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CONVERGENCE OF FINITE ELEMENT SOLUTIONS REPRESENTED BY A NON-CONFORMING BASIS

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1. Introduction

The application of finite element method to plate bending problems (boundary value problems of biharmonic equations) has a special difficulty, since the construction of the trial function belonging to C^1-class and having a simple form is very difficult. For this reason, engineers employ frequently the non-conforming trial functions which belong only to C^0-class.

But such procedures can not be justified by the theory of the classical variational methods. Nevertheless, it is known that some non-conforming trial functions can give good approximate solutions [1, 3].

In the present paper we justify the use of the Adini-Clough-Melosh's trial function which is known as a typical non-conforming trial function to define the rectanguler stiffness. It may be possible to apply our method to the justification of the use of some non-conforming trial functions defined on the trianguler elements. But we have not yet succeeded in such attempt.

2. ACM-basis

Let Ω be a bounded domain in (x, y)-plane. Through the present paper we assume that the boundary of Ω is sufficiently smooth. Let Ω_h be a subdomain of Ω satisfying the following 2 conditions.

(1) Ω_h consists of the square elements of side length h. Here we assume that the square decomposition is done by the lines parallel to the coordinate axes and at least 2 sides of any element are in the interior of Ω_h.

(2) Measure$(\Omega-\Omega_h)=O(h)$ as $h\to 0$.

Let $u_h(x, y)$ be a function defined on Ω_h such that

(i) in each element e it is expressed as

$$u_h(x, y)= a_1^{(e)}+a_2^{(e)}x+a_3^{(e)}y+a_4^{(e)}x^2+a_5^{(e)}xy+a_6^{(e)}y^2 + a_7^{(e)}x^3+a_8^{(e)}x^2y+a_9^{(e)}xy^2+a_{10}^{(e)}y^3+a_{11}^{(e)}x^3y+a_{12}^{(e)}xy^3. \tag{2.1}$$

(ii) $(u_h(p), \partial_x u_h(p), \partial_y u_h(p))^*$ takes the prescrived value $(\alpha_p, \beta_p, \gamma_p)$ at each grid point $p \in \Omega_h$, and equals to (0, 0, 0) for the point $p \in \partial\Omega_h$.

The polynomial defined by (2.1) is the "*shape function*" which has been used by Adini-Clough and Melosh in defining the rectanguler element stiffness. For the theoretical treatment we transform this expression into another one.

Consider the following functions for each point $p \in \Omega_h$.

(1) $\varphi_p^{(0)}$;

$$(\varphi_p^{(0)},\ \partial_x\varphi_p^{(0)},\ \partial_y\varphi_p^{(0)})=(1,\ 0,\ 0) \text{ at } p$$
$$=(0,\ 0,\ 0) \text{ at other grids,}$$

(2) $\varphi_p^{(1)}$;

$$(\varphi_p^{(1)},\ \partial_x\varphi_p^{(1)},\ \partial_y\varphi_p^{(1)})=(0,\ 1,\ 0) \text{ at } p$$
$$=(0,\ 0,\ 0) \text{ at other grids,}$$

(3) $\varphi_p^{(2)}$;

$$(\varphi_p^{(2)},\ \partial_x\varphi_p^{(2)},\ \partial_y\varphi_p^{(2)})=(0,\ 0,\ 1) \text{ at } p$$
$$=(0,\ 0,\ 0) \text{ at other grids,}$$

where $\varphi_p^{(k)}$ has the expression like (2.1) in each square element. Then it is evident that the function $w_h(x, y)$ can be expressed also in the following form.

(2.2) $$w_h(x,y)=\sum_{p=1}^{M}\left\{w_h(p)\varphi_p^{(0)}+\partial_x w_h(p)\varphi_p^{(1)}+\partial_y w_h(p)\varphi_p^{(2)}\right\}.$$

We call the system $\{\varphi_p^{(0)},\ \varphi_p^{(1)},\ \varphi_p^{(2)}\}$ ($p=1, 2, \ldots, M$), where M is the number of grids in Ω_h, Adini-Clough-Melosh's (ACM) basis. As easily seen ACM-basis is continuous on Ω_h but not continuously differentiable.

3. Finite element method for biharmonic equations

The problem considered in this paper is

(3.1) $$\Delta^2 w=f \quad \text{in } \Omega,$$

(3.2) $$w=\frac{dw}{dn}=0 \quad \text{on } \partial\Omega,$$

where Δ^2 denotes the biharmonic operator and n is the outward normal to $\partial\Omega$. We assume that f is sufficiently smooth in $\bar{\Omega}$. Therefore, as well known, this boundary value problem has a unique solution w which is sufficiently smooth in $\bar{\Omega}$.

* $\partial_x w$ denotes $\partial w/\partial x$.

In finite element method, the unknowns $\{a_k^{(e)}\}$ in (2.1) are determined so as to minimize the following functional.

(3.3) $$F(w_h)=\sum_e \|w_h\|_{2,e}^2-2(f,w_h)_{\Omega_h},$$

where $(f,g)_{\Omega_h}=\int_{\Omega_h} fg\,dxdy$ and

(3.4) $$(u,v)_{2,e}=(\partial_{xx}u,\ \partial_{xx}v)_e+2(\partial_{xy}u,\ \partial_{xy}v)_e+(\partial_{yy}u,\ \partial_{yy}v)_e,$$
$$\|u\|_{2,e}^2=(u,u)_{2,e}.$$

THEOREM 1. *The above procedure is equivalent to the following one; Determine the unknowns* $\{w_p^{(0)},\ w_p^{(1)},\ w_p^{(2)}\}$ *in*

(3.5) $$w_h=\sum_{p=1}^{M}\{w_p^{(0)}\varphi_p^{(0)}+w_p^{(1)}\varphi_p^{(1)}+w_p^{(2)}\varphi_p^{(2)}\}$$

by solving the system of equations

(3.6) $$\sum_e (w_h,\ \varphi_p^{(k)})_{2,e}=(f,\ \varphi_p^{(k)})\qquad (k=0,\ 1,\ 2)$$
$$(p=1,\ 2,\ \dots,\ M).$$

To prove this we provide

LEMMA 1. *For any function* w *of the form* (3.5) *holds the following inequality.*

(3.7) $$\operatorname*{Max}_{\bar{\Omega}_h}|w|\leq d\|w_{xy}\|_{\Omega_h}\qquad (d\text{: diameter of }\Omega_h),$$

where $\|\cdot\|_{\Omega_h}$ *denotes the usual* L^2*-norm.*

PROOF. Let the upper and lower parts of $\partial\Omega_h$ be denoted by the equations $y=y_1(x)$, $y=y_0(x)$ and the right and left parts by $x=x_1(y)$, $x=x_0(y)$ respectively. Since w is piecewise smooth in Ω_h and Ω_h is partitioned by the lines paralell to the coordinate axes, w_x exists and piecewise smooth with respect to y except finite number of x. Therefore, the function w_{xy} exists as the function in $L^2(\Omega_h)$ and we can write that

(3.8) $$w_x(x,y)=\int_{y_0}^{y} w_{xy}(x,y)dy\qquad (\text{except finite } x),$$

since $w_x(x,y_0)=0$ (if necessary, we extend w to whole space by setting $w=0$ in $R^2-\Omega_h$). Hence by Schwarz' inequality

(3.9) $$w_x^2(x,y)\leq d\int_{y_0}^{y_1} w_{xy}^2(x,y)dy.$$

On the other hand,

$$w^2(x, y) \leqq d\int_{x_0}^{x_1} u_x^2(x, y)dx. \tag{3.10}$$

Substituting (3.9) into (3.10) we obtain (3.7).

PROOF OF THEOREM 1. First we note that we can obtain the same function whether we minimize $F(u_h)$ by $\{a_k^e\}$ or by $\{w_p^{(0)}, w_p^{(1)}, w_p^{(2)}\}$, since the two expressions are equivalent. Now multiplying the first, second and third equations in (3.6) by $w_p^{(0)}$, $w_p^{(1)}$ and $w_p^{(2)}$ respectively, summing first each equation on p and then the resulting three equations, we obtain

$$\sum_e \|u_h\|_{2,e}^2 = (f, w_h)_{\Omega_h}. \tag{3.11}$$

By the inequality (3.7), if the left-hand side of the above equation equals to zero then $u_h=0$, and consequently the equation (3.6) has a unique solution. On the other hand, $F(u_h)$ is a quadratic form on $\{w_p^{(k)}\}$, and moreover this form is positive definite by the inequality (3.7). Since the stationary condition of $F(u_h)$ is nothing but the equation (3.6), the theorem is now completely proved.

The behaviour of u_h as $h \to 0$ is a little complicated compared with those of the approximate solutions obtained by the usual variational methods. First we observe that by Green's identity

$$\sum_e (w, w_h)_{2,e} = (f, w_h)_{\Omega_h} + \mathrm{G}(w, u_h), \tag{3.12}$$

where

$$\mathrm{G}(w, w_h) = \sum_e \int_{\partial e} \{\partial_{xx} w \cos(n, x)\partial_x w_h + \partial_{xy} w [\cos(n, x)\partial_y w_h + \cos(n, y)\partial_x w_h] + \partial_{yy} w \cos(n, y)\partial_y w_h\}\, ds. \tag{3.13}$$

Therefore, the functional $F(u_h)$ is expressed also by

$$F(u_h) = \sum_e \|w - w_h\|_{2,e}^2 + 2G(w, u_h) - \|w\|_{2,\Omega_h}^2 \tag{3.14}$$

The approximate solution is thus so determined as to minimize *the sum of the first and second terms* of the right hand side of (3.14). Therefore we have to estimate the quantity $G(w, w_h)$ for the obtained approximate solution u_h to prove the convergence of the approximate solutions. This will be done in the next section.

4. Estimation of $G(w, \varphi_p^{(k)})$

Without loss of generality we can assume that p is the origin of the coordinate. In the present section we thus drop the suffix p of $\varphi_p^{(k)}$. Let e_k $(k=1,2,3,4)$ be the square element which is in the k-th quadrant and has the origin as a corner.

(a) *Estimation of* $G(w,\varphi^{(0)})$: Consider the function

$$\Psi_0(x,y)=1-\frac{3}{h^2}x^2-\frac{1}{h^2}xy-\frac{3}{h^2}y^2+\frac{2}{h^3}x^3+\frac{3}{h^3}x^2y+\frac{3}{h^3}xy^2+\frac{2}{h^3}y^3-\frac{2}{h^4}x^3y-\frac{2}{h^4}xy^3. \tag{4.1}$$

By using this function $\varphi^{(0)}$ can be written explicitly as follows.

$$\varphi^{(0)}(x,y)=\begin{cases}\Psi_0(x,y) & \text{in } e_1\\ \Psi_0(-x,y) & \text{in } e_2\\ \Psi_0(-x,-y) & \text{in } e_3\\ \Psi_0(x,-y) & \text{in } e_4.\end{cases} \tag{4.2}$$

First we find that

$$\sum_{i=1}^{4}\int_{\partial e_i}\partial_{xy}w\cos(n,x)\partial_y\varphi^{(0)}ds=0,$$

$$\sum_{i=1}^{4}\int_{\partial e_i}\partial_{xy}w\cos(n,y)\partial_x\varphi^{(0)}ds=0.$$

Therefore, by symmetry it suffices to estimate the quantity

$$E^0=\sum_{i=1}^{4}\int_{\partial e_i}\partial_{xx}w\cos(n,x)\partial_x\varphi^{(0)}ds. \tag{4.3}$$

Define

$$\beta(y)=-\frac{1}{h^2}y+\frac{3}{h^3}y^2-\frac{2}{h^4}y^3. \tag{4.4}$$

Then by (4.1) we see that, for example, in e_1

$$\lim_{\varepsilon\to+0}\cos(n,x)\partial_x\varphi^{(0)}(\varepsilon,y)=-\beta(y),$$

$$\lim_{\varepsilon\to+0}\cos(n,x)\partial_x\varphi^{(0)}(h-\varepsilon,y)=\beta(y)$$

for any $0\leqq y\leqq h$, where n denotes the outward normal to ∂e_1 from the inside of e_1. The similar relations hold for the other integrands in (4.3), and hence E^0 can be written as follows.

$$E^0=\int_0^h [\partial_{xx}w(h,y)-2\partial_{xx}w(0,y)+\partial_{xx}w(-h,y)]\beta(y)dy$$
$$+\int_{-h}^0 [\partial_{xx}w(h,y)-2\partial_{xx}w(0,y)+\partial_{xx}w(-h,y)]\beta(-y)dy.$$

By using the Taylor expansion and the relations

$$\int_0^h \beta(y)dy=0,\quad \int_0^h y\beta(y)dy=\frac{1}{60}h,\quad \int_0^h y^2\beta(y)dy=\frac{1}{60}h^2,$$

we can prove that

$$E^0=\partial_{4x2y}w(0,0)\frac{h^4}{60}+O(h^6)\quad (\partial_{4x2y}=\partial_{xxxxyy}),$$

and consequently we have

LEMMA 2.

$$G(w,\varphi_p^{(0)})=[\partial_{4x2y}w(p)+\partial_{2x4y}w(p)]\frac{h^4}{60}+O(h^6). \tag{4.5}$$

(b) *Estimation of* $G(w,\varphi^{(1)})$: Define

$$\Psi_1(x,y)=x-\frac{2}{h}x^2-\frac{1}{h}xy+\frac{1}{h^2}x^3+\frac{2}{h^2}x^2y-\frac{1}{h^3}x^3y. \tag{4.6}$$

Then $\varphi^{(1)}$ can be written as follows.

$$\varphi^{(1)}(x,y)=\begin{cases}\Psi_1(x,y) & \text{in} \quad e_1\\ -\Psi_1(-x,y) & \text{in} \quad e_2\\ -\Psi_1(-x,-y) & \text{in} \quad e_3\\ \Psi_1(x,-y) & \text{in} \quad e_4.\end{cases} \tag{4.7}$$

Since

$$\sum_{i=1}^4 \int_{\partial e_i} \partial_{xy}w\ \cos(n,x)\partial_y\varphi^{(1)}ds=0,$$
$$\sum_{i=1}^4 \int_{\partial e_i} \partial_{xy}w\ \cos(n,y)\partial_x\varphi^{(1)}ds=0,$$
$$\sum_{i=1}^4 \int_{\partial e_i} \partial_{xx}w\ \cos(n,x)\partial_x\varphi^{(1)}ds=0,$$

we see that

$$G(w,\varphi^{(1)})=\sum_{i=1}^4 \int_{\partial e_i} \partial_{yy}w\ \cos(n,y)\partial_y\varphi^{(1)}ds$$
$$=\int_0^h [\partial_{yy}w(x,h)-2\partial_{yy}w(x,0)+\partial_{yy}w(x,-h)]\beta'(x)dx$$
$$-\int_{-h}^0 [\partial_{yy}w(x,h)-2\partial_{yy}w(x,0)+\partial_{yy}w(x,-h)]\beta'(-x)dx,$$

where

$$\beta'(x) = -\frac{1}{h}x + \frac{2}{h^2}x^2 - \frac{1}{h^3}x^3,$$

and hence we can prove

LEMMA 3.

$$G(w, \varphi_p^{(1)}) = -\partial_{4yx}w(p)\frac{1}{15}h^4 + O(h^6),$$
(4.8)
$$G(w, \varphi_p^{(2)}) = -\partial_{4xy}w(p)\frac{1}{15}h^4 + O(h^6).$$

Let us put

$$M_0 = \operatorname*{Max}_p |w_p^{(0)}|, \ M_1 = \operatorname*{Max}_p (|w_p^{(1)}|, \ |w_p^{(2)}|).$$

Then by Lemma 2 and Lemma 3 we obtain

THEOREM 2. *For any w_h of the form* (3.5) *it holds that*

(4.9) $$|G(w, w_h)| \leqq \text{const} \cdot \{M_0 + M_1\} h^2,$$

where the constant depends only on the exact solution w.

5. An interpolation theorem

We shall prove

LEMMA 4. *Let w be a sufficiently smooth function defined on $e = \{(x,y);$ $0 \leqq x, y \leqq h\}$. Then there exists a function $\tilde{w}_h$ of the form* (2.1) *such that*

(5.1) $$\|w - \tilde{w}_h\|_{2,e}^2 \leqq \text{const} \cdot h^4 \quad \text{as } h \to 0,$$

where the constant depends only on w.

PROOF. Let the points $(0,0)$, $(0,h)$, $(h,0)$ and (h,h) be denoted by 1, 2, 3 and 4 respectively. Then the interpolating function

(5.2) $$\tilde{u}_h = \sum_{p=1}^{4} (w_p \varphi_p^{(0)} + \partial_x w_p \varphi_p^{(1)} + \partial_y w_p \varphi_p^{(2)})$$

is the desired one. To prove this we first note that by the definition of $\varphi_p^{(k)}$ the following equalities hold.

$$\sum_{p=1}^{4} w_p \partial_{xx}\varphi_p^{(0)} = -\frac{6}{h^2}(w_1-w_2)+\frac{12}{h^3}(w_1-w_2)x$$

$$+\frac{6}{h^3}(w_1-w_2-w_3+w_4)y-\frac{12}{h^4}(w_1-w_2-w_3+w_4)xy,$$

$$\sum_{p=1}^{4} \partial_x w_p \partial_{xx}\varphi_p^{(1)} = -\frac{2}{h}(2\partial_x w_1+\partial_x w_2)+\frac{6}{h^2}(\partial_x w_1+\partial_x w_2)x$$

$$+\frac{2}{h^2}(2\partial_x w_1+\partial_x w_2-2\partial_x w_3-\partial_x w_4)y$$

$$-\frac{6}{h^3}(\partial_x w_1+\partial_x w_2-\partial_x w_3-\partial_x w_4)xy,$$

$$\sum_{p=1}^{4} \partial_y w_p \partial_{xx}\varphi_p^{(2)}=0.$$

Therefore it is easy to see that

$$\partial_{xx}\tilde{w}_h=\partial_{xx}w+O(h) \qquad \text{in } e.$$

By the same way we have

$$\partial_{xy}\tilde{w}_h=\partial_{xy}w+O(h) \qquad \text{in } e,$$

$$\partial_{yy}\tilde{w}_h=\partial_{yy}w+O(h) \qquad \text{in } e.$$

The estimation (5.2) is thus evident.

By this lemma we have

THEOREM 3. *If w is sufficiently smooth in $\bar{\Omega}_h$, then there exists a function $\tilde{w}_h$ of the form* (2.1) *such that*

$$\sum_e \|w-\tilde{w}_h\|_{2,e}^2 \leqq \text{const}\cdot h^2 \quad \text{as } h\to 0.$$

6. Convergence of the approximate solutions

By the preceding results we can obtain some a priori estimates necessary for proving the convergence of approximate solutions. Let w and w_h be the exact and approximate solutions respectively.

LEMMA 5. *w_h is uniformly bounded as $h\to 0$.*

PROOF. By (3.11) and Schwarz' inequality

$$\sum_e \|w_h\|_{2,e}^2 \leqq \|f\|_{\Omega_h}\|w_h\|_{\Omega_h}, \tag{6.1}$$

and by the lemma 1

$$\|w_h\|_{\Omega_h} \leqq \text{const}\cdot \operatorname*{Max}_{\bar{\Omega}_h} |w_h| \leqq \text{const}\cdot \sqrt{\sum_e \|w_h\|_{2,e}^2}. \tag{6.2}$$

Therefore by (6.1)

$$\sqrt{\sum_e \|w_h\|_{2,e}^2} \leqq \text{const}\cdot \|f\|_{\Omega_h}, \tag{6.3}$$

and again by lemma 1 we have

$$\operatorname*{Max}_{\bar{\Omega}_h} |w_h| \leqq \text{const}\cdot \sqrt{\sum_e \|w_h\|_{2,e}^2} \leqq \text{const}\cdot \|f\|_{\Omega_h},$$

which completes the proof.

LEMMA 6.

(i) $\operatorname*{Max}_p |w_p^{(1)}|$, $\operatorname*{Max}_p |w_p^{(2)}| \leqq \text{const}\cdot h^{-1}$,

(ii) $|G(w, w_h)| \leqq \text{const}\cdot h$,

(iii) $\sum_e \|w - w_h\|_{2,e}^2 \leqq \text{const}\cdot h$,

PROOF. (i) follows from lemma 5 and the Markov's inequality [2]. (ii) is the immediate consequence of the theorem 2 and (i) of this lemma. Proof of (iii). For the interpolationg function $\tilde{w}_h$ given in theorem 3 holds

$$\sum_e \|w - \tilde{w}_h\|_{2,e}^2 \leqq \text{const}\cdot h^2. \tag{6.4}$$

Therefore by theorem 2

$$\begin{aligned} F(\tilde{w}_h) &= \sum_e \|\tilde{w}_h\|_{2,e}^2 - 2(f, \tilde{w}_h) \\ &= \sum_e \|w - \tilde{w}_h\|_{2,e}^2 + 2G(w, \tilde{w}_h) - \|w\|_{2,\Omega_h}^2 \\ &\leqq -\|w\|_{2,\Omega_h}^2 + \text{const}\cdot h^2. \end{aligned}$$

On the other hand, it holds that

$$F(w_h) \leqq F(\tilde{w}_h),$$

so that

$$\begin{aligned} &\sum_e \|w - w_h\|_{2,e}^2 + 2G(w, w_h) - \|w\|_{2,\Omega_h}^2 \\ &\quad = F(w_h) \leqq -\|w\|_{2,\Omega_h}^2 + \text{const}\cdot h^2, \end{aligned}$$

and consequently by (ii) we have the estimation (iii).

Finally we get

THEOREM 4. *If the exact solution w is sufficiently smooth in $\bar{\Omega}$, then the approximate solution w_h converges to w on $\bar{\Omega}$, and the order of convergence is given by*

$$\text{(6.5)} \qquad \underset{\bar{\Omega}}{\text{Max}}\, |w-w_h| \leqq \text{const}\cdot\sqrt{h} \quad \text{as } h\to 0,$$

where if $\Omega-\Omega_h\neq\phi$ we extend w_h to $\Omega-\Omega_h$ by setting $w_h=0$ in $\Omega-\Omega_h$.

PROOF. By the similar way to the proof of lemma 1, it can be shown that

$$\underset{\bar{\Omega}}{\text{Max}}\, |w-w_h| \leqq d\|\partial_{xy}(w-w_h)\|_{\Omega}.$$

Therefore, by the assumption (2) and the estimate (iii) of the lemma 6, we have

$$\underset{\bar{\Omega}}{\text{Max}}\, |w-w_h|^2 \leqq \text{const}\cdot\Big(\sum_e \|w-w_h\|^2_{2,e} + \|w-w_h\|^2_{2,\Omega-\Omega_h}\Big)$$
$$\leqq \text{const}\cdot h,$$

which completes the proof.

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