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ON A S. S. CHERN, M. DO CARMO, S. KOBAYASHI THEOREM

Teruo IWATANI

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S. S. Chern-M. do Carmo-S. Kobayashi [1] proved the following theorem:
Theorem A. *The Veronese surface in* S^4 *and the naturally imbedded submanifolds* $S^m(\sqrt{m/n})\times S^{n-m}(\sqrt{(n-m)/n})$ $(0<m<n)$ *in* S^{n+1} *are the only compact minimal submanifolds of dimension* n *in* S^{n+p} *satisfying* $\|A\|^2=n/(2-1/p)$.

In the present paper, we shall prove this theorem applying the method of J. Simons [2].

1. The method of J. Simons.

Let $\bar{M}$ be an $(n+p)$-dimensional Riemannian manifold and let f be an immersion of an n-dimensional manifold M into $\bar{M}$. Let $T(M)$ and $T(M)^\perp$ denote the tangent and normal bundles of M. The connection $\bar{\nabla}$ and metric $<,>$ on $T(\bar{M})$ lead to connections ∇ and invariant inner products $<,>$ on $T(M), T(M)^\perp$ and the tensor product of them. Let $X\in\mathfrak{X}(M)$, $V\in\mathfrak{X}(M)^\perp$ and $m\in M$, and put

$$(\bar{\nabla}_X V)_m=-(A^V(X))_m+(\nabla_X V)_m,$$

where $-(A^V(X))_m\in T_m(M)$ and $(\nabla_X V)_m\in T_m(M)^\perp$. Then the mapping $(X,V)\in\mathfrak{X}(M)\times\mathfrak{X}(M)^\perp\to A^V(X)\in\mathfrak{X}(M)$ induces a bilinear mapping $(X_m,V_m)\in T_m(M)\times T_m(M)^\perp\to A^{V_m}(X_m)\in T_m(M)$. Let $S(M)$ be the bundle whose fibre at each point is the space of symmetric linear transformations of $T_m(M)$. The second fundamental form A of the immersion f is the cross-section in $\mathrm{Hom}(T(M)^\perp, S(M))$ which is defined by

$$A(v)=A^v\in S_m(M),\ v\in T_m(M)^\perp.$$

If $\{v_\alpha\}$ is an orthonormal frame for $T_m(M)^\perp$, then $(1/n)\sum_{\alpha=1}^{p}(\mathrm{tr}\ A^{v_\alpha})v_\alpha$ is called the mean curvature normal of M at m. M is called a minimal submanifold in $\bar{M}$ if the mean curvature normal of M vanishes at each point. By using the second fundamental form A, we define two cross-sections. We define one cross-section $\tilde{A}$ in Hom$(T(M)^{\perp}, T(M)^{\perp})$ by

$$<\tilde{A}(v), w> = <A^{v}, A^{w}>, \quad v, w \in T_m(M)^{\perp}.$$

The other cross-section $\underset{\sim}{A}$ in Hom$(S(M), S(M))$ is defined by

$$\underset{\sim}{A} = \sum_{\alpha=1}^{p} ad\ A^{v_\alpha} \circ ad\ A^{v_\alpha}.$$

Under these notations, J. Simons [2] proved the following theorem:

THEOREM 1. *Let M be an n-dimensional minimal submanifold in an $(n+p)$-dimensional space of constant curvature c. Then the second fundamental form satisfies*

$$\nabla^2 A = ncA - A \circ \tilde{A} - \underset{\sim}{A} \circ A,$$

where ∇^2 is the Laplace operator.

Let M be as in Theorem 1 and let $\{E_i\}$ be a local orthonormal frame field on M which is covariant constant with respect to ∇ at m. Then we have

$$(1) \qquad <\nabla^2 A, A>(m) = \sum_{i=1}^{n} <\nabla_{E_i}\nabla_{E_i} A, A>(m)$$
$$= (1/2)\sum_{i=1}^{n} \nabla_{E_i}\nabla_{E_i} <A, A>(m) - \sum_{i=1}^{n} <\nabla_{E_i} A, \nabla_{E_i} A>(m).$$

Now, we assume M is compact. Integrating the both sides of (1) over M and applying Green's theorem, we obtain

$$\int_M <\nabla^2 A, A> = -\int_M <\nabla A, \nabla A> \leq 0.$$

Using Theorem 1, we obtain

$$<\nabla^2 A, A>(m) = <ncA - A \circ \tilde{A} - \underset{\sim}{A} \circ A, A>(m)$$
$$= nc\|A\|^2(m) - <A \circ \tilde{A} + \underset{\sim}{A} \circ A, A>(m).$$

Since $\tilde{A}$ is a symmetric, positive semi-definite operator at each point, we may choose a frame $\{v_\alpha\}$ for $T_m(M)^{\perp}$ such that

$$\tilde{A}(v_\alpha) = \lambda_\alpha v_\alpha, \quad \lambda_\alpha \geq 0, \quad \alpha = 1, 2, \ldots, p.$$

Then we have

(2) $$\langle A\circ\tilde{A}+\underset{\sim}{A}\circ A, A\rangle(m)=\sum_{\alpha=1}^{p}\lambda_\alpha^2+\sum_{\alpha,\beta=1}^{p}\|[A^{v_\alpha}, A^{v_\beta}]\|^2\leq\sum_{\alpha=1}^{p}\lambda_\alpha^2+2\sum_{\alpha\neq\beta}\|A^{v_\alpha}\|^2\|A^{v_\beta}\|^2$$
$$=\sum_{\alpha=1}^{p}\lambda_\alpha^2+2\sum_{\alpha\neq\beta}\lambda_\alpha\lambda_\beta\leq(2-1/p)\|A\|^4(m).$$

Therefore we obtain

(3) $$\langle\nabla^2 A, A\rangle(m)\geq(nc-(2-1/p)\|A\|^2)\|A\|^2(m).$$

Thus, we have

THEOREM 2 ([2]). *If M is an n-dimensional compact minimal submanifold in an $(n+p)$-dimensional space of constant curvature c, then we have the inequality*

$$\int_M (\|A\|^2-nc/(2-1/p))\|A\|^2\geq 0.$$

The lower bound for this estimate in the special case where $\bar{M}=S^{n+p}$ is, of cource, achieved when $\|A\|^2\equiv 0$ or $\|A\|^2\equiv n/(2-1/p)$. If $\|A\|^2\equiv 0$, then M is a totally geodesic submanifold S^n. We shall determine all minimal submanifolds M of S^{n+p} satisfying $\|A\|^2\equiv n/(2-1/p)$.

Let M be an n-dimensional manifold which is minimally immersed in S^{n+p} and satisfies $\|A\|^2\equiv n/(2-1/p)$. Setting $\|A\|^2=$constant in (1), we obtain

$$\langle\nabla^2 A, A\rangle(m)=-\sum_{i=1}^{n}\langle\nabla_{E_i}A, \nabla_{E_i}A\rangle(m).$$

Using (3), we have

$$\sum_{i=1}^{n}\langle\nabla_{E_i}A, \nabla_{E_i}A\rangle(m)\leq((2-1/p)\|A\|^2-n)\|A\|^2(m).$$

This shows that if $\|A\|^2\equiv n/(2-1/p)$, then A is parallel. Since two inequalities in (2) are actually equalities in the case $\|A\|^2=n/(2-1/p)$, we have

(4) $$\sum_{\alpha,\beta=1}^{p}\|[A^{v_\alpha}, A^{v_\beta}]\|^2=2\sum_{\alpha\neq\beta}\|A^{v_\alpha}\|^2\|A^{v_\beta}\|^2,$$

(5) $$\lambda_1=\cdots=\lambda_p\neq 0.$$

2. Proof of Theorem A.

Chern-do Carmo-Kobayashi [1] gave the following algebraic lemma:

LEMMA. *For non-zero symmetric matrices A_1 and A_2, the equality*

$$\|[A_1, A_2]\|^2=2\|A_1\|^2\|A_2\|^2$$

holds if and only if A_1 *and* A_2 *can be transformed simultaneously by an orthogonal matrix into scalar multiples of*

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Moreover, if A_1, A_2 *and* A_3 *are symmetric matrices and if*

$$\|[A_\alpha, A_\beta]\|^2 = 2\|A_\alpha\|^2\|A_\beta\|^2,\ 1 \leq \alpha, \beta \leq 3,$$

then at least one of matrices A_α *must be zero.*

From (4), (5) and Lemma, we conclude that p must be 1 or 2. We consider the cases $p=1$ and $p=2$ separately.

The case $p=1$. Let V be a local unit normal vector field and let $\{e_i\}$ be an orthonormal frame for $T_m(M)$ such that $A^{V_m} e_i = \rho_i e_i$. We extend it to a local frame field $\{E_i\}$ by parallel translation of $\{e_i\}$ along geodesics issued from m. Since A is parallel and V is also parallel, the representation of A^V with respect to $\{E_i\}$ is

$$\begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{pmatrix},\ \rho_i = \text{constant},\ 1 \leq i \leq n.$$

We choose all of the different elements from $\rho_1, \cdots, \rho_n$, and assume them $\rho_1, \cdots, \rho_r$. If we define distributions $T_{\rho_1}, \cdots, T_{\rho_r}$ by

$$T_{\rho_i}(m) = \{X \in T_m(M) \colon A^V X = \rho_i X\},\ i=1, \cdots, r,$$

then each T_{ρ_i} is involutive, totally geodesic and parallel. Using this parallelism and Gauss equation, we have

$$1 + \rho_i \rho_j = 0,\ i \neq j.$$

Hence, we get that $r=2$ and if we put $\rho_1 = \rho$, then $\rho_2 = -1/\rho$. Thus the two distributions T_ρ and $T_{-1/\rho}$ give a local decomposition of M, that is, M is locally a Riemannian product of spaces of constant curvatures $(1+\rho^2)$ and $(1+1/\rho^2)$.

The case $p=2$. Let $\{v_1, v_2\}$ be an orthonormal frame for $T_m(M)^\perp$ such that $\tilde{A} v_\alpha = 2\lambda^2 v_\alpha$, $\alpha = 1, 2$. From Lemma, the matrix representations of A^{v_1} and A^{v_2} with respect to an adapted frame $\{e_i\}$ for $T_m(M)$ are

$$\lambda\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right) \text{ and } -\lambda\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array}\right).$$

We extend $\{v_\alpha\}$ and $\{e_i\}$ to local frame fields $\{V_\alpha\}$ and $\{E_i\}$ by parallel translation with respect to the connections of $T(M)^\perp$ and $T(M)$. Since A is parallel, from the constructions of $\{V_\alpha\}$ and $\{E_i\}$, the matrix representations of A^{V_1} and A^{V_2} with respect to $\{E_i\}$ are

$$\lambda\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right) \text{ and } -\lambda\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array}\right), \quad \lambda=\text{const.}.$$

We assume $n\geq 3$ and consider the distribution

$$T_0=\{X: A^{V_1}X=0\}=\{X: A^{V_2}X=0\}\neq\{0\}.$$

Then T_0 is parallel. We therefore obtain

$$0=\langle E_1, R(E_1,X)X\rangle=\langle E_1,E_1\rangle\langle X,X\rangle,\ X\in T_0.$$

This is a contradiction. Hence, $n=2$. We may therefore assume that $\lambda=1/\sqrt{3}$, since $n/(2-1/2)=\|A\|^2=4\lambda^2$. Using Gauss equation, we have that M is a space of constant curvature $1/3$.

By the actual calculation, we know that the second fundamental forms and the normal connection forms in the cases $p=1$ and $p=2$ coincide with those of $S^m(\sqrt{m/n})\times S^{n-m}(\sqrt{(n-m)/n})$ in S^{n+1} and those of the Veronese surface of S^4 with respect to such frame fields as the predescribed ones respectively. We may therefore conclude that minimal submanifolds in S^{n+p} satisfying $\|A\|^2=n/(2-1/p)$ coincide locally with $S^m(\sqrt{m/n})\times S^{n-m}(\sqrt{(n-m)/n})$ in S^{n+1} or the Veronese surface of S^4. If such manifolds are compact, they coincide globally. This completes the proof of Theorem A.

References

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Department of Mathematics,
Faculty of Science,
Kumamoto University