{2+\alpha}^{B} \leqq K(\|f\|_{\alpha}^{M} + \|u_m\|_{0}^{M}),$$

where $K$ does not depend on $m$. Using the estimate (3.4) in Lemma 3, it follows that for $m \geqq N$ the sequence $\{u_m\}$ is equicontinuous and uniformly bounded on $B$. Hence we can extract a subsequence $\{u_{m_\nu}\}$ of $\{u_m\}$ which converges uniformly on $B$ together with the derivatives up to second order. Since $M$ and $B$ are subdomains of $\Omega$, we obtain the limit function $u$ which satisfies $Lu=f$ in $\Omega$ and belongs to class $C^{2+\alpha}(\Omega)$.

Next, we shall observe that the function $u$ also satisfies the boundary condition (3.6). We consider any bounded subdomain $B'$ of $\Omega_r$ such that $\dot{B}' \cap (\dot{\Omega}_r \cap \dot{\Omega})$ is not void. Then we can select a number $N'$ such that $\bar{D}_{m'} \supset B'$ for each $m' \geqq N'$. Let $M' = D_{N'} \cap \Omega_r$. For $m' \geqq N'$ we denote by $\varphi'_m$ the restriction of $u'_m$ on the $(n-1)$-dimensional hypersurface $\dot{M}' \cap \Omega$. Then $u_{m'}$ $(m' \geqq N')$ satisfies the following relations:

$$Lu_{m'} = f \quad \text{in} \quad M',$$

$$u_{m'} = 0 \quad \text{on} \quad \dot{M}' \cap \dot{\Omega}, \qquad = \varphi_{m'} \text{ on } \dot{M}' \cap \Omega,$$

where we may consider that $u_{m'} \in C^{2+\alpha}(M')$ and $\dot{M}' \in C^{2+\alpha}$ (see [4]). Therefore, by the assumptions (A) and the above properties of $u_{m'}$, we can apply the Schauder's boundary estimate (3.2) in Lemma 1 to each functions $u_{m'}(m' \geqq N')$:

$$\|u_{m'}\|_{2+\alpha}^{B'} \leqq K'(\|f\|_{\alpha}^{M'} + \|u_{m'}\|_{0}^{M'} + \|\varphi_{m'}\|_{2+\alpha}^{\dot{M}'}),$$

where $K'$ does not depend on $m'$. Consequently, we can obtain a subsequence $\{u_{m'_\nu}\}$ of $\{u_{m'}\}$ converging uniformly in the neighborhood of $\dot{M}'\cap\dot{\Omega}$ in $\bar{\Omega}$ to a function which agrees with the limit function $u$ obtained above. Since $u_m(x)=0$ on $\dot{\Omega}$ for every $m\geqq 1$ and $r$ can be chosen arbitrarily small, it follows that $u(x)=0$ on the part of $\dot{\Omega}$ which is not on the hyperplane $x_n=0$ and $u$ is continuous on $\bar{\Omega}$ except the part on $x_n=0$.

Furthermore, we have to estimate the function $u$ in a neighborhood of the part of $\dot{\Omega}$ which is on the hyperplane $x_n=0$. By Lemma 3, we have for each $u_m$

$$|u_m(x)|\leqq h(x_n) \quad \text{in } D_m. \tag{3.9}$$

Since by the definition (1.5) $h(t)$ is continuous for all $t\geqq 0$ and converges monotonically to zero as $t\to 0$, $\{u_m(x)\}$ converges to zero uniformly as $x_n\to 0$ in (3.9) in any small neighborhood of the part of $\dot{\Omega}$ which lies on the hyperplane $x_n=0$. On the other hand, $u_m(x)=0$ on $\dot{\Omega}$ for every $m\geqq 1$. Therefore, the limit function which necessarily agrees with $u$ is continuous in any neighborhood of the part of $\dot{\Omega}$ which lies on the hyperplane $x_n=0$, and vanishes on the part of $\dot{\Omega}$ lying on the hyperplane $x_n=0$. Thus, the function $u$ satisfies the problem (3.5)–(3.6) and belongs to class $C^{2+\alpha}(\Omega)\cap C(\bar{\Omega})$.

Finally, we shall observe that the function $u$ obtained above is bounded in $\bar{\Omega}$. Now, we choose a domain $\Omega_r$ suitably and fix it. And we divide the domain $D_m$ into the two domains such that $D_m-\Omega_r$ and $D_m\cap\Omega_r$. Then we shall try to estimate the boundedness of $u_m$ in each domain.

First, from the estimate (3.9), we can observe the uniform boundedness for $u_m$ in the domain $D_m-\Omega_r$:

$$|u_m(x)|\leqq h(r) \text{ in } \overline{D_m-\Omega_r} \text{ for all } m\geqq 1. \tag{3.10}$$

Next, let $\tilde{u}_m$ be the trace of $u_m$ on $D_m\cap\dot{\Omega}_r$. Then we consider the following boundary value problems in the domain $D_m\cap\Omega_r$:

$$\begin{aligned} &Lu_m=f \quad \text{in } D_m\cap\Omega_r,\\ &u_m=\tilde{u}_m \text{ on } D_m\cap\dot{\Omega}_r,\ =0 \text{ on } \dot{D}_m\cap\bar{\Omega}_r. \end{aligned} \tag{3.11}$$

Here, since from the condition (1.3) $a_{ij}(x)$, $a_i(x)$, $a(x)$ and $f(x)$ are bounded in $\bar{\Omega}_r$, we can apply the ordinary maximum principle (cf. [6]) to this problem (3.11) and obtain the following estimate for $u_m$ in the domain $D_m\cap\Omega_r$:

(3.12) $$|u_m(x)|\leqq\max\left\{\sup_{D_m\cap\Omega_r}|f|/\inf_{D_m\cap\Omega_r}|a|,\ \sup_{D_m\cap\dot{\Omega}_r}|\tilde{u}_m(x)|\right\}$$

$$\leqq\max\left\{\sup_{\overline{\Omega}_r}|f|/\inf_{\Omega_r}|a|,\ h(r)\right\}\ \text{in}\ \overline{D_m\cap\Omega_r}\ \text{for all}\ m\geqq1,$$

because $u_m=0$ on $\dot{D}_m\cap\overline{\Omega}_r$ and $|\tilde{u}_m(x)|\leqq h(r)$ in $D_m\cap\dot{\Omega}_r$ for all $m\geqq1$. Hence, from the estimates (3.10) and (3.12), we can obviously observe that $u_m$ is uniformly bounded for all $m\geqq1$. Consequently, it follows that the function $u$ is bounded in $\overline{\Omega}$. We complete the proof of Theorem 1.

## 4. The proof of Theorem 2.

In order to prove Theorem 2, we shall first prepare the extended maximum principle, which is the extended from of the maximum principle in [5], for the following homogeneous case of the boundary value problem (1.1)-(1.2):

(4.1) $$Lu=0 \quad \text{in}\ \Omega,$$

(4.2) $$u=0 \quad \text{on}\ \dot{\Omega}.$$

LEMMA 4. *Let the assumptions* (A) *be fulfilled. Then every bounded solution $u$ of the problem* (4.1)-(4.2) *belonging to class $C^{2+\alpha}(\Omega)\cap C(\overline{\Omega})$ satisfies the extended maximum principle*:

(4.3) $$\max_{x\in\Sigma_R}|u(x)|\leqq\max_{x\in\dot{\Sigma}_R}|u(x)|,$$

*where $R$ is sufficiently large* (*cf. Section* 2).

PROOF. Since, from the example in Section 2, there exists an anti-barrier at infinity for the operator $L$ in $\Sigma_R$, we denote by $V(x)$ an anti-barrier in $\Sigma_R$. Then, consider the following function in $\overline{\Sigma}_R$

$$v(x)=u(x)-\varepsilon V(x)-M,$$

where $M=\max_{x\in\dot{\Sigma}_R}|u(x)|$ and $\varepsilon$ is an arbitrary positive number. Since $R$ is sufficiently large, by the definition of anti-barrier at infinity and conditions (4.1)-(4.2), we have

(4.4) $$Lv=Lu-\varepsilon LV-aM\geqq0 \quad \text{in}\ \Sigma_R,$$

(4.5) $$v=u-\varepsilon V-M\leqq0 \quad \text{on}\ \dot{\Sigma}_R.$$

It follows that $v\leqq0$ in $\Sigma_R$. In fact, let us suppose that there is a point $x_0$ in $\Sigma_R$ such that $v(x_0)>0$. Since $v\to-\infty$ as $|x|\to\infty$, $v$ must attain its positive maximum at some bounded point $x^*$ in $\Sigma_R$. Then by this fact and the relation (4.4), it follows that, from E. Hopf's strong maximum principle, $v(x)\equiv v(x^*)=\text{const}$ in $\Sigma_R$. This is obviously absurd. Thus we obtain

$$u\leqq M \quad \text{in } \bar{\Sigma}_R, \tag{4.6}$$

since $\varepsilon$ is arbitrary. Also replacing $u$ by $-u$, we have in the same way

$$u\geqq -M \quad \text{in } \bar{\Sigma}_R. \tag{4.7}$$

Thus, by (4.6) and (4.7) we obtain the result (4.3).

Now we are ready to prove Theorem 2.

PROOF OF THEOREM 2. Let $u_1$ and $u_2$ are two solutions of the problem (1.1)–(1.2). And set $w=u_1-u_2$. Then, in order to prove Theorem 2, it is sufficient to show $w\equiv 0$ in $\bar{\Omega}$.

First, we can easily observe that $w$ satisfies the homogeneous boundary value problem (4.1)–(4.2). Next, let us assume that there is a point $x_0$ in $\Omega$ such that $w(x_0)\neq 0$. Then, by the continuity of $w$ in $\bar{\Omega}$ and the extended maximum principle for $w$ obtained by Lemma 4, we may consider that, without loss of generality, $w$ must attain its positive maximum at some bounded point $x^*$ in $\Omega$. Take $R$ sufficiently large that $x^*$ is contained within the domain $\hat{\Omega}_R=\Omega-\bar{\Sigma}_R$. Furthermore, consider any bounded subdomain $B$ of $\hat{\Omega}_R$ such that it contains $x^*$ in its interior and does not meet the hyperplane $x_n=0$. Then, by the fact that $Lw=0$ in $B$ and $w(x^*)$ is a positive maximum in $B$, it follows that, from E. Hopf's strong maximum principle, $w(x)\equiv w(x^*)$ $=\text{const.}$ in $B$. Since $B$ is arbitrary, we have $w\equiv 0$ in $\hat{\Omega}_R$, because $w$ is continuous on $\bar{B}$ and vanishes on $\dot{\Omega}$. Moreover, since we can apply Lemma 4 to $w$ in the domain $\Sigma_R$, we obtain the following:

$$w(x)\equiv 0 \quad \text{in } \bar{\Omega}.$$

Thus, the bounded solution of the boundary value problem (1.1)–(1.2) is unique. We complete the proof of Theorem 2.

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