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A FINITE ELEMENT METHOD FOR THE SOLUTIONS OF FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS

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0. Introduction

In seeking accurate approximate solutions to fourth order partial differential equations, one meets a special difficulty due to fourth order derivatives appearing in the equations.

The methods employed usually for these equations can be classified into two classes. The methods in one class are based on variational principles and those in other class are based on finite differences.

A typical method in the former class is, as well known, a method of Ritz-Galerkin. In this method one chooses a suitable system of functions satisfying the given boundary conditions-which is called a basis or a coordinate system-and seeks an approximate solution in the form of a linear combination of a finite number of functions of the basis.

A weak point of this method is the difficulty of the construction of a basis, in particular, for a domain with a complicate boundary. This weak point, however, seems to have been overcome by the use of finite element method which is based on the same principle. Finite element method, however, has yielded another difficulty, that is, the difficulty due to the explosive increase of unknown parameters to be determined. This difficulty may be avoided by the use of "non-conforming basis", but by the use of it the accuracy of the approximate solution obtained is decreased in general (see [6]).

The difficulty of the treatment of boundary conditions appears also in finite difference methods. Moreover, finite difference methods seem to be inferior considerably to finite element methods in the efficiency when these methods are practised on a computer.

In the present paper we propose a new finite element method for fourth order equations. Though our method may be regarded as a new mixed finite element method, our motivation comes from the coupled equation approach to biharmonic equation. In our method the smoothness of the basis is not required since the fourth order equation is reduced to a system of second order equations. Moreover numerical experiments show that the accuracy of the approximate solutions obtained is acceptable in proportion to the number of unknown parameters employed. It should be noted that in a certain case our method can be regarded as a generalized finite difference method for fourth order equations.

Although only biharmonic equations are treated in the present paper, our method can be applied to more general fourth order equations without difficulty.

Chapter (I) is concerned with a static problem for biharmonic operator. Our algorithm is proposed and the solvability of an approximate linear system and the convergence of the solution of this linear system are proved. Chapter (II) is devoted to a dynamic problem. By giving some stability criteria and convergence proof, it is shown that our algorithm for static problems can be extended to dynamic problems in a natural way. The first half of this chapter is motivated by the work [2] of H. Fujii. The appendix shows the results of some numerical experiments.

Chapter (I). Static problems

1. A weak representation of the biharmonic boundary value problem.

Let Ω be a bounded domain in (x_1, x_2)-plane with sufficiently smooth boundary Γ. Consider the biharmonic equation

$$\Delta^2 u = \Delta(\Delta u) = \partial_{1111} u + 2\partial_{1122} u + \partial_{2222} u = f(x_1, x_2) \quad \text{in } \Omega \ (\partial_{1111} = \partial^4/\partial x_1 x_1 x_1 x_1 \text{ etc.}), \tag{1.1}$$

under the following mixed boundary condition.

$$u \Big|_{\Gamma_1} = 0 \qquad \frac{du}{dn} \Big|_{\Gamma_1} = 0, \tag{1.2}$$

$$u \Big|_{\Gamma_2} = 0 \qquad M(u) \Big|_{\Gamma_2} = 0, \tag{1.3}$$

$$M(u) \Big|_{\Gamma_3} = 0 \qquad V(u) \Big|_{\Gamma_3} = 0, \tag{1.4}$$

where

(1.5) $$M(u)=\nu\Delta u+(1-\nu)(\partial_{11}u\cos^2(n,x_1)+2\partial_{12}u\cos(n,x_1)\cos(n,x_2)+\partial_{22}u\cos^2(n,x_2)),$$

(1.6) $$V(u)=(1-\nu)\frac{d}{ds}[(\partial_{11}u-\partial_{22}u)\cos(n,x_1)\cos(n,x_2)-\partial_{12}u(\cos^2(n,x_1)-\cos^2(n,x_2)]-\frac{d}{dn}\Delta u$$

In the above expressions ν denotes a positive constant less than unity and $\cos(n,x_i)$ denotes the directional cosine of outward normal n to Γ. d/ds denotes the differentiation along the boundary Γ. Γ_i (i=1, 2, 3) denotes a portion of Γ such that $\Gamma=\sum_i\oplus\Gamma_i$. Through the present paper we assume, for brevity, that the length of Γ_1 is positive.

To derive our method we shall rewrite this boudary value problem in a more "weak" form. We use the notation (u,v) for the inner product of functions u and v in $L^2(\Omega)$-space. Let u be the exact solution of the present problem. It is evident that

(1.7) $$(\Delta^2u,\varphi)+\int_{\Gamma_{2,3}}M(u)\frac{d\varphi}{dn}ds+\int_{\Gamma_3}V(u)\varphi\,ds=(f,\varphi)$$

for any smooth function φ satisfying $\varphi|_{\Gamma_{1,2}}=0$ ($\Gamma_{i,j}=\Gamma_i\oplus\Gamma_j$). This equality can be written by Green's theorem also as follows.

$$\begin{aligned}
&-(\partial_1\Delta u,\partial_1\varphi)-(\partial_2\Delta u,\partial_2\varphi)+\int_{\Gamma_{2,3}}M(u)\frac{d\varphi}{dn}ds+\int_{\Gamma_3}\frac{d}{ds}Q(u)\varphi ds\\
&=-(\partial_1[\partial_{11}u+\nu\partial_{22}u],\partial_1\varphi)\\
&\quad-(1-\nu)[(\partial_1\partial_{12}u,\partial_2\varphi)+(\partial_2\partial_{12}u,\partial_1\varphi)]\\
&\quad-(\partial_2[\nu\partial_{11}u+\partial_{22}u],\partial_2\varphi)\\
&\quad+\int_{\Gamma_{2,3}}M(u)\frac{d\varphi}{dn}ds-\int_{\Gamma_3}Q(u)\frac{d\varphi}{ds}ds\\
&=(f,\varphi),
\end{aligned}$$

where

$$Q(u)=(1-\nu)[(\partial_{11}u-\partial_{22}u)\cos(n,x_1)\cos(n,x_2)-\partial_{12}u(\cos^2(n,x_1)-\cos^2(n,x_2))].$$

Define

(1.8) $$U_{ij}=\partial_{ij}u \qquad (i,j=1,2),$$

and

(1.9) $$K_{ij}(\varphi,\psi)=\frac{1}{2}\{(\partial_i\varphi,\partial_j\psi)+(\partial_j\varphi,\partial_i\psi)$$
$$-\int_{\Gamma_{2,3}}(\partial_i\varphi\cos(n,x_j)+\partial_j\varphi\cos(n,x_i))\psi ds\}$$
$$(i,j=1,2)$$

for sufficiently smooth φ and ψ.
Then the left side of the equality (1.7) can be expressed as

$$-(\partial_1[U_{11}+\nu U_{22}],\partial_1\varphi)$$
$$-(1-\nu)[(\partial_1 U_{12},\partial_2\varphi)+(\partial_2 U_{12},\partial_1\varphi)]$$
$$-(\partial_2[\nu U_{11}+U_{22}],\partial_2\varphi)$$
$$+\int_{\Gamma_{2,3}}M(U)\frac{d\varphi}{dn}ds-\int_{\Gamma_3}Q(U)\frac{d\varphi}{ds}ds$$
$$=-K_{11}(\varphi,U_{11}+\nu U_{22})-2(1-\nu)K_{12}(\varphi,U_{12})-K_{22}(\varphi,\nu U_{11}+U_{22}),$$

where $M(U)$ and $Q(U)$ denote $M(u(U))$ and $Q(u(U))$ respectively. Therefore if we denote the right side of the above equality by $-K(\varphi,U)$, then the exact solution $\{u,U\}$ satisfies the following equation.

(1.10) $$\begin{cases} K_{ij}(u,\psi)+(U_{ij},\psi)=0 & \text{for any } \psi \ (i,j=1,2), \\ K(\varphi,U)+(f,\varphi)=0 & \text{for any } \varphi;\varphi|_{\Gamma_{1,2}}=0. \end{cases}$$

This representation of the solution u is very convenient for the finite element approximation since in this expression only the first derivatives are employed. Our method proposed in this paper is a Galerkin-type appoximation method for this system of equations.

We use the following bi-linear form as the "*energy form*" of the present problem.

(1.11) $$E(U,V)=\iint_\Omega[U_{11}V_{11}+U_{22}V_{22}+\nu(U_{11}V_{22}+U_{22}V_{11})$$
$$+2(1-\nu)U_{12}V_{12}]dx_1dx_2,$$

where $U=(U_{11},U_{12},U_{22})$ and $V=(V_{11},V_{12},V_{22})$ are vectors defined by (1.8) and consisting of the second derivatives of u and v. Note that if v satisfies the boundary condition (1.2), then since $K_{ij}(v,\varphi)=-(V_{ij},\varphi)$ it holds that

(1.12) $$K(v,W)+E(V,W)=0 \qquad \text{for any } W.$$

2. Approximations of consistent and lumped mass type

In what follows we assume, for brevity, that Ω is a polygon.

DEFINITION 1. We say a sequence of triangulations Ω_h of Ω is regular as $h\to 0$ (or briefly, triangulation Ω_h is regular) if,
(*a*) each end point of Γ_i is a vertex of a certain triangle of Ω_h,
(*b*) the smallest angle $\theta(\Omega_h)$ of all the triangles of Ω_h is bounded below by a positive constant θ_0 as $h\to 0$.

Hereafter we assume that Ω_h denotes always a regular decomposition of Ω. Let p be a vertex in Ω_h and $\triangle_{pk}$ $(k=1,2,\ldots,K_p)$ be the set of all the triangles in Ω_h with vertex p. Let the three vertexes of $\triangle_{pk}$ be denoted by p,q and r and the center of gravity by c. Let us denote by $\square_{pk}$ the quadrilateral with four vertexes $\{p$, *centers of* $\overline{pq}$ *and* $\overline{pr}$, $c\}$. Let

$$T_p=\bigcup_k \triangle_{pk}$$

$$S_p=\bigcup_k \square_{pk}.$$

DEFINITION 2. $\{\varphi_p(x_1,x_2)\}$ (p moves all the vertexes of Ω_h) is a system of linearly independent functions such that (i) linear in each triangle and continuous on Ω_h (ii) $\varphi_p=0$ on Ω_h-T_p and $=1$ at p. $\{\chi_p(x_1,x_2)\}$ (p moves all the vertexes of Ω_h) is a system of the characteristic function of S_p.

REMARK. We can, of course, use more "accurate" but little complicated basis (see, for example [7]). Though the following discussion is valid for such cases we restrict the basis to the so called "piecewise linear" basis (See the last remark of Chapter (II)).

We shall number the vertexes in Ω_h as follows. The first n vertexes (denoted by N_0) is in $\Omega_h-\partial\Omega_h$. The next n_1 vertexes (denoted by N_1) is on $\overline{\Gamma}_1$. The third n_2 vertexes (denoted by N_2) is on $\overline{\Gamma}_2-\overline{\Gamma}_1$ and the last n_3 vertexes (denoted by N_3) is in the interior of Γ_3.
Now let

(2.1) $$\hat{u}(x_1,x_2)=\sum_{N_0,N_3} u_p\ \varphi_p(x_1,x_2),$$

(2.2) $$\bar{u}(x_1,x_2)=\sum_{N_0,N_3} u_p\ \chi_p(x_1,x_2),$$

$$\hat{U}_{ij}(x_1, x_2)=\sum_{\Omega_h} U_p^{ij}\ \varphi_p(x_1, x_2), \tag{2.3}$$

$$\bar{U}_{ij}(x_1, x_2)=\sum_{\Omega_h} U_p^{ij}\ \chi_p(x_1, x_2), \tag{2.4}$$

where $\{u_p\}$, $\{U_p^{ij}\}$ are the unknown coefficients.

Algorithm (C): In this procedure these unknowns are determined by solving the following system of linear equations.

$$\begin{cases} K_{ij}(\hat{u}, \varphi_p)+(\hat{U}_{ij}, \varphi_p)=0 & p\in\Omega_h\ (i, j=1, 2) \\ K(\varphi_p, \hat{U})+(f, \varphi_p)=0 & p\in N_0, N_3, \end{cases} \tag{2.5}$$

where $\hat{U}=(\hat{U}_{11}, \hat{U}_{12}, \hat{U}_{22})$.

Algorithm (L): In this procedure the equation to be solved is

$$\begin{cases} K_{ij}(\hat{u}, \varphi_p)+(\bar{U}_{ij}, \chi_p)=0 & p\in\Omega_h\ (i, j=1, 2) \\ K(\varphi_p, \hat{U})+(f, \varphi_p)=0 & p\in N_0, N_3. \end{cases} \tag{2.6}$$

REMARK. In algorithm (C) $\hat{U}_{ij}$ is given in a implicit form, but in algorithm (L) explicitly, since $\chi_p\chi_q=0$ if $p\neq q$. Note that in a certain triangulation, for example, see Friedrichs-Keller [1], the system (2.6) is essentially equivalent to a finite difference approximation.

Through the present paper we put the following assumption.

ASSUMPTION: Let $\hat{U}_{ij}(\bar{U}_{ij})$ be determined by the equation (2.5) ((2.6)). Then holds the following inequality.

$$\|\hat{u}\|\leqq C\sum_{i,j}\|\hat{U}_{ij}\| \quad (C\sum_{i,j}\|\bar{U}_{ij}\|), \tag{2.7}$$

where C is a positive constant independent of h as $h\to 0$.(*)

If $\Gamma_3=\phi$ this inequality can be easily shown, that is, we have the following more strong inequality.

LEMMA 2.1. *Let $\Gamma_3=\phi$ then holds*

$$\|\partial u\|\leqq C\sum_{i,j}\|\hat{U}_{ij}\| \quad (C\sum_{i,j}\|\bar{U}_{ij}\|). \tag{2.8}$$

(*) In what follows we use C as generic constants which are not nocessarily the same.

where

$$\|\partial u\|=(\partial u,\partial u)^{\frac{1}{2}}=(\sum_i(\partial_i u,\partial_i u))^{\frac{1}{2}}.$$

PROOF. By the equality (2.5) we have

$$K_{ij}(\hat{u},\hat{u})+(\hat{U}_{ij},\hat{u})=0.$$

But, since $\Gamma_3=\phi$, $\hat{u}$ vanishes on Γ and hence

$$K_{ij}(\hat{u},\hat{u})=(\partial_i\hat{u},\partial_j\hat{u})\ (i,j=1,2),$$

so follows the inequality (2.8) by Poincare's inequality. (The second inequality follows from the fact that the quantities $\|\hat{U}_{ij}\|$ and $\|\bar{U}_{ij}\|$ are of the same order).

If $\Gamma_3\neq\phi$, though it is not clear whether the inequality (2.7) holds or not, but in a certain situation we can easily prove it. For example, suppose that Γ_3 can be represented by a function of one variable, for example, by

$$x_2=g(x_1),\ s_{i_0}\leqq x_1\leqq s_{i_k}.$$

Let $p_i=(s_i,g(s_i))\,(i=i_0\sim i_k)$ be the vertex points on Γ_3. If the intersection of Ω_h and the lines $x_1=s_i(i=i_0\sim i_k)$ is represented by a union of the sides of some triangles in Ω_h and if the strip $(s_{i_0},s_{i_k})\times x_2$ does not contain Γ_2, then holds the inequality (2.7). To show this let us define a function $\hat{\psi}(x_1,x_2)$ by

$$\hat{\psi}(x_1,x_2)=\begin{cases}0 & x_1\leqq s_{i_0}\\ \hat{u}\,|_{\Gamma_3} & s_{i_0}\leqq x_1\leqq s_{i_k}\\ 0 & s_{i_k}\leqq x_1\end{cases}$$

Clearly $\hat{\psi}(x_1,x_2)$ is continuous in the whole plane and can be represented by $\{\varphi_p\}$ in the form

$$\hat{\psi}(x_1,x_2)=\sum_p\hat{\psi}(p)\varphi_p(x_1,x_2)$$

Since $\partial_2\hat{\psi}=0$ we have by (2.5)

$$-(\partial_2\hat{u},\partial_2\hat{u})=(\hat{U}_{22},\hat{u}-\hat{\psi}). \tag{2.9}$$

On the other hand it is easy to prove that

$$\|\hat{\psi}\|^2\leqq C\int_{\Gamma_3}(\hat{u})^2ds\leqq C\int_{\Omega_h}(\partial_2\hat{u})^2\,dx_1dx_2,$$

and also the inequality

$$\|\hat{u}\| \leqq C\|\partial_2 \hat{u}\|.$$

Therefore (2.9) implies

$$\|\hat{u}\| \leqq C\|\hat{U}_{22}\|.$$

Which is the desired inequality.

Under this assumption we have

THEOREM I–1. *Both in algorithm (C) and in algorithm (L) the unknowns are uniquely determined.*

PROOF. Let us first consider the algorithm (C). Let $(f, \varphi_p)=0$ for $p \in N_0+N_3$. Multiplying the second equality of (2.5) by u_p and summing up on p we have

$$K(\hat{u}, \hat{U})=0. \tag{2.10}$$

On the other hand as an analogue of (1.12) we have

$$K(\hat{u}, \hat{U})+E(\hat{U}, \hat{U})=0.$$

Therefore (2.10) implies $\hat{U}_{ij}=0$ $(i, j=1, 2)$ and this implies $\hat{u}=0$ by the inequality (2.7). Therefore the unknowns are uniquely determined. The unique solvability in algorithm (L) can be proved by the same way.

The system (2.5) or (2.6) is, in a certain sense, a Galerkin-type approximation of the original problem. As well known Galerkin's method coincides with Rayleigh-Ritz's method in many cases. The following theorem shows that the our method can be understood as a conditional Rayleigh-Ritz's method.

THEOREM I–2.

(A) *Algorithm(C) is equivalent to the following algolithms.*

(i) *Seek the minimizing function of the functional*

$$F_1(\hat{u})=E(\hat{U}, \hat{U})-2(f, \hat{u}) \tag{2.11}$$

when $\hat{u}$ *and* $\hat{U}$ *are connected by the following equations.*

$$K_{ij}(\hat{u}, \varphi_p)+(\hat{U}_{ij}, \varphi_p)=0 \qquad p \in \Omega_h \quad (i, j=1, 2) \tag{2.12}$$

(ii) *Seek the stationary function of the functional*

(2.13) $$B_1[\hat{u}, \hat{U}]=2K(\hat{u}, \hat{U})+E(\hat{U}, \hat{U})+2(f, \hat{u}).$$

(B) *Algorithm (L) is equivalent to the following algorithms.*

(i) *Seek the minimizing function of the functional*

(2.14) $$F_2(\hat{u})=E(\bar{U}, \bar{U})-2(f, \hat{u})$$

when $\hat{u}$ *and* $\bar{U}$ *are connected by the following equation.*

(2.15) $$K_{ij}(\hat{u}, \varphi_p)+(\bar{U}_{ij}, \chi_p)=0 \qquad p\in\Omega_h \quad (i, j=1, 2)$$

(ii) *Seek the stationary function of the functional*

(2.16) $$B_2[\hat{u}, \bar{U}]=2K(\hat{u}, \bar{U})+E(\bar{U}, \bar{U})+2(f, \hat{u}).$$

PROOF. (A): First we see that

$$\frac{\partial}{\partial u_q} E(\hat{U}, \hat{U})=2E(\hat{U}, \frac{\partial}{\partial u_q}\hat{U})$$

On the other hand, since

$$K_{ij}(\varphi_q, \hat{U}_{kl})+(\frac{\partial}{\partial u_q}\hat{U}_{ij}, \hat{U}_{kl})=0 \qquad (k, l=1, 2) \quad (i, j=1, 2)$$

we have

$$K(\varphi_q, \hat{U})+E(\hat{U}, \frac{\partial}{\partial u_q}\hat{U})=0,$$

and hence

$$\frac{\partial}{\partial u_q} E(\hat{U}, \hat{U})=-2K(\varphi_q, \hat{U}).$$

Therefore the stationary condition of $F_1(\hat{u})$ is nothing else but the equation (2.5). This proves (i) of (A). The second assertion is almost evident. (B) can be also proved by the same way.

REMARK. This theorem shows that our algorithms are regarded as a kind of mixed method in finite element method. See the survey paper of Pian [5].

3 Convergence of approximate solutions

DEFINITION 2. Let h be the length of the largest sides of the triangles in Ω_h. We say the sequence $\{\Omega_h\}$ is a nearly consistent triangulation of Ω (or briefly, decomposition Ω_h is nearly consistent) if decomposition Ω_h is regular and there exists a sequence of subdomains Ω'_h of Ω_h such that

(i) there is a definite constant C such that (the number of vertexes in $\Omega_h-\bar{\Omega}'_h)\leqq Ch^{-1}$.

(ii) for any sufficiently smooth function w satisfying the boundary condition (1.2) holds

$$K_{ij}(\hat{w}, \varphi_p)=\begin{cases} -\partial_{ij}w\big|_p \cdot (1, \varphi_p)+Ch^3 & \text{if } p\in\bar{\Omega}'_h \\ Ch^2 & \text{if } p\in\Omega_h-\bar{\Omega}'_h \end{cases} \tag{3.1}$$

where $\hat{w}$ is the interpolating function of w represented by using the basis $\{\varphi_p\}$.

We remark that there are some triangulations having the above properties. For example, see [1]. We further remark that, if Ω_h is a nearly consistent triangulation, we can assume the following equalities.

$$K_{ij}(\varphi_p, \hat{w})=\begin{cases} -\partial_{ij}w\big|_p \cdot (1, \varphi_p)+Ch^3 & \text{if } p\in\bar{\Omega}'_h, \\ \frac{1}{2}\int_{\Gamma_{2,3}}\Big\{(\partial_i\hat{w}\cos(n, x_j)+\partial\hat{w}_j\cos(n, x_i))\varphi_p \\ \quad -(\partial_i\varphi_p\cos(n, x_j)+\partial_j\varphi_p\cos(n, x_i))\hat{w}\Big\}\,ds \\ \quad +Ch^2 & \text{if } p\in\Gamma. \end{cases} \tag{3.2}$$

Now we shall prepare some lemmas necessary for the following discussion.

LEMMA I-1. *Let $\hat{u}$ be a linear function on a triangle $\triangle$ in Ω_h. Then holds the following inequality.*

$$\operatorname*{Max}_{\triangle}|\hat{u}|\leqq Ch^{-1}\|\hat{u}\|_{\triangle} \tag{3.3}$$

PROOF. Since Ω_h is regular triangulation we can assume without loss of generality that $\triangle$ is the triangle with vertexes $(0,0), (0,h)$ and $(h,0)$. Then by a direct calculation we have

(3.4)
$$\|\hat{u}\|_{\triangle}^2=\frac{h^2}{24}\Big\{[\hat{u}(0,0)+\hat{u}(0,h)+\hat{u}(h,0)]^2 + \hat{u}^2(0,0)+\hat{u}^2(0,h)+\hat{u}^2(h,0)\Big\},$$

which proves (3.3).

LEMMA I–2 *Let $\hat{u}$ be a function represented by $\{\varphi_p\}$. Then holds the following inequality.*

(3.5)
$$\sqrt{\sum_{p\in\Omega_h} u_p^2h^2}\leqq C\|\hat{u}\|.$$

PROOF. Let $\triangle$ be a triangle in Ω_h and p,q and r be the three vertexes of $\triangle$. Then by the Lemma I–1 we have

(3.6)
$$\|\hat{u}\|_{\triangle}^2\geqq Ch^2\underset{\triangle}{\mathrm{Max}}|\hat{u}|^2\geqq Ch^2\frac{1}{3}(u_p^2+u_q^2+u_r^2).$$

Summing up this inequality for all triangles in Ω_h we obtain the desired inequality.

The finite element method using conforming basis (the classical Ritz-Galerkin's method) can be regarded as a least square method in energy norm. Therefore we can use the results of approximation theory directly to prove the convergence of approximate solutions. But this property does not exist in our case, since the functional $F(\hat{u})$ is minimized under an additional condition. The next theorem is basic for our convergence proof. Let $\{u,U\}$ be the exact solution of the present problem, and $\hat{U}$ and $\bar{U}$ be the interpolations of U represented by $\{\varphi_p\}$ and $\{\chi_p\}$ respectively.

THEOREM I–3. *$F_1(\hat{u})$ and $F_2(\hat{u})$ can be represented also as follows.*

(3.7)
$$F_1(\hat{u})=E(U-\hat{U},U-\hat{U})-E(U,U)-2E(U-\hat{U},U-\hat{\hat{U}}) -2K(u-\hat{u},U-\hat{\hat{U}}),$$

(3.8)
$$F_2(\hat{u})=E(U-\bar{U},U-\bar{U})-E(U,U)-2E(U-\bar{U},U-\bar{\hat{U}}) -2K(u-\hat{u},U-\hat{\hat{U}})-2E(U,\bar{\hat{U}}-\hat{\hat{U}}).$$

PROOF. Proof of (3.7). We observe that

$$K(u, U) = -E(U, U),$$
$$K(u, \hat{U}) = -E(U, \hat{U}),$$
$$K(\hat{u}, \hat{U}) = -E(\hat{U}, \hat{U}),$$
$$K(\hat{u}, U) = -(f, \hat{u}).$$

Using these representations of "K", the right side of (3.7) becomes

$$E(\hat{U}, \hat{U}) - 2(f, \hat{u})$$

which proves (3.7).
Equality (3.8) can be also proved by the same way.

Let us determine $\mathring{\hat{U}}_{ij}$ and $\mathring{\bar{U}}_{ij}$ by the following equations respectively.

(3.9) $$K_{ij}(\hat{u}, \varphi_p) + (\mathring{\hat{U}}_{ij}, \varphi_p) = 0 \qquad p \in \Omega_h,$$

(3.10) $$K_{ij}(\hat{u}, \varphi_p) + (\mathring{\bar{U}}_{ij}, \chi_p) = 0 \qquad p \in \Omega_h, \quad (i, j = 1, 2)$$

where $\hat{u}$ is the interpolation of the exact solution u(*).

LEMMA I–4. *Let* $\{\hat{u}, \hat{U}\}$ *be the approximate solution. Then*

(3.11) $$\|u - \hat{u}\| \leqq C(h^{\frac{1}{2}} + E(U - \hat{U}, U - \hat{U})^{\frac{1}{2}})$$

in algorithm(C), *and*

(3.12) $$\|u - \hat{u}\| \leqq C(h^{\frac{1}{2}} + E(U - \bar{U}, U - \bar{U})^{\frac{1}{2}})$$

in algorithm(L).

PROOF. (algorithm(C)). Since

$$K_{ij}(\hat{\hat{u}} - \hat{u}, \varphi_p) + (\mathring{\hat{U}}_{ij} - \hat{U}_{ij}, \varphi_p) = 0 \qquad p \in \Omega_h,$$

we have

$$\|\hat{\hat{u}} - \hat{u}\| \leqq C \sum_{i,j} \|\mathring{\hat{U}}_{ij} - \hat{U}_{ij}\|,$$

by inequality (2.7), and hence

*) We assume that the solution u is sufficiently smooth in $\bar{\Omega}$.

(3.13) $$\|u-\hat{u}\| \leqq \|u-\hat{\mathring{u}}\| + C\sum_{i,j}(\|U_{ij}-\hat{\mathring{U}}_{ij}\| + \|U_{ij}-\hat{U}_{ij}\|).$$

The second term of the right side can be estimated as follows. First we see

$$K_{ij}(u-\hat{u}, \varphi_p) + (U_{ij}-\hat{\mathring{U}}_{ij}, \varphi_p) = 0,$$

where

$$K_{ij}(u-\hat{u}, \varphi_p) = \begin{cases} Ch^3 & p \in \bar{\Omega}'_h \\ Ch^2 & p \in \Omega_h - \bar{\Omega}'_h. \end{cases}$$

Therefore by Lemma I–2

$$\begin{aligned}(U_{ij}-\hat{\mathring{U}}_{ij}, \hat{U}_{ij}-\hat{\mathring{U}}_{ij}) \\ &= C[h^3 \sum_{p \in \bar{\Omega}'_h} (\hat{U}_{ij}(p)-\hat{\mathring{U}}_{ij}(p)) \\ &\qquad + h^2 \sum_{p \in \Omega_h - \bar{\Omega}'_h} (\hat{U}_{ij}(p)-\hat{\mathring{U}}_{ij}(p))] \\ &\leqq Ch^{\frac{1}{2}} \|\hat{U}_{ij}-\hat{\mathring{U}}_{ij}\|,\end{aligned}$$

that is,

$$\|U_{ij}-\hat{\mathring{U}}_{ij}\| \leqq Ch^{\frac{1}{2}}.$$

In the above calculation we used a well known interpolation theorem. This proves (3.11). (3.12) can be proved by the same way.

THEOREM I–4. *Let $\hat{U}$ and $\bar{U}$ be the approximate solutions obtained by algorithm (C) and algorithm(L) respectively. Then*

(3.14) $$E(U-\hat{U}, U-\hat{U})^{\frac{1}{2}}, E(U-\bar{U}, U-\bar{U})^{\frac{1}{2}} \leqq Ch^{\frac{1}{2}}$$

as $h \to 0$.

PROOF. By Theorem I–2 we know $F_1(\hat{u}) \leqq F_1(\hat{\mathring{u}})$, therefore Theorem I–3 implies

(3.15) $$\begin{aligned}&E(U-\hat{U}, U-\hat{U}) - E(U, U) - 2E(U-\hat{U}, U-\hat{U}) \\ &\qquad -2K(u-\hat{u}, U-\hat{U}) \\ &\leqq E(U-\hat{\mathring{U}}, U-\hat{\mathring{U}}) - E(U, U) - 2E(U-\hat{\mathring{U}}, U-\hat{U}) + Ch^2.\end{aligned}$$

The fourth term of the left side is bounded by Ch^2 plus, by (3.2),

$$
\begin{aligned}
|2K(\hat{u}-\hat{u}, U-\hat{U})| &\leqq C \sum_{p\in\bar{\Omega}'_h} h^3 |\hat{u}(p)-\hat{u}(p)| \\
&\quad + C \sum_{p\in\Omega_h-\bar{\Omega}'_h} h^2 |\hat{u}(p)-\hat{u}(p)| \\
&\leqq C\sqrt{\textstyle\sum_1 h^4}\ \sqrt{\textstyle\sum_1 (\hat{u}(p)-\hat{u}(p))^2 h^2} \\
&\quad + C\sqrt{\textstyle\sum_2 h^2}\ \sqrt{\textstyle\sum_2 (\hat{u}(p)-\hat{u}(p))^2 h^2} \\
&\leqq Ch^{\frac{1}{2}} \|\hat{u}-\hat{u}\| \leqq Ch^{\frac{1}{2}}(\|u-\hat{u}\|+\|u-\hat{u}\|) \\
&\leqq C(h+h^{\frac{1}{2}}E(U-\hat{U}, U-\hat{U})^{\frac{1}{2}}) \quad \text{(by Lemma I–4).}
\end{aligned} \tag{3.16}
$$

Now define

$$\lambda = E(U-\hat{U}, U-\hat{U})^{\frac{1}{2}}, \quad \mu = E(U-\hat{\mathring{U}}, U-\hat{\mathring{U}})^{\frac{1}{2}}.$$

Then (3.15), (3.16) imply

$$\lambda^2 \leqq \mu^2 + 2Ch^2\mu + 2Ch^{\frac{1}{2}}\lambda + Ch.$$

Since $\mu \leqq Ch^{\frac{1}{2}}$ is shown in the proof of Lemma I–4, the above inequality implies the first inequality of the present theorem. The second one is also proved by the same way.

From Lemma I–4 and Theorem I–4 follows the next theorem, which is the conclusion of this chapter.

THEOREM I–5. *Let Ω be a polygon and the triangulation is nearly consistent. If the exact solutoin $\{u, U\}$ is sufficiently smooth in $\bar{\Omega}$, then the approximate solutions converge to the exact solution both in algorithm(C) and (L), and the order of convergence is given by*

$$\|u-\hat{u}\|,\ E(U-\hat{U}, U-\hat{U}) = O(h^{\frac{1}{2}}) \quad (\textit{in algorithm(C)}),$$

$$\|u-\hat{u}\|,\ E(U-\bar{U}, U-\bar{U}) = O(h^{\frac{1}{2}}) \quad (\textit{in algorithm(L)}).$$

REMARK. The term (f, φ_p) in (2.6) can be replaced by (f, χ_p). Really, let the approximate solution obtained by this replacement be $\{\hat{u}^{(*)}, \bar{U}^{(*)}\}$ and let $\hat{z} = \hat{u} - \hat{u}^{(*)}$ and $\bar{Z} = \bar{U} - \bar{U}^{(*)}$. If f is sufficiently smooth then holds

$$
\begin{cases}
K_{ij}(\hat{z}, \varphi_p) + (\bar{Z}_{ij}, \chi_p) = 0 \\
K(\varphi_p, \hat{Z}) + C_p h^3 = 0,
\end{cases}
$$

where C_p is a constant bounded by the maximum value of the first derivatives of f. Therefore we have

$$E(\bar{Z},\bar{Z})\leqq C\sum_p C_p h^3|z_p|$$
$$\leqq Ch\|\hat{z}\|,$$

and hence holds

$$\|\hat{z}\|\leqq CE(\bar{Z},\bar{Z})^{\frac{1}{2}}\leqq Ch.$$

Chapter (II) Dynamic problems

1. Approximations of consistent and lumped mass type

In this chapter we consider the following initial-boundary value problem.

(1.1) $$\partial_{tt}u+\Delta^2 u=f \qquad \text{in } \Omega\times(0,T),$$

(1.2) $$u|_{t=0}=a, \qquad \partial_t u|_{t=0}=b,$$

(1.3)
(i) $$u\Big|_{\Gamma_1}=0, \qquad \frac{du}{dn}\Big|_{\Gamma_1}=0,$$
(ii) $$u|_{\Gamma_2}=0, \qquad M(u)|_{\Gamma_2}=0,$$
(iii) $$M(u)|_{\Gamma_3}=0, \qquad V(u)|_{\Gamma_3}=0.$$

Also for this problem we can apply the same algorithms as before. Let $\triangle t=T/N(N;$positive integer) and $\hat{u}^{(m)},\bar{u}^{(m)},\hat{U}^{(m)}_{ij},\bar{U}^{(m)}_{ij}$ be denoted the values at $t=m\triangle t$ of the functions defined like (2.1)~(2.4) in chapter(I).

Algolithm(C′): Solve the following system for $m=1,2,\ldots,N-1$.

(1.4) $$\begin{cases} K_{ij}(\hat{u}^{(m)},\varphi_p)+(\hat{U}^{(m)}_{ij},\varphi_p)=0 & p\in\Omega_h \\ \qquad (i,j=1,2) & \\ (D_tD_{\bar{t}}\hat{u}^{(m)},\varphi_p)-K(\varphi_p,\hat{U}^{(m)})=(f^{(m)},\varphi_p) & (p\in N_0,N_3)\end{cases}$$

under the initial condition

(1.5) $$\hat{u}^{(0)}=\hat{a},\quad D_t\hat{u}^{(0)}=\hat{b},$$

where D_t and $D_{\bar{t}}$ are, respectively, the forward and backward difference operators on t.

Algorithm(L′): Solve the following system for $m=1,2,\ldots,N-1$.

(1.6) $$\begin{cases} K_{ij}(\hat{u}^{(m)},\varphi_p)+(\hat{U}^{(m)}_{ij},\chi_p)=0 & p\in\Omega_h \\ \qquad (i,j=1,2) & \\ (D_tD_{\bar{t}}\bar{u}^{(m)},\chi_p)-K(\varphi_p,\hat{U}^{(m)})=(f^{(m)},\varphi_p) & (p\in N_0,N_3)\end{cases}$$

under the initial condition

$$\bar{u}(0)=\bar{a},\ D_t\bar{u}(0)=\bar{b}. \tag{1.7}$$

REMARK: We can take other various approximations of $\partial_{tt}u$. See [2].

THEOREM II-1. *Let $\hat{w}^{(m)}$ be of the same form as $\hat{u}^{(m)}$.*
(i). *Algorithm(C′) is equivalent to the following algorithm:*

Determine $\underline{\hat{U}}_{ij}^{(m+1)}$ and $\underline{\hat{w}}^{(m+1)}$ by solving

$$\left\{\begin{aligned} &(D_t\underline{\hat{U}}_{ij}^{(m)}, \varphi_p)+K_{ij}(\underline{\hat{w}}^{(m)}, \varphi_p)=0 \qquad p\in\Omega_h\\ &\qquad (i, j=1, 2)\\ &(D_t\underline{\hat{w}}^{(m)}, \varphi_p)-K(\varphi_p, \underline{\hat{U}}^{(m+1)})=(f^{(m+1)}, \varphi_p)\\ &\qquad (p\in N_0, N_3)\end{aligned}\right. \tag{1.8}$$

for $m=0,1,2,\dots,N-1$ and define $\underline{\hat{u}}^{(m+2)}$ by

$$D_t\underline{\hat{u}}^{(m+1)}=\underline{\hat{w}}^{(m+1)} \quad (m=0, 1, 2, \dots, N-2).$$

Here the initial condition is

$$\left\{\begin{aligned} &\underline{\hat{u}}^{(0)}=\hat{a},\ D_t\underline{\hat{u}}^{(0)}=\hat{b},\ \underline{\hat{w}}^{(0)}=\hat{b},\\ &K_{ij}(\underline{\hat{u}}^{(0)}, \varphi_p)+(\underline{\hat{U}}_{ij}^{(0)}, \varphi_p)=0 \qquad p\in\Omega_h.\\ &\qquad (i, j=1, 2)\end{aligned}\right. \tag{1.9}$$

(ii). *Algorithm(L′) is equivalent to the following algorithm:*

Determine $\underline{\hat{U}}_{ij}^{(m+1)}$ and $\underline{\hat{w}}^{(m+1)}$ by solving

$$\left\{\begin{aligned} &(D_t\underline{\hat{U}}_{ij}^{(m)}, \chi_p)+K_{ij}(\underline{\hat{w}}^{(m)}, \varphi_p)=0 \qquad p\in\Omega_h\\ &(D_t\underline{\hat{w}}^{(m)}, \chi_p)-K(\varphi_p, \underline{\hat{U}}^{(m+1)})=(f^{(m+1)}, \varphi_p)\\ &\qquad (p\in N_0, N_3)\end{aligned}\right. \tag{1.10}$$

for $m=0,1,2,\dots,N-1$ and define $\underline{\hat{u}}^{(m+2)}$ by

$$D_t\underline{\hat{u}}^{(m+1)}=\underline{\hat{w}}^{(m+1)} \quad (m=0, 1, 2, \cdots, N-2),$$

where

$$\left\{\begin{aligned} &\underline{\hat{u}}^{(0)}=\hat{a},\ D_t\underline{\hat{u}}^{(0)}=\hat{b},\ \underline{\hat{w}}^{(0)}=\hat{b},\\ &K_{ij}(\underline{\hat{u}}^{(0)}, \varphi_p)+(\underline{\hat{U}}_{ij}^{(0)}, \chi_p)=0 \qquad p\in\Omega_h.\\ &\qquad (i, j=1, 2)\end{aligned}\right. \tag{1.11}$$

PROOF. We show that $\underline{\hat{u}}^{(m)}=\hat{u}^{(m)}$ and $\underline{\hat{U}}_{ij}^{(m)}=\hat{U}_{ij}^{(m)}$ $(m=0, 1, 2, \dots, N,$ and $i, j=1,$

2).

(i) Clearly $\underline{\hat{u}}^{(k)}=\hat{u}^{(k)}$ for $k=0,1$ and $\underline{\hat{U}}^{(0)}=\hat{U}^{(0)}$. On the other hand, since

$$D_t\underline{\hat{u}}^{(m)}=\underline{\hat{w}}^{(m)} \quad (m=0,1,2,\dots,N-1),$$

we see that

$$D_tD_{\bar{t}}\underline{\hat{u}}^{(m+1)}=D_t\underline{\hat{w}}^{(m)} \quad (m=0,1,2,\dots,N-2).$$

Substituting this into the second equation of (1.8) we get

$$(D_tD_{\bar{t}}\underline{\hat{u}}^{(m+1)},\varphi_p)-K(\varphi_p,\underline{\hat{U}}^{(m+1)})=(f^{(m+1)},\varphi_p) \qquad p\in N_0,N_3. \quad (m=0,1,2,\dots,N-2) \tag{1.12}$$

Since

$$D_t\partial_i\underline{\hat{u}}^{(m)}=\partial_i\underline{\hat{w}}^{(m)} \quad \text{(almost every where)},$$

substituting this into the first equation of (1.8) we get

$$(D_t\underline{\hat{U}}_{ij}^{(m)},\varphi_p)+K_{ij}(D_t\underline{\hat{u}}^{(m)},\varphi_p)=0 \qquad p\in\Omega_h.$$

By the initial condition this equation implies

$$(\underline{\hat{U}}_{ij}^{(m)},\varphi_p)+K_{ij}(\underline{\hat{u}}^{(m)},\varphi_p)=0 \qquad p\in\Omega_h. \quad (m=0,1,2,\dots,N) \tag{1.13}$$

Equations (1.12) and (1.13) are identical to the equations for $\hat{u}$ and $\hat{U}$. Since the initial conditions are the same for the both systems, we see that the two algorithms are equivalent. The proof of (ii) is the same for (i).

REMARK. This theorem shows that the approximation of the system

$$\begin{cases} \partial_{ij}u=U_{ij} \\ \partial_{tt}u+\sum\partial_{ij}U_{ij}=f \end{cases}$$

is equivalent to the approximation of the system

$$\begin{cases} \partial_t u=w, \quad \partial_{ij}u=U_{ij} \\ \partial_t U_{ij}-\partial_{ij}w=0 \\ \partial_t w+\sum\partial_{ij}U_{ij}=f, \end{cases}$$

in our method.

2. Energy inequalities

In this section we derive some energy inequalities for the approximating schemes presented in the preceding section. We employ the method used by Fujii [2] for second order equations. The main tool in [2] is the following lemma. Let κ be the length of the minimum perpendicular of Ω_h. In this section we do not assume the regularity of the triangulation.

LEMMA II-1. *For any $\hat{u}$ hold the following inequalities.*

(2.1) $$\|\partial\hat{u}\|^2 \leqq \frac{\mu_1^2}{\kappa^2}\|\hat{u}\|^2 \quad (\mu_1^2=48),$$

(2.2) $$\|\partial\hat{u}\|^2 \leqq \frac{\mu_2^2}{\kappa^2}\|\bar{u}\|^2 \quad (\mu_2^2=12).$$

For the proof, see [2].

The following lemma is easily proved by taking a pollor coordinate.

LEMMA II-2. *Take a triangle $\triangle$ of vertexes p, q, r and assume that the smallest angle of $\triangle$ is θ. Then*

$$\int_{qr}\hat{u}^2 ds \leqq 2h_1'\|\partial\hat{u}\|_\triangle^2 + \frac{4}{h_0'}\|\hat{u}\|_\triangle^2,$$

where $h_1'=\mathrm{Max}(pq, pr)/\sin\theta$, $h_0'=\mathrm{Min}(pq, pr)/\sin\theta$.

As a corollary of this lemma we have

LEMMA II-3. *Suppose that at most 2 sides of any triangles in Ω_h is on $\Gamma_{2,3}$. Then for any $\hat{u}$ hold*

(2.3) $$\int_{\Gamma_{2,3}}\hat{u}^2\, ds \leqq 4h_1\|\partial\hat{u}\|^2 + \frac{8}{h_0}\|\hat{u}\|^2,$$

(2.4) $$\int_{\Gamma_{2,3}}\hat{u}^2\, ds \leqq 4h_1\|\partial\hat{u}\|^2 + \frac{24}{h_0}\|\bar{u}\|^2,$$

where

$$h_1 = (\textit{length of the largest side of } \Omega_h)/\sin\theta_0,$$
$$h_0 = (\textit{length of the smallest side of } \Omega_h)/\sin\theta_0.$$

LEMMA II–4. *For any $\hat{u}$ and $\hat{U}$ hold*

$$|K(\hat{u},\hat{U})|\leqq\frac{\mu_1^2}{\kappa^2}\,\xi_1(\|\hat{u}\|^2+E(\hat{U},\hat{U})), \tag{2.5}$$

$$|K(\hat{u},\hat{U})|\leqq\frac{\mu_2^2}{\kappa^2}\,\xi_2(\|\bar{u}\|^2+E(\bar{U},\bar{U})), \tag{2.6}$$

where

$$\xi_i=1+\frac{1}{2}\left\{\frac{2\nu-1}{2(1-\nu)}+\sqrt{\left(\frac{2\nu-1}{2(1-\nu)}\right)^2+\frac{16\delta_i}{\kappa}}\right\},$$

$$\left(\begin{array}{l}\delta_1=\dfrac{1}{1-\nu}\left(4h_1+\dfrac{8\kappa^2}{\mu_1^2h_0}\right)\\[2ex] \delta_2=\dfrac{1}{1-\nu}\left(4h_1+\dfrac{24\kappa^2}{\mu_2^2h_0}\right)\end{array}\right)$$

PROOF. By the definition of $K(\hat{u},\hat{U})$ and by the Lemma II–3, We have for any positive ε

$$\begin{aligned}|K(\hat{u},\hat{U})|&\leqq\|\partial\hat{u}\|^2+\frac{1}{2}\left\{\{\|\partial\hat{U}_{11}\|^2+(1-\nu)\|\partial\hat{U}_{12}\|^2+\|\partial\hat{U}_{22}\|^2\right\}\\&\qquad+\varepsilon\int_{\Gamma_{2,3}}(\partial\hat{u})^2ds+\frac{1}{\varepsilon}\int_{\Gamma_{2,3}}[\hat{U}_{11}^2+(1-\nu)\hat{U}_{12}^2+\hat{U}_{22}^2]ds\\&\leqq\frac{\mu_1^2}{\kappa^2}\,(1+\frac{4}{\kappa}\,\varepsilon)\|\hat{u}\|^2\\&\qquad+\frac{\mu_1^2}{\kappa^2}\left(\frac{1}{2(1-\nu)}+\frac{\delta_1}{\varepsilon}\right)E(\hat{U},\hat{U}).\end{aligned}$$

By choosing ε as

$$1+\frac{4}{\kappa}\,\varepsilon=\frac{1}{2(1-\nu)}+\frac{\delta_1}{\varepsilon},$$

we have the estimate (2.5). The other estimate can be also proved by the same way.

REMARK. If $\Gamma_{2,3}=\phi$ then, for example, we have

$$|K(\hat{u},\hat{U})|\leqq\frac{\mu_1^2}{\kappa^2}\,(\|\hat{u}\|^2+\frac{1}{2(1-\nu)}\,E(\hat{U},\hat{U})).$$

Therefore in this case we can take

$$\xi_i = \text{Max}\left(1, \frac{1}{2(1-\nu)}\right)$$

Now let $\{\hat{u}^{(k)}\}$ and $\{\hat{U}^{(k)}\}$ be the approximate solutions obtained by the algorithm(C′).

LEMMA II–5. *The following inequality holds for* $2 \leqq k \leqq M$.

(2.7)
$$\left(1-\frac{\lambda_1^2 \triangle t}{\kappa^2}\right)(\|D_{\bar{t}}\hat{u}^{(k)}\|^2 + E(\hat{U}^{(k)}, \hat{U}^{(k)}))$$
$$\leqq \left(1+\frac{\lambda_1^2 \triangle t}{\kappa^2}\right)(\|D_t\hat{u}^{(0)}\|^2 + E(\hat{U}^{(0)}, \hat{U}^{(0)}))$$
$$+\sum_{m=1}^{k} \triangle t\|f^{(m)}\|^2 + \sum_{m=1}^{k} \triangle t\|D_{\bar{t}}\hat{u}^{(m)}\|^2,$$

where $\lambda_1 = \mu_1\sqrt{\xi_1}$.

PROOF. By (1.4) we see that

(2.8)
$$\sum_1^{k-1} (D_tD_{\bar{t}}\hat{u}^{(m)}, D_t\hat{u}^{(m)} + D_{\bar{t}}\hat{u}^{(m)})\triangle t$$
$$-\sum_1^{k-1} K(D_t\hat{u}^{(m)} + D_{\bar{t}}\hat{u}^{(m)}, \hat{U}^{(m)})\triangle t$$
$$= \sum_1^{k-1} (f^{(m)}, D_t\hat{u}^{(m)} + D_{\bar{t}}\hat{u}^{(m)})\triangle t.$$

Since $D_tD_{\bar{t}}\hat{u}^{(m)}(D_t\hat{u}^{(m)} + D_{\bar{t}}\hat{u}^{(m)}) = D_t(D_{\bar{t}}\hat{u}^{(m)})^2$, the first term of the left side becomes

$$\|D_{\bar{t}}u^{(k)}\|^2 - \|D_tu^{(0)}\|^2.$$

On the other hand, since

$$K_{ij}(D_t\hat{u}^{(m)}, \hat{U}^{(m)}_{kl}) + (D_t\hat{U}^{(m)}_{ij}, \hat{U}^{(m)}_{kl}) = 0$$

it holds that

$$K(D_t\hat{u}^{(m)}, \hat{U}^{(m)}) + E(D_t\hat{U}^{(m)}, \hat{U}^{(m)}) = 0,$$

and hence

$$\sum_1^{k-1} K(D_t\hat{u}^{(m)} + D_{\bar{t}}\hat{u}^{(m)}, \hat{U}^{(m)})\triangle t$$
$$= -\sum_1^{k-1} E(D_t\hat{U}^{(m)} + D_{\bar{t}}\hat{U}^{(m)}, \hat{U}^{(m)})\triangle t$$
$$= -E(\hat{U}^{(k)}, \hat{U}^{(k)}) + E(\hat{U}^{(0)}, \hat{U}^{(0)}) + E(\hat{U}^{(k)}, D_{\bar{t}}\hat{U}^{(k)})\triangle t$$
$$+ E(\hat{U}^{(0)}, D_{\bar{t}}\hat{U}^{(0)})\triangle t.$$

Therefore equality (2.8) can be represented as follows.

$$\begin{aligned}(2.9)\qquad &\|D_{\bar{t}}\hat{u}^{(k)}\|^2+E(\hat{U}^{(k)},\hat{U}^{(k)})\\ &=\|D_t\hat{u}^{(0)}\|^2+E(\hat{U}^{(0)},\hat{U}^{(0)})+\sum_1^{k-1}(f^{(m)},D_t\hat{u}^{(m)}+D_{\bar{t}}\hat{u}^{(m)})\triangle t\\ &\quad+E(\hat{U}^{(k)},D_{\bar{t}}\hat{u}^{(k)})\triangle t+E(\hat{U}^{(0)},D_t\hat{U}^{(0)})\triangle t.\end{aligned}$$

On the other hand by Lemma II-4 we have

$$\begin{aligned}E(\hat{U}^{(k)},D_{\bar{t}}\hat{U}^{(k)})\triangle t&=-K(D_t\hat{u}^{(k)},\hat{U}^{(k)})\triangle t\\ &\leqq\frac{\mu_1^2\xi_1}{\kappa^2}\triangle t(\|D_t\hat{u}^{(k)}\|^2+E(\hat{U}^{(k)},\hat{U}^{(k)}))\end{aligned}$$

Since similar inequality holds for $E(\hat{U}^{(0)},D_t\hat{U}^{(0)})\triangle t$, substituting these inequalities into (2.9) we obtain (2.7).

LEMMA II-6. *The following inequality holds for* $2\leqq k\leqq M$.

$$\begin{aligned}(2.10)\qquad &(1-\rho)\|D_{\bar{t}}\hat{u}^{(k)}\|^2\\ &\leqq\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{k-2}\{(c+\triangle t)\|D_t\hat{u}^{(0)}\|^2+cE(\hat{U}^{(0)},\hat{U}^{(0)})\\ &\quad+\sum_1^M\triangle t\|f^{(m)}\|^2\}+\triangle t\|D_{\bar{t}}\hat{u}^{(k)}\|^2,\end{aligned}$$

where $\rho=\lambda_1^2\triangle t/\kappa^2$, $c=1+\rho$.

PROOF. For $k=2$ this is correct by Lemma II-5. Suppose that this inequality holds until $k(\leqq M-1)$. Let the quantity in the blanket { } of (2.10) be denoted by Y. Then for any $2\leqq m\leqq k$,

$$\|D_{\bar{t}}\hat{u}^{(m)}\|^2\leqq\frac{1}{1-\rho-\triangle t}\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{m-2}\{Y\}$$

and therefore

$$(2.11)\qquad \sum_{m=2}^{k}\triangle t\|D_{\bar{t}}\hat{u}^{(m)}\|^2\leqq\left[\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{k-1}-1\right]\{Y\}.$$

On the other hand (2.7) implies

$$\begin{aligned}&(1-\rho)\|D_{\bar{t}}\hat{u}^{(k+1)}\|^2\\ &\leqq c\|D_t\hat{u}^{(0)}\|^2+cE(\hat{U}^{(0)},\hat{U}^{(0)})+\sum_1^M\triangle t\|f^{(m)}\|^2\\ &\quad+\triangle t\|D_t\hat{u}^{(0)}\|^2+\sum_2^k\triangle t\|D_{\bar{t}}\hat{u}^{(m)}\|^2+\triangle t\|D_{\bar{t}}\hat{u}^{(k+1)}\|^2.\end{aligned}$$

Substituting (2.11) into this inequality we have

$$(1-\rho)\|D_{\bar{t}}\hat{u}^{(k+1)}\|^2$$
$$\leqq\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{k-1}\{(c+\triangle t)\|D_t\hat{u}^{(0)}\|^2+cE(\hat{U}^{(0)},\hat{U}^{(0)})$$
$$+\sum_1^M\triangle t\|f^{(m)}\|^2\}+\triangle t\|D_{\bar{t}}\hat{u}^{(k+1)}\|^2,$$

which proves that the inequality (2.10) holds for $k+1$. This completes the proof.

From the proof of this lemma we have

COROLLARY. *The following inequality holds.*

$$\sum_{m=2}^M\triangle t\|D_{\bar{t}}\hat{u}^{(m)}\|^2$$
$$\leqq\left[\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{M-1}-1\right]\{(c+\triangle t)\|D_t\hat{u}^{(0)}\|^2$$
$$+cE(\hat{U}^{(0)},\hat{U}^{(0)})+\sum_1^M\triangle t\|f^{(m)}\|^2\}.$$

THEOREM II-2. *Let $\sqrt{\triangle t}/\kappa<1/\lambda_1$. Then for the approximate solution $(\hat{u}^{(M)}, \hat{U}^{(M)})$ obtained by the algorithm (C') holds the following inequality.*

$$\|D_{\bar{t}}\hat{u}^{(M)}\|^2+E(\hat{U}^{(M)},\hat{U}^{(M)})$$
$$\leqq C(T)\sum_{m=1}^M\triangle t\|f^{(m)}\|^2+C'(T)[\|D_t\hat{u}^{(0)}\|^2+E(\hat{U}^{(0)},\hat{U}^{(0)})].$$

PROOF. For fixed ρ

$$\left(1+\frac{\triangle t}{1-\rho-\triangle t}\right)^{M-1}\rightarrow\exp\left(\frac{T}{1-\rho}\right)\quad\text{as }\triangle t\rightarrow 0.$$

Therefore by the above corollary we have

$$\sum_{m=1}^M\triangle t\|D_{\bar{t}}\hat{u}^{(m)}\|^2\leqq\tilde{C}(T)\{\|D_t\hat{u}^{(0)}\|^2+E(\hat{U}^{(0)},\hat{U}^{(0)})\}$$
$$+\tilde{C}'(T)\sum_{m=1}^M\triangle t\|f^{(m)}\|^2.$$

Therefore the theorem follows from (2.7) by setting $k=M$ and substituting the above inequality.

THEOREM II-3. *Let $\sqrt{\triangle t}/\kappa<1/\lambda_2$. Then for the approximate solutions* $\{\hat{u}^{(M)},$

$\hat{U}^{(M)}\}$ *obtained by the algorithm*(*L'*) *holds the following inequality.*

$$\|D_{\bar{t}}\bar{u}^{(M)}\|^2+E(\bar{U}^{(M)},\bar{U}^{(M)})$$
$$\leqq C(T)\sum_{m=1}^{M}\triangle t\|f^{(m)}\|^2+C'(T)\{\|D_t\bar{u}^{(0)}\|^2+E(\bar{U}^{(0)},\bar{U}^{(0)})\}.$$

Proof is exactly same for Theorem II–2

3. Convergence of the approximate solutions

In this section we assume, for brevity, that $f=0$. First we consider the algorithm (C′):

LEMMA II–7. *Suppose that* ε *belongs to* $L^2(\Omega)$. *If* α, *satisfying* $\alpha|_{\Gamma_{1,2}}=0$ *and* $d\alpha/dn|_{\Gamma_1}=0$, *and* β *satisfy the equation*

$$\begin{cases} K_{ij}(\alpha,\varphi)+(\beta_{ij},\varphi)=0 \quad (i,j=1,2) \\ K(\varphi,\beta)+(\varepsilon,\varphi)=0 \end{cases}$$

for any smooth φ *satisfying* $\varphi|_{\Gamma_{1,2}}=0$, *then hold the following inequalities.*

$$\|\partial\alpha\|\leqq C\sum_{i,j}\|\beta_{ij}\|,\quad \sum_{i,j}\|\beta_{ij}\|\leqq C\|\varepsilon\|^{(*)}.$$

Now let $u^{(m)}=u(m\triangle t,x_1,x_2)$ and determine $\mathring{\hat{u}}^{(m)}$ and $\mathring{\hat{U}}^{(m)}$ by solving

$$(3.1)\qquad \begin{cases} K_{ij}(\mathring{\hat{u}}^{(m)},\varphi_p)+(\mathring{\hat{U}}_{ij}^{(m)},\varphi_p)=0 \qquad p\in\Omega_h \\ \qquad (i,j=1,2) \\ (\partial_{tt}u^{(m)},\varphi_p)-K(\varphi_p,\mathring{\hat{U}}^{(m)})=0, \end{cases}$$

where $\mathring{\hat{u}}^{(m)}$ and $\mathring{\hat{U}}_{ij}^{(m)}$ are functions of the same form as $\hat{u}^{(m)}$ and $\hat{U}_{ij}^{(m)}$ respectively.

LEMMA II–8. *The following estimate holds.*

$$(3.2)\qquad \|D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}-\partial_{tt}u^{(m)}\|\leqq Ch^{\frac{1}{2}}$$

as $h\to0$ $(1\leqq m\leqq M-1)$.

PROOF. Since $u^{(m)}$ is the exact solution, the pair $\{w^{(m)}=D_tD_{\bar{t}}u^{(m)},\ W^{(m)}=D_tD_{\bar{t}}U^{(m)}\}$ satisfies

*) We assume, of course, that α and β are sufficiently smooth.

$$(3.3)\quad \begin{cases} K_{ij}(w^{(m)}, \varphi)+(W_{ij}^{(m)}, \varphi)=0 \\ -K(\varphi, W^{(m)})=-(\partial_{tttt}u^{(m)}, \varphi) \\ \qquad\qquad +(\partial_{tttt}u^{(m)}-D_tD_{\bar{t}}\partial_{tt}u^{(m)}, \varphi) \end{cases}$$

for any smooth φ satisfying $\varphi|_{\Gamma_{1,2}}=0$.

If the second term of the right side of the second equation is omitted, then $\{\partial_{tt}u^{(m)}, \partial_{tt}U^{(m)}\}$ satisfies the resulting equation. Since $\|\partial_{tttt}u^{(m)}-D_tD_{\bar{t}}\partial_{tt}u^{(m)}\|\leqq C\triangle t^2$, we have by Lemma II-7

$$(3.4)\qquad \|\partial_{tt}u^{(m)}-w^{(m)}\|\leqq C\triangle t^2 \quad (1\leqq m\leqq M-1).$$

On the other hand, operating $D_tD_{\bar{t}}$ to the both side of (3.1) we have

$$\begin{cases} K_{ij}(D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}, \varphi_p)+(D_tD_{\bar{t}}\mathring{\hat{U}}_{ij}^{(m)}, \varphi_p)=0 & p\in\Omega_h \\ \qquad (i, j=1, 2) & \\ -K(\varphi_p, D_tD_{\bar{t}}\mathring{\hat{U}}^{(m)})=-(D_tD_{\bar{t}}\partial_{tt}u^{(m)}, \varphi_p) & p\in N_0, N_3 \\ \qquad (1\leqq m\leqq M-1) & \end{cases}$$

which is a discrete analogue of (3.3). Therefore by the result of chap(I) we have

$$(3.5)\qquad \|w^{(m)}-D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}\|\leqq Ch^{\frac{1}{2}} \qquad \text{as } h\to 0.$$

Inequalities (3.4) and (3.5) implies inequality (3.2).

THEOREM II-3 *If the triangulation is nearly consistent and the assumption of Theorem* II-2 *is satisfied, then the approximate solutions obtained by algorithm* (C') *converge to the exact solution as* $h\to 0$, *and hold the following estimates.*

$$(3.6)\qquad \|u^{(M)}-\hat{u}^{(M)}\|,\ E(U^{(M)}-\hat{U}^{(M)}, U^{(M)}-\hat{U}^{(M)})^{\frac{1}{2}}\leqq Ch^{\frac{1}{2}}.$$

PROOF. Rewrite (3.1) as follows.

$$(3.7)\quad \begin{cases} K_{ij}(\mathring{\hat{u}}^{(m)}, \varphi_p)+(\mathring{\hat{U}}_{ij}^{(m)}, \varphi_p)=0 & p\in\Omega_h \\ \qquad (i, j=1, 2) & \\ (D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}, \varphi_p)-K(\varphi_p, \mathring{\hat{U}}^{(m)})=(D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}-\partial_{tt}u^{(m)}, \varphi_p) & \\ \qquad p\in N_0, N_3 & \end{cases}$$

By Lemma II-8 we see that the L^2-norm of the quantity $E^{(m)}=D_tD_{\bar{t}}\mathring{\hat{u}}^{(m)}-\partial_{tt}u^{(m)}$ is bounded by $Ch^{1/2}$ as $h\to 0$.

Let $\hat{\alpha}^{(m)}=\mathring{\hat{u}}^{(m)}-\hat{u}^{(m)}$, $\hat{\beta}_{ij}^{(m)}=\mathring{\hat{U}}_{ij}^{(m)}-\hat{U}_{ij}^{(m)}$. Then they satisfy the following equality for $1\leqq m\leqq M-1$.

$$(3.8)\quad \begin{cases} K_{ij}(\hat{\alpha}^{(m)}, \varphi_p)+(\hat{\beta}_{ij}^{(m)}, \varphi_p)=0 & p\in\Omega_h \\ \qquad (i, j=1, 2) & \\ (D_t D_{\bar{t}}\hat{\alpha}^{(m)}, \varphi_p)-K(\varphi_p, \hat{\beta}^{(m)})=(E^{(m)}, \varphi_p) & \\ \qquad p\in N_0, N_3 & \end{cases}$$

(1) Estimation of $E(\hat{\beta}^{(0)}, \hat{\beta}^{(0)})$. The solution of the "original problem" for the equation (3.7) is $\{u^{(m)}, U^{(m)}\}$. Therefore, applying the result of Chap(I) for $m=0$, we obtain

$$\|U_{ij}^{(0)}-\hat{\mathring{U}}_{ij}^{(0)}\|\leqq Ch^{\frac{1}{2}}.$$

On the other hand, since $\hat{u}^{(0)}$ is the interpolating function of $u^{(0)}$, we know

$$\|U_{ij}^{(0)}-\hat{U}_{ij}^{(0)}\|\leqq Ch^{\frac{1}{2}}$$

which implies

$$(3.9)\quad \|\hat{\beta}_{ij}^{(0)}\|=\|\hat{\mathring{U}}_{ij}^{(0)}-\hat{U}_{ij}^{(0)}\|\leqq Ch^{\frac{1}{2}}$$

or $E(\hat{\beta}^{(0)}, \hat{\beta}^{(0)})=O(h)$.

(2) Estimation of $\|D_t\hat{\alpha}^{(0)}\|$. First we see

$$(3.10)\quad \|D_t\hat{u}^{(0)}-\partial_t u^{(0)}\|\leqq Ch^{\frac{1}{2}},$$

Since the exact solution of the "original problem" of the system

$$(3.11)\quad \begin{cases} K_{ij}(D_t\hat{\mathring{u}}^{(0)}, \varphi_p)+(D_t\hat{\mathring{U}}_{ij}^{(0)}, \varphi_p)=0, & p\in\Omega_h \\ \qquad (i, j=1, 2) & \\ (D_t\partial_{tt}u^{(0)}, \varphi_p)-K(\varphi_p, D_t\hat{\mathring{U}}^{(0)})=0, & \\ \qquad p\in N_0, N_3 & \end{cases}$$

is clearly $(D_t u^{(0)}, D_t U^{(0)})$, the result of chap(I) implies

$$(3.12)\quad \|D_t u^{(0)}-D_t\hat{\mathring{u}}^{(0)}\|\leqq Ch^{\frac{1}{2}}.$$

By (3.10), (3.11) and the assumption of the theorem we obtain

$$(3.13)\quad \|D_t\hat{\alpha}^{(0)}\|=\|D_t(\hat{\mathring{u}}^{(0)}-\hat{u}^{(0)})\|\leqq Ch^{\frac{1}{2}}.$$

(3) Equation (3.8), estimations (3.9) and (3.13) and Theorem II–2 imply

$$(3.14)\quad \|D_{\bar{t}}\hat{\alpha}^{(M)}\|^2+E(\hat{\beta}^{(M)}, \hat{\beta}^{(M)})\leqq Ch,$$

and thus

$$(3.15)\quad \|\hat{\mathring{u}}^{(M)}-\hat{u}^{(M)}\|^2,\ E(\hat{\mathring{U}}^{(M)}-\hat{U}^{(M)}, \hat{\mathring{U}}^{(M)}-\hat{U}^{(M)})\leqq Ch.$$

Now by (3.7) and the result of chap(I) we have

$$\|\hat{u}^{(M)}-u^{(M)}\|^2,\quad E(\hat{\hat{U}}^{(M)}-U^{(M)},\hat{\hat{U}}^{(M)}-U^{(M)})\leqq Ch. \tag{3.16}$$

Therefore by (3.15) we see

$$\|u^{(M)}-\hat{u}^{(M)}\|^2,\quad E(U^{(M)}-\hat{U}^{(M)},U^{(M)}-\hat{U}^{(M)})\leqq Ch,$$

which is the desired estimates.

THEOREM II-4 *The conclusion of the Theorem* II-3 *is, replacing* $\hat{U}^{(M)}$ *by* $\bar{U}^{(M)}$ *valid for the approximate solutions obtained by algorithm*(L').

Proof is almost the same for Theorem II-3 and hence we ommit it.

REMARK. *Approximations of high order accuracy.*

The assumption "nearly consistent decomposition" is obliged by the use of piecewise linear basis. But if we employ more accurate basis this assumption will be unnecessary. In fact the authour have obtained the following result [3].

Let us call the trial functions treated by Zlámal [7] k-th order basis if the used polynomial is of k-th order ($k\leqq 3$). We consider the first boundary value problem and approximate the exact solution u and Δu by

$$\hat{u}(x_1,x_2)=\sum_{p\in N_0} u_p\varphi_p(x_1,x_2)$$

$$\hat{U}(x_1,x_2)=\sum_{p\in\Omega_h} U_p\varphi_p(x_1,x_2),$$

respectively, where $\{\varphi_p\}$ is the basis of order k. Note that the trial function in [7] can be represented by a suitable system of functions $\{\varphi_p\}$ as above. Then if the decomposition is regular, the convergence rate of the approximate solutions of order k is given by

$$\|\partial(u-\hat{u})\|,\quad \|\Delta u-\hat{U}\|\leqq Ch^{k-1}$$

in static problem.

The proof is almost the same as in the previous algorithms and thus we omit this. Further, we add that the dynamic problems can be also solved by this method and the similar theorems to Theorem II-1~Theorem II-4 hold. The details is stated in [3].

APPENDIX. Numerical examples

Let Ω be a unit square and consider the following problems.

(A)
$$\Delta^2 u = 1.0 \qquad in\ \Omega,$$
$$u = \frac{du}{dn} = 0 \qquad on\ \partial\Omega.$$

(B)
$$\Delta^2 u = 1.0 \qquad in\ \Omega,$$
$$\begin{cases} u = \dfrac{du}{dn} = 0 & on\ \Gamma_1, \\ u = M(u) = 0 & on\ \Gamma_2, \\ M(u) = V(u) = 0 & on\ \Gamma_3, \end{cases}$$

where Γ_i is the portion of $\partial\Omega$ illustrated in Figure 1.

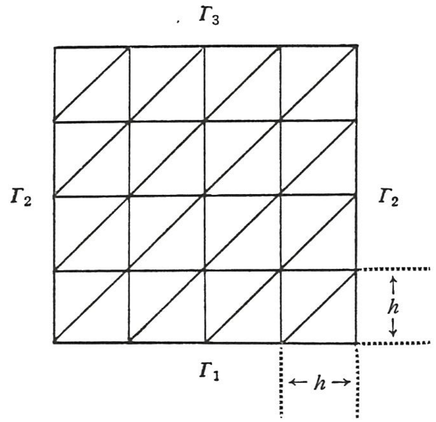

Figure 1.

The element used in our computation is right-angled equilateral triangle (see Figure 1).

To solve these static problems we applied the algorithm(L′) to the equation

$$u_{tt} + \alpha u_t + \Delta^2 u = 1.0,$$

under the corresponding boundary condition, where α is a suitable positive number (dumping factor). Therefore, our computational procedure is as follows.

For (A):

$$\hat{u}^{(m)}=\sum_{p\in N_0} u_p^{(m)}\varphi_p,$$

$$\hat{U}^{(m)}=\sum_{p\in\Omega_h} U_p^{(m)}\varphi_p,$$

$$\begin{cases} (\partial\hat{u}^{(m)},\partial\varphi_p)+(\bar{U}^{(m)},\chi_p)=0 & p\in\Omega_h, \\ (D_tD_{\bar{t}}\bar{u}^{(m)},\chi_p)+\dfrac{\alpha}{2}([D_t+D_{\bar{t}}]\bar{u}^{(m)},\chi_p)-(\partial\hat{U}^{(m)},\partial\varphi_p) & \\ \qquad =(1.0,\varphi_p) & p\in N_0, \end{cases}$$

$$\hat{u}^{(0)}, D_t\hat{u}^{(0)}: \text{ given.}$$

For (B):

$$\hat{u}^{(m)}=\sum_{p\in N_0+N_3} u_p\varphi_p$$

$$\hat{U}_{ij}^{(m)}=\sum_{p\in\Omega_h} U_{ij}(p)\varphi_p \quad (i,j=1,2),$$

$$\begin{cases} K_{ij}(\hat{u}^{(m)},\varphi_p)+(\bar{U}_{ij}^{(m)},\chi_p)=0 & p\in\Omega_h\ (i,j=1,2), \\ (D_tD_{\bar{t}}\bar{u}^{(m)},\chi_p)+\dfrac{\alpha}{2}([D_t+D_{\bar{t}}]\bar{u}^{(m)},\chi_p)-K(\varphi_p,\hat{U}^{(m)}) & \\ \qquad =(1.0,\varphi_p) & p\in N_0, N_3, \end{cases}$$

$$\hat{u}^{(0)}, D_t\hat{u}^{(0)}: \text{ given.}$$

The time-mesh $\triangle t$ was taken according to the Theorem II–2, and α was chosen 50 for (A) and 30 for (B), since the smallest eigenvalue of simply supported unit square plate is $4\pi^4$. If our theory of this paper is correct, the above schemes must be stable, and if the above schemes converge, then the limit function must be good approximate solution of the corresponding static problem. The results are described in Table 1 and 2. Table 1 is the central value of $\hat{u}^{(M)}$ in problem (A). For comparison we cited the result obtained by using Adini-Clough-Melosh's basis from [6]. Table 2 is the value of $\hat{u}^{(M)}$ at the middle point of the free edge (Γ_3), and the maximum values of $\bar{M}_x$ and $\bar{M}_y$, where

	Algorithm (L)	ACM
h=1/4	1.800×10^{-3} (M=20)	1.403×10^{-3}
h=1/8	1.425×10^{-3} (M=80)	1.304×10^{-3}
h=1/16	1.309×10^{-3} (M=300)	1.275×10^{-3}
exact	1.26×10^{-3}	
number of parameters	3	12

Table 1.

	$\hat{u}^{(M)}(1/2,1)$	$\bar{M}_x(1/2,1)$	$\bar{M}_y(1/2,0)$	M
h=1/4	1.021×10^{-2}	-0.982×10^{-1}	0.880×10^{-1}	100
h=1/8	1.103×10^{-2}	-1.084×10^{-1}	1.097×10^{-1}	300
h=1/16	1.173×10^{-2}	-1.138×10^{-1}	1.182×10^{-1}	800
(B)	$1.13\ \times10^{-2}$	-0.972×10^{-1}	$1.19\ \times10^{-1}$	

Table 2. ($\nu=0.3$)

$$\bar{M}_x=\bar{U}_{11}^{(M)}+\nu\bar{U}_{22}^{(M)},$$
$$\bar{M}_y=\nu\bar{U}_{11}^{(M)}+\bar{U}_{22}^{(M)}.$$

(B) denotes the value calculated by Boobnov about 70 years ago (cited from [6]).

It is easily forecast that the accuracy of $\bar{M}_x$ and $\bar{M}_y$ will decrease on the boundary. Really, the "finite difference equation"

$$K_{22}(\hat{u},\varphi_p)+(\bar{U}_{22},\chi_p)=0$$

is not "consistent", for example, on the boundary Γ_3 (see the equation (2.6) in chapter (I)). In such case we can take a suitable interpolating function using the values of the internal points. Table 3 shows the result of such interpolations. The first row is the value of the function

	h=1/8	h=1/16
Modified $\bar{M}_x$. (a)	-1.055×10^{-1}	-1.078×10^{-1}
Modified $\bar{M}_x$. (b)	-1.050×10^{-1}	-1.076×10^{-1}

Table 3.

(a) $$\bar{U}_{11}^{(M)}\left(\frac{1}{2},1\right)+\nu\bar{U}_{22}^{(M)}\left(\frac{1}{2},1-h\right),$$

and the second is for

(b) $$\bar{U}_{11}^{(M)}\left(\frac{1}{2},1\right)+\nu\left\{\bar{U}_{22}^{(M)}\left(\frac{1}{2},1-h\right)+h\left(\bar{U}_{22}^{(M)}\left(\frac{1}{2},1-h\right)-\bar{U}_{22}^{(M)}\left(\frac{1}{2},1-2h\right)\right)\right\}.$$

These tables shows that the accuracy of $\hat{u}^{(M)}$ is very good in spite of the small number of used parameters and the accuracy of $\bar{M}_x$ and $\bar{M}_y$ is also not bad considering the number of parameters.

It was further observed that the criterion given by Theorem II-3 is certainly a sufficient condition, though about 6 times overestimated for (A) and 44 times for (B). It seems that the constant ξ_2 in Lemma II-4 can be regarded as unity in practical cumputation. But the reason of this phenomenon is not clear for the present.

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