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A NOTE ON BLOCKS AND DEFECT GROUPS OF A FINITE GROUP

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Let $\mathfrak{G}$ be a group of finite order and $\Lambda[\mathfrak{G}]$ the group algebra of $\mathfrak{G}$ over a field Λ of characteristic p, p a rational prime, and Z the center of $\Lambda[\mathfrak{G}]$. R. Brauer, in his paper [3], considered certain subspaces of the dual spaces of the block ideals in Z and, in terms of them, gave important results concerning blocks, defect groups and p-sections of $\mathfrak{G}$ (cf. also Brauer [1, 2]). One of the authors, in [5], has approached some of the results in the way which was adopted in Osima [7], Iizuka [4] and Iizuka-Sasaki [6]. In the present paper, we shall work connecting with the properties (i) and (ii) in Introduction of [3] or with Proposition [2. B] of [5].

1. Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_n$ be the classes of conjugate elements in $\mathfrak{G}$ and denote by K_α the sum of all elements belonging to $\mathfrak{K}_\alpha$; $K_1, K_2, \cdots, K_n$ form a Λ-basis of Z. Let

$$1=\eta_1+\eta_2+\cdots\cdots+\eta_s \tag{1.1}$$

be the decomposition of the unit element 1 of Z to the mutually orthogonal primitive idempotents of Z. Corresponding to (1.1), we have the block decomposition of Z:

$$Z=B_1\oplus B_2\oplus\cdots\cdots\oplus B_s, \tag{1.2}$$

where each B_τ is a nonzero indecomposable ideal in Z and the notation $\oplus$ means an internal direct sum.

Let $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_r$ be a complete system of representatives for the classes of conjugate p-subgroups in $\mathfrak{G}$. If $\mathfrak{P}_i$ is a Sylow p-subgroup of the normalizer of an element of $\mathfrak{K}_\alpha$, we refer $\mathfrak{P}_i$ as the *defect group* of the class $\mathfrak{K}_\alpha$ and denote it by $D(\mathfrak{K}_\alpha)$. We denote by $V(\mathfrak{P}_i)$ the linear subspace of Z which is spanned by all K_β with $\mathfrak{P}_i=D(\mathfrak{K}_\beta)$. We set

$$V^{\tau}_{\mathfrak{P}_i}=B_\tau\cap\sum_{\mathfrak{P}_j\le\mathfrak{P}_i}\oplus V(\mathfrak{P}_j),$$

where $\mathfrak{P}_j\le\mathfrak{P}_i$ means that $\mathfrak{P}_j$ is conjugate to a subgroup of $\mathfrak{P}_i$, and set

$$W^{\tau}_{\mathfrak{P}_i}=B_\tau\cap\sum_{\mathfrak{P}_j<\mathfrak{P}_i}\oplus V(\mathfrak{P}_j),$$

where $\mathfrak{P}_j < \mathfrak{P}_i$ means that $\mathfrak{P}_j$ is conjugate to a *proper* subgroup of $\mathfrak{P}_i$.

Let x_τ be the Λ-dimension of B_τ. As was shown in [6], with each B_τ, we can associate x_τ classes $\mathfrak{K}_\beta$ as follows: (i) Each $\mathfrak{K}_\alpha$ is associated with one and only one block. (ii) If $\mathfrak{K}_1^\tau, \mathfrak{K}_2^\tau, \cdots, \mathfrak{K}_{x_\tau}^\tau$ are the classes associated with B_τ, then $K_1^\tau\eta_\tau, K_2^\tau\eta_\tau, \cdots, K_{x_\tau}^\tau\eta_\tau$ form a Λ-basis of B_τ. (Cf. [4, 5]) In the same way as in the proof of [2. A] in [5], we obtain the following:

[**1. A**] *Let $\mathfrak{K}_1^\tau, \mathfrak{K}_2^\tau, \cdots, \mathfrak{K}_{x_\tau}^\tau$ be the classes associated with B_τ in the above sense. If $\mathfrak{K}_{i\mu}^\tau$ $(\mu = 1, 2, \cdots, m(\tau, i))$ are the classes $\mathfrak{K}_\beta^\tau$ with $\mathfrak{P}_i = D(\mathfrak{K}_\beta^\tau)$, then $K_{j\nu}^\tau\eta_\tau$ $(\nu = 1, 2, \cdots, m(\tau, j);\ \mathfrak{P}_j \leq \mathfrak{P}_i)$ form a Λ-basis of $V_{\mathfrak{P}_i}^\tau$ and the residue classes modulo $W_{\mathfrak{P}_i}^\tau$, $K_{i\mu}^\tau\eta_\tau + W_{\mathfrak{P}_i}^\tau$ $(\mu = 1, 2, \cdots, m(\tau, i))$, form a Λ-basis of the factor space $V_{\mathfrak{P}_i}^\tau / W_{\mathfrak{P}_i}^\tau$.*

Let $\hat{Z}$ be the dual space of Z, *i.e.*, $\hat{Z}$ is the linear space which consists of all linear mappings of Z to Λ. Corresponding to (1.2), the space $\hat{Z}$ is decomposed in

$$\hat{Z} = \hat{B}_1 \oplus \hat{B}_2 \oplus \cdots\cdots \oplus \hat{B}_s, \tag{1.3}$$

where each $\hat{B}_\tau$ may be regarded as the dual space of B_τ, *i.e.*, $\hat{B}_\tau$ consists of all elements f of $\hat{Z}$ such that f vanishes on all B_λ with $\lambda \neq \tau$. Let U_i^τ be the linear subspace of B_τ spanned by those elements $K_{i\nu}^\tau\eta_\tau$ $(\nu = 1, 2, \cdots, m(\tau, i))$:

$$B_\tau = U_1^\tau \oplus U_2^\tau \oplus \cdots\cdots \oplus U_r^\tau. \tag{1.4}$$

Corresponding to (1.4), we have

$$\hat{B}_\tau = \hat{U}_1^\tau \oplus \hat{U}_2^\tau \oplus \cdots\cdots \oplus \hat{U}_r^\tau, \tag{1.5}$$

where each $\hat{U}_i^\tau$ may be regarded as the dual space of U_i^τ. Then, in the same way as in the proof of [2. B] of [5], we obtain the following proposition:

[**1. B**] (1) *For any nonzero element f of $\hat{U}_i^\tau$, there exists at least one class $\mathfrak{K}_\alpha$ with $\mathfrak{P}_i = D(\mathfrak{K}_\alpha)$ and with $f(K_\alpha) \neq 0$.*

(2) *If $\mathfrak{K}_\beta$ is a class such that the defect group $D(\mathfrak{K}_\beta)$ does not contain any subgroup conjugate to $\mathfrak{P}_i$, then $f(K_\beta)$ vanishes for all elements f of $\hat{U}_i^\tau$.*

This proposition is a supplement to [2. B] of [5], refered in the following:

I. *For $V = \hat{U}_i^\tau$, the following two conditions are satisfied:*

(1) *For any nonzero element f of V, there exists at least one class $\mathfrak{K}_\alpha$ with $\mathfrak{P}_i = D(\mathfrak{K}_\alpha)$ and with $f(K_\alpha) \neq 0$.*

(2) *If the order of $D(\mathfrak{K}_\beta)$ is smaller than that of $\mathfrak{P}_i$, then $f(K_\beta)$ vanishes for all elements f of V.*

II. *$\hat{U}_i^\tau$ has the maximum dimension of all linear subspaces V of $\hat{B}_\tau$ which satisfy the conditions* (1) *and* (2) *of* I.

I and II play an important rôle in [3]. In the later sections, we shall study the conditions (1) and (2) of I.

2. We shall keep the notation of the preceding section. We denote by $|\mathfrak{H}|$ the order of a subgroup $\mathfrak{H}$ of $\mathfrak{G}$. In the chain of p-subgroups $\mathfrak{P}_i$,

$$\mathfrak{P}: \mathfrak{P}_1, \mathfrak{P}_2, \cdots\cdots, \mathfrak{P}_r,$$

we assume that $|\mathfrak{P}_i| \leq |\mathfrak{P}_j|$ for $1 \leq i < j \leq r$. Let $h_1, h_2, \cdots, h_e$ be the numbers such that $|\mathfrak{P}_{h_i}| < |\mathfrak{P}_{h_{i+1}}|$, $i = 1, 2, \cdots, e-1$, and that $|\mathfrak{P}_j| = |\mathfrak{P}_{h_i}|$ for $h_{i-1} < j \leq h_i$, $i = 1, 2, \cdots, e$, where $h_0 = 0$. Let

$$\mathfrak{X}: X_1, X_2, \cdots\cdots, X_e$$

be a system of linear subspaces of B_τ. Assume that the system $\mathfrak{X}$ satisfies the following condition:

(*a*) *For* $i = 1, 2, \cdots, e$,

$$X_1 \oplus X_2 \oplus \cdots\cdots \oplus X_i = U_1^\tau \oplus U_2^\tau \oplus \cdots\cdots \oplus U_{h_i}^\tau \quad (= V_{h_i}^\tau) \tag{2.1}$$

holds.

From (2.1) for $i = e$, we get

$$B_\tau = X_1 \oplus X_2 \oplus \cdots\cdots \oplus X_e. \tag{2.2}$$

Corresponding to (2.2), we have

$$\hat{B}_\tau = \hat{X}_1 \oplus \hat{X}_2 \oplus \cdots\cdots \oplus \hat{X}_e, \tag{2.3}$$

where each $\hat{X}_i$ may be regarded as the dual space of X_i. Then the condition (*a*) is equivalent to the following:

($\hat{a}$) *For* $i = 0, 1, 2, \cdots, e-1$,

$$\hat{X}_{i+1} \oplus \hat{X}_{i+2} \oplus \cdots\cdots \oplus \hat{X}_e = \hat{U}_{h_i+1}^\tau \oplus \hat{U}_{h_i+2}^\tau \oplus \cdots\cdots \oplus \hat{U}_r^\tau \tag{2.4}$$

holds.

We denote by $p^{d(\mathfrak{K}_\alpha)}$ the order of $D(\mathfrak{K}_\alpha)$ and by p^{d_i} the order of $\mathfrak{P}_i$; $d(\mathfrak{K}_\alpha)$ is called the *defect* of $\mathfrak{K}_\alpha$. We then have the following:

[**2. A**] I. *For $V = \hat{X}_i$, the following two conditions are satisfied.*

(1) *For any nonzero element f of V, there exists at least one class $\mathfrak{K}_\alpha$ with $d(\mathfrak{K}_\alpha) = d_i$ and with $f(K_\alpha) \neq 0$.*

(2) *If $\mathfrak{K}_\beta$ is a class such that $d(\mathfrak{K}_\beta) < d_i$, then $f(K_\beta)$ vanishes for all elements f of V.*

II. *$\hat{X}_i$ has the maximum dimension of all linear subspaces V of $\hat{B}_\tau$ which* *satisfy the conditions* (1) *and* (2) *of* I *in this proposition.*

PROOF. We first show I. By (2.4) for $i=i-1$, i, we have

$$\hat{X}_i \oplus A_2 = A_1 \oplus A_2, \tag{2.5}$$

where

$$A_1 = \sum_{h_{i-1}<k\leq h_i}\oplus \hat{U}^\tau_k,\ A_2 = \sum_{j>i}\oplus \sum_{h_{j-1}<k\leq h_j}\oplus \hat{U}^\tau_k.$$

If f is an element of $\hat{X}_i$, then f belongs to $A_1 \oplus A_2$ hence, by [1. B], f vanishes on $V^\tau_{h_{i-1}}$. Let $f=f_1+f_2$ ($f_1 \in A_1$, $f_2 \in A_2$) and $f \neq 0$. Then $f_1 \neq 0$ hence, by [1. B], there exists a class $\mathfrak{K}_\alpha$ with defect d_i and with $f_1(K_\alpha) \neq 0$. On the other hand, $f_2(K_\beta)=0$ for all $\mathfrak{K}_\beta$ with $d(\mathfrak{K}_\beta) \leq d_i$. Therefore, we see that $f(K_\alpha)=f_1(K_\alpha) \neq 0$. We next show II. Let V be a linear subspace of $\hat{B}_\tau$ satisfying (1) and (2) of I. Since we have $V \subset \sum_{j>h_{i-1}}\oplus \hat{U}^\tau_j$ by (2) and $V \cap \sum_{j>h_i}\oplus \hat{U}^\tau_j = \{0\}$ by (1), we have

$$V \oplus \sum_{j>h_i}\oplus \hat{U}^\tau_j \subset \sum_{j>h_{i-1}}\oplus \hat{U}^\tau_j.$$

Hence we see that $\dim V \leq \dim \hat{X}_i$.

As a converse of [2. A], we obtain the following:

[2. B] *Let* $\mathfrak{X}$ *be a system for which* (2.2) *holds. If each* $\hat{X}_i$ *satisfies* I *of* [2. A], *then the system* $\mathfrak{X}$ *satisfy the condition* (a).

PROOF. From (2) of I, we see that

$$\sum_{j>i}\oplus \hat{X}_j \subset \sum_{j>i}\oplus \sum_{h_{j-1}<k\leq h_j}\oplus \hat{U}^\tau_k.$$

On the other hand, II of [2. A] for $\hat{X}_i = \sum_{h_{i-1}<k\leq h_i}\oplus \hat{U}^\tau_k$ and (2.3) yield that $\dim X_j = \dim \sum_{h_{j-1}<k\leq h_j}\oplus U^\tau_k$ holds for $j=1, 2, \cdots, e$. Therefore, $\mathfrak{X}$ satisfies the condition ($\hat{a}$) and hence the condition (a).

3. We shall keep the notation of the preceding sections. Let

$\mathfrak{Y}$: $Y_1, Y_2, \cdots\cdots, Y_r$

be a system of linear subspaces of B_τ. Assume that $\mathfrak{Y}$ satisfies the following condition:

(b) *For* $i=1, 2, \cdots, e$,

$$Y_1 \oplus Y_2 \oplus \cdots\cdots \oplus Y_{h_i} = U^\tau_1 \oplus U^\tau_2 \oplus \cdots\cdots \oplus U^\tau_{h_i}\ (=V^\tau_{h_i}) \tag{3.1}$$

and, for $h_{i-1} < k \leq h_i$, $i=1, 2, \cdots, e$, (3.2) $$Y_1 \oplus \cdots\cdots \oplus Y_{k-1} \oplus U_k^\tau \oplus Y_{k+1} \oplus \cdots\cdots \oplus Y_{h_i} = U_1^\tau \oplus \cdots\cdots \oplus U_{k-1}^\tau \oplus U_k^\tau \oplus U_{k+1}^\tau \oplus \cdots\cdots \oplus U_{h_i}^\tau$$

hold.

We see that $\dim Y_j = \dim U_j^\tau = m(\tau, j)$ holds for $j = 1, 2, \cdots, r$ and that

(3.3) $$B_\tau = Y_1 \oplus Y_2 \oplus \cdots\cdots \oplus Y_r$$

holds. Corresponding to (3.3), we have

(3.4) $$\hat{B}_\tau = \hat{Y}_1 \oplus \hat{Y}_2 \oplus \cdots\cdots \oplus \hat{Y}_r,$$

where each $\hat{Y}_j$ may be regarded as the dual space of Y_j. It is easy to see that (b) is equivalent to the following condition:

$(\hat{b})$ *For* $i = 0, 1, 2, \cdots, e-1,$

(3.5) $$\hat{Y}_{h_i+1} \oplus \hat{Y}_{h_i+2} \oplus \cdots\cdots \oplus \hat{Y}_r = \hat{U}_{h_i+1}^\tau \oplus \hat{U}_{h_i+2}^\tau \oplus \cdots\cdots \oplus \hat{U}_r^\tau$$

and, for $h_i < k \leq h_{i+1}$, $i = 0, 1, 2, \cdots, e-1,$

(3.6) $$\hat{U}_{h_i+1}^\tau \oplus \cdots\cdots \oplus \hat{U}_{k-1}^\tau \oplus \hat{Y}_k \oplus \hat{U}_{k+1}^\tau \oplus \cdots\cdots \oplus \hat{U}_r^\tau = \hat{U}_{h_i+1}^\tau \oplus \cdots\cdots \oplus \hat{U}_{k-1}^\tau \oplus \hat{U}_k^\tau \oplus \hat{U}_{k+1}^\tau \oplus \cdots\cdots \oplus \hat{U}_r^\tau$$

hold.

Then we have the following:

[**3. A**] I. *For* $V = \hat{Y}_k$, *the following two conditions are satisfied.*

(1) *For any nonzero element* f *of* V, *there exists at least one class* $\mathfrak{K}_\alpha$ *with* $\mathfrak{P}_k = D(\mathfrak{K}_\alpha)$ *and with* $f(K_\alpha) \neq 0$.

(2) *If* $\mathfrak{K}_\beta$ *is a class such that* $d(\mathfrak{K}_\beta) < d_k$, *then* $f(K_\beta)$ *vanishes for all elements* f *of* V.

II. $\hat{Y}_k$ *has the maximum dimension of all linear subspaces* V *of* $\hat{B}_\tau$ *which satisfy* (1) *and* (2) *of* I *in this proposition.*

PROOF. For $\hat{X}_i = \sum_{h_{i-1}<k\leq h_i}\oplus \hat{Y}_k$, we have (2.4) and, for $\hat{X}_i = \hat{Y}_k$, $A_1 = \hat{U}_k^\tau$, $A_2 = \sum_{\substack{j>h_i \\ j\neq k}}\oplus \hat{U}_j^\tau$, we have (2.5). Then we can verify [3. A] by the same way as [2. A] is proved.

As a converse of [3. A], we have the following:

[**3. B**] *Let* $\mathfrak{Y}$ *be a system for which* (3.3) *holds. If each* $\hat{Y}_k$ *satisfies* I *of* [3. A], *then the system* $\mathfrak{Y}$ *satisfies the condition* (b).

PROOF. For $\hat{X}_i = \sum_{h_{i-1}<k\leq h_i}\oplus \hat{Y}_k$, the statement I of [2. A] remains valid,

hence we have (3.5) for these $\hat{Y}_k$. Therefore, by (1) of I, we get

$$\hat{Y}_k \oplus \sum_{\substack{j>h_i \\ j\neq k}}\oplus \hat{U}_j^\tau \subset \sum_{j>h_i}\oplus \hat{U}_j^\tau.$$

Then, by the same way as (2.4) is deduced, we see that (3.6) holds.

4. The notation of the preceding sections will be used throughout, unless stated. Now, assume that the system $\mathfrak{Y}$ satisfies the following condition:

(*c*) *For* $i = 1, 2, \cdots, r$,

$$\sum_{\mathfrak{P}_j<\mathfrak{P}_i}\oplus Y_j \oplus Y_i = \sum_{\mathfrak{P}_j<\mathfrak{P}_i}\oplus U_j^\tau \oplus U_i^\tau \quad (= V_{\mathfrak{P}_i}^\tau) \tag{4.1}$$

holds.

From the condition (*c*), we see that, for $i = 1, 2, \cdots, r$,

$$\sum_{\mathfrak{P}_j<\mathfrak{P}_i}\oplus Y_j = \sum_{\mathfrak{P}_j<\mathfrak{P}_i}\oplus U_j^\tau \quad (= W_{\mathfrak{P}_i}^\tau) \tag{4.2}$$

holds and that the system $\mathfrak{Y}$ satisfies the following condition:

(*d*) *For* $i = 1, 2, \cdots, r$,

$$W_{\mathfrak{P}_i}^\tau \oplus Y_i = W_{\mathfrak{P}_i}^\tau \oplus U_i^\tau \tag{4.3}$$

holds.

Conversely, from the condition (*d*), we see that (4.2) holds for $i = 1, 2, \cdots, r$ and that the system $\mathfrak{Y}$ satisfies the condition (*c*).

As a special case of (4.1), we have

$$B_\tau = Y_1 \oplus Y_2 \oplus \cdots\cdots \oplus Y_r. \tag{4.4}$$

Corresponding to (4. 4), we have

$$\hat{B}_\tau = \hat{Y}_1 \oplus \hat{Y}_2 \oplus \cdots\cdots \oplus \hat{Y}_r, \tag{4.5}$$

where each $\hat{Y}_i$ may be regarded as the dual space of Y_i. It is easy to see that (*c*) is equivalent to the following condition:

($\hat{c}$) *For* $i = 0, 1, 2, \cdots, r-1$,

$$\hat{Y} \oplus \sum_{\mathfrak{P}_j>\mathfrak{P}_i}\oplus \hat{Y}_j = \hat{U}_i^\tau \oplus \sum_{\mathfrak{P}_j>\mathfrak{P}_i}\oplus \hat{U}_j^\tau \tag{4.6}$$

holds.

($\hat{c}$) yields that

$$\sum_{\mathfrak{P}_j>\mathfrak{P}_i}\oplus \hat{Y}_j = \sum_{\mathfrak{P}_j>\mathfrak{P}_i}\oplus \hat{U}_j^\tau \tag{4.7}$$

holds for $i = 0, 1, 2, \cdots, r-1$. ($\hat{c}$) is equivalent to the following condition:

($\hat{d}$) *For* $i = 0, 1, 2, \cdots, r-1$,

$$\hat{Y}_i \oplus \sum_{\mathfrak{P}_j > \mathfrak{P}_i}\oplus \hat{U}_j^T = \hat{U}_i^T \oplus \sum_{\mathfrak{P}_j > \mathfrak{P}_i}\oplus \hat{U}_j^T \tag{4.8}$$

holds.

Then we obtain the following:

[**4. A**] *Under the assumption* (4.4), *the system* $\mathfrak{Y}$ *satisfies one of the conditions* (c) *and* (d) *if and only if each* Y_i *of* $\mathfrak{Y}$ *satisfies the following two conditions.*

(1) *For any nonzero element* f *of* $\hat{Y}_i$, *there exists at least one class* $\mathfrak{K}_\alpha$ *with* $\mathfrak{P}_i = D(\mathfrak{K}_\alpha)$ *and with* $f(\mathfrak{K}_\alpha) \neq 0$.

(2) *If* $\mathfrak{K}_\beta$ *is a class such that* $D(\mathfrak{K}_\beta)$ *does not contain any subgroup conjugate to* $\mathfrak{P}_i$, *then* $f(K_\beta)$ *vanishes for all elements* f *of* $\hat{Y}_i$.

REMARK 1. In stead of (*c*), assume the following:

(*e*) *For* $h_{i-1} < k \leq h_i$, $i = 1, 2, \cdots, e$,

$$\sum_{j \leq h_{i-1}}\oplus Y_j \oplus Y_k = \sum_{j \leq h_{i-1}}\oplus U_j^T \oplus U_k^T \tag{4.9}$$

olds.

Then (3.1), (4.4), (4.5) and (3.5) hold; (e) is equivalent to

($\hat{e}$) *For* $h_{i-1} < k \leq h_i$, $i = 1, 2, \cdots, e$,

$$\hat{Y}_k \oplus \sum_{j > h_i}\oplus \hat{Y}_j = \hat{U}_k^T \oplus \sum_{j > h_i}\oplus \hat{U}_j^T \tag{4.10}$$

holds.

In this case, we have to replace (2) of [4. A] by the following:

(2') *If* $\mathfrak{K}_\beta$ *is a class with* $d(\mathfrak{K}_\beta) \leq d_i$ *such that* $D(\mathfrak{K}_\beta)$ *does not contain any subgroup conjugate to* $\mathfrak{P}_i$, *then* $f(K_\beta)$ *vanishes for all elements* f *of* $\hat{Y}_i$.

REMARK 2. In the left side of (4.1) or (4.9), we may replace $\sum\oplus$ by $\sum$.

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