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A NOTE ON KÄHLERIAN HYPERSURFACES OF SPACES OF CONSTANT CURVATURE

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1. Introduction.

It is known that a Sasakian manifold M^{2n+1} which is isometrically immersed in a Riemannian manifold $\tilde{M}^{2n+2}$ of constant curvature $\tilde{c}\neq 1$ is of constant curvature 1 ([4]). In the case when $\tilde{c}=1$, the Sasakian manifold is of constant curvature 1 if and only if it is of constant scalar curvature $2n(2n+1)$ ([3]). In this note, we study the Kählerian analogues.

For notations and fundamental facts, we refer to [1] and [2].

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2. Fundamental formulas.

Let (M^{2n}, g, J) be a Kählerian manifold which is isometrically immersed in a Riemannian manifold $(\tilde{M}^{2n+1}, \tilde{g})$ of constant curvature $\tilde{c}$. Then we have the equation of Gauss:

$$R(X, Y) = \tilde{c}X\wedge Y + AX\wedge AY. \tag{2.1}$$

In particular, we have

$$\begin{aligned} R(X, Y)JZ = {} & \tilde{c}\{g(Y, JZ)X - g(X, JZ)Y\} \\ & + g(AY, JZ)AX - g(AX, JZ)AY. \end{aligned} \tag{2.2}$$

On the other hand, since the Riemannian connection of a Kählerian manifold is almost complex, taking account of (2.1), we get

$$\begin{aligned} R(X, Y)JZ = {} & \tilde{c}\{g(Y, Z)JX - g(X, Z)JY\} \\ & + g(AY, Z)JAX - g(AX, Z)JAY. \end{aligned} \tag{2.3}$$

According to (2.1), we get

$$S(X, Y) = (2n-1)\tilde{c}g(X, Y) + g(AX, Y)\mathrm{Tr}A - g(AX, AY). \tag{2.4}$$

On the other hand, since we have $S(X, Y) = 1/2\{\text{trace of } J\circ R(X, JY)\}$ for a

Kählerian manifold, taking account of (2.1), we get

$$S(X, Y)=\bar{c}g(X, Y)+g(JAX, AJY). \tag{2.5}$$

Now, let $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ be the principal curvatures at a point of M^{2n} and let $e_1, e_2, \cdots, e_{2n}$ be the corresponding principal directions: $Ae_j=\lambda_j e_j$, $1\leqq j\leqq 2n$. For $i\neq j$, (2.2) and (2.3) respectively imply

$$R(e_i, e_j)Je_i=(\bar{c}+\lambda_i\lambda_j)g(e_j, Je_i)e_i,$$

$$R(e_i, e_j)J\bar{c}_i=-(\bar{c}+\lambda_i\lambda_j)Je_j.$$

Thus we get

$$(\bar{c}+\lambda_i\lambda_j)\{g(e_j, Je_i)e_i+Je_j\}=0, \tag{2.6}$$

and consequently, we get either (I) or (II):

(I) $\bar{c}+\lambda_i\lambda_j=0$ *for all* $i\neq j$,

(II) $e_j=Je_i$ *for some* $i\neq j$.

Case (I): $\bar{c}=0$ implies that rank $A\leqq 1$, and hence M^{2n} is of constant curvature 0 at the point in consideration. $\bar{c}\neq 0$ implies that $\lambda_1=\lambda_2=\cdots=\lambda_{2n}=\lambda\neq 0$ in the case when $n\geqq 2$. In this case, M^{2n} is of constant curvature $\bar{c}+\lambda^2$ at the point in consideration. In general, if a Kählerian manifold of complex dimension $\geqq 2$ is of constant curvature at a point, then it is flat at the point in consideration (see, for example, [5]). Hence M^{2n} is flat at the point. In particular, we have $\bar{c}<0$.

Case (II): (2.4) implies

$$S(e_i, e_j)=(2n-1)\bar{c}\delta_{ij}+\lambda_i(\sum\lambda_k)\delta_{ij}-\lambda_i^2\delta_{ij}. \tag{2.7}$$

On the other hand, (2.5) implies

$$S(e_i, e_j)=\bar{c}\delta_{ij}+\lambda_i g(Je_i, AJe_j). \tag{2.8}$$

Hence, according to (2.7) and (2.8), we get

$$(2n-2)\bar{c}+\lambda_i(\sum\lambda_k)-\lambda_i^2=\lambda_i g(Je_i, AJe_i). \tag{2.9}$$

Now, suppose we have $Je_i=e_j$, then (2.9) implies

$$\lambda_i(\sum_{k\neq i,j}\lambda_k)=-(2n-1)\bar{c}, \tag{2.10}$$

$$\lambda_j(\sum_{k\neq i,j}\lambda_k)=-(2n-1)\bar{c}. \tag{2.10$'$}$$

Thus, if $n\geqq 2$ and $\bar{c}\neq 0$, then $\lambda_i=\lambda_j\neq 0$ holds good.

3. The case $\tilde{c}=0$.

In this section, we assume $\tilde{c}=0$. Suppose there were 3 non-zero principal curvatures, say λ_1, λ_2 and λ_3. Then (2.6) implies $e_1=\pm Je_2$ and $e_1=\pm Je_3$, and hence $e_2=\pm e_3$. This is a contradiction. Thus there are at most two non-zero principal curvatures, say λ_1 and λ_2. In this case, (2.7) implies that the scalar curvature of M^{2n} is equal to $2\lambda_1\lambda_2$. Hence we get

THEOREM 1. *A Kählerian manifold which is a hypersurface of a flat Riemannian manifold is flat if and only if it has a constant scalar curvature* 0.

4. The case $\tilde{c}\neq 0$.

In this section, we assume $\tilde{c}\neq 0$ and $n\geqq 2$. Suppose (I) holds at a point. Then there is only one principal curvature at the point in consideration.

For a Kählerian manifold, we have $R(JX, JY)=R(X, Y)$. Hence, taking account of (2.1), we get

$$(4.1)\qquad \begin{aligned}(\tilde{c}+\lambda_i\lambda_j)\{g(e_j, e_k)e_i-g(e_i, e_k)e_j\}\\ = c\{g(Je_j, e_k)Je_i-g(Je_i, e_k)Je_j\}\\ +g(AJe_j, e_k)AJe_i-g(AJe_i, e_k)AJe_j.\end{aligned}$$

Now, suppose (II) holds at a point. Then we may assume that $Je_1=e_2$ holds. In this case, (4.1) with $i=1$ and $j=k\neq 1, 2$ implies

$$(4.2)\qquad \tilde{c}+\lambda_1\lambda_j=0,$$

and hence we get $\lambda_3=\lambda_4=\cdots=\lambda_{2n}$. Thus we may suppose $Je_3=e_4$. If $n\geqq 3$, we get $\lambda_1=\lambda_2=\lambda_3=\cdots=\lambda_{2n}$. If $n=2$, we get $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$ and $\tilde{c}+\lambda_1\lambda_3=0$.

THEOREM 2. *A Kählerian manifold M^{2n}, $n\geqq 3$, which is isometrically immersed in a Riemannian manifold $\tilde{M}^{2n+1}$ of constant curvature $\tilde{c}\neq 0$ is flat. Moreover, in this case, we have $\tilde{c}<0$.*

PROOF. By the above argument, M^{2n} is totally umbilic and hence of constant curvature. Thus M^{2n} is flat. Q.E.D.

THEOREM 3. *If a Kählerian manifold M^4 is isometrically immersed in a real space form $\tilde{M}^5$ of constant curvature $\tilde{c}\neq 0$,* then *M^4 has a non-negative scalar curvature, and M^4 has a constant scalar curvature* 0 *if and only if M^4 is flat. In the latter case, we have $\tilde{c}<0$.*

PROOF. By the argument above, we see that there are two cases. The first case is that all the principal curvatures are the same at a point, and the second case is that $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$ and $\bar{c}+\lambda_1\lambda_3=0$ hold at a point. In the first case, M^4 is of constant curvature and hence flat at the point in consideration. In the second case, (2.7) implies that the scalar curvature is equal to $2(\lambda_1-\lambda_3)^2$ at the point. Thus these two cases imply Theorem 3. Q. E. D.

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