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# ON THE SOLUTIONS OF THE MODIFIED FRANKL' PROBLEM

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## 1. Introduction.

The Frankl' problem for partial differential equations of mixed type was posed by F. I. Frankl' [4] (see also [9]) so as to investigate the transonic flow past a profile with shocks. After that many authors in USSR studied the problem to establish the maximum principles, the uniqueness and the existence for the solution of the problem under the limitation of various hypotheses ([3] [5] [6] [7] [8]).

The present paper is concerned with the modified Frankl' problem [12] which is proposed as the modification of the Frankl's original problem in order to be able to utilize the maximum principle of Agmon, Nirenberg and Protter [1]. Under the definitions and assumptions in Section **2**, there is proved the maximum principle for our problem in Section **3**. Then as its applications Sections **4** and **5** are devoted to establish the uniqueness and some estimations for the solutions of the problem of the linear equation for which the boundary condition of the third kind is given on the elliptic boundary, and of the nonlinear equation for which the nonlinear boundary condition is given on the elliptic boundary.

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## 2. Definitions and assumptions.

Let $K(y)$ be a function in $C^2(-y_1, y_2)$ for $y_1, y_2>0$ and have the property $yK(y)>0$ for $y\neq 0$. Consider a domain $\Omega$ which is contained in the strip $R^1\times(-y_1, y_2)$ of the $x,y$-plane and defined as follows. Let $A(a, 0)$, $B(b, 0)$, $D(d, 0)$ and $E(e, 0)$ be four points on the $x$-axis with $d<a<b<e$ and let $C$ be the intersection point of two arcs in $y<0$ which issue from $A$ and $B$ and have the slopes $0\geqslant dx/dy>-\sqrt{-K(y)}$ and $0\leqslant dx/dy<\sqrt{-K(y)}$, respectively. We shall denote the arcs $AC$ and $BC$ by $\gamma_1$ and $\gamma_2$, respectively. Let $\sigma$ be a smooth Jordan arc in $y>0$ joining $D$ and $E$ where it is assumed that the length of $\sigma$ is not less than the length $l$ of $\gamma_1$. Let $\sigma_0$ be the part of $\sigma$ whose length is equal to $l$ having the end points $F$ and $G$. $\Omega$ shall be the domain enclosed with the curve $ACBEGFDA$. Let $\Omega_1=\Omega\cap\{y>0\}$ and $\Omega_2=\Omega\cap\{y<0\}$.

Let us consider the following differential operators on functions $u(x, y)$ defined in $\bar{\Omega}$[1]:

$$Tu = K(y)u_{xx} + u_{yy}$$
$$Lu = Tu + a(x, y)u_x + b(x, y)u_y + c(x, y)u,$$

where $a(x, y)$, $b(x, y) \in C^2(\Omega)\cap C^0(\bar{\Omega})$ and $c(x, y)\in C^0(\bar{\Omega})$.

In $\bar{\Omega}_2$, two characteristic derivatives $v_\xi = \sqrt{-K(y)}\, v_x + v_y$ and $v_\eta = -\sqrt{-K(y)}\, v_x + v_y$ for a function $v(x, y)$ are defined. Let $v_\tau$ on $\sigma\cup\overline{DA}\cup\overline{BE}$ denote the directional derivative for a function $v(x, y)$ in the direction of the vector $\tau$ whose inner product with the inner normal vector to $\sigma$, $\overline{DA}$ or $\overline{BE}$ is positive.

ASSUMPTION I. Let the coefficients of $L$ be satisfy the following conditions:

$$\begin{cases} c\leqslant 0 \text{ in } \Omega, \\ a + b\sqrt{-K} + (\sqrt{-K})_y < 0 \text{ in } \Omega_2, \\ 4(-K)c + [a - b\sqrt{-K} + 3(\sqrt{-K})_y]\cdot[a + b\sqrt{-K} + (\sqrt{-K})_y] \\ \qquad - 2\sqrt{-K}\,[a + b\sqrt{-K} + (\sqrt{-K})_y]_\xi \geqslant 0 \text{ in } \Omega_2. \end{cases}$$

DEFINITION 1. Consider the following ten functions which are continuous and bounded on each part of the boundary of $\Omega$:

$\alpha_{11}(x)$, $\alpha_{12}(x)$, $\varphi_1(x)$ on $\overline{DA}\cup\overline{BE}$; $\alpha_{21}(x, y)$, $\alpha_{22}(x, y)$, $\varphi_2(x, y)$ on $\sigma$;
$\beta_1(x, y)$, $\beta_2(x, y)$, $\varphi_3(x, y)$ on $\bar{\gamma}_1$; $\varphi_4(x, y)$ on $\gamma_1$,

where $\alpha_{i1}\leqslant 0$, $\alpha_{i2}\geqslant 0$, $-\alpha_{i1}+\alpha_{i2}\geqslant a_i>0$ $(i=1, 2)$ and $\beta_2\geqslant\beta_1>0$, $\beta_1+\beta_2\geqslant a_3>0$ for some positive constants $a_j$ $(j=1, 2, 3)$. If, moreover, the boundary of $\Omega$ is smooth at the point $D$ or $E$, we shall assume that $\alpha_{1i}=\alpha_{2i}$ $(i=1, 2)$, $\varphi_1=\varphi_2$ there and if not we shall set $\alpha_{12}=0$ there. We shall say a function $u(x, y)$ defined on $\bar{\Omega}$ satisfies the boundary condition $u\in B_1(A, B, \Phi)$ where $A=(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$, $B=(\beta_1, \beta_2)$ and $\Phi=(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, if it satisfies

[1] In what follows the domain with a bar means the closure of it and the arc or interval with a bar contains its end points, but the one without a bar does not.

$$\alpha_{11} u + \alpha_{12} u_\tau = \varphi_1 \text{ on } \overline{DA} \cup \overline{BE}$$
$$\alpha_{21} u + \alpha_{22} u_\tau = \varphi_2 \text{ on } \sigma$$
$$\beta_1 u(x, y) - \beta_2 u(X, Y) = \varphi_3$$
$$u_\eta = \varphi_4 \text{ on } \gamma_1,$$

where in the third relation $(x, y) \in \bar{\sigma}_3$ corresponds to $(X, Y) \in \bar{\gamma}_1$ in such a way that the length of the arc from $F$ to $(x, y)$ is equal to the length of the arc from $A$ to $(X, Y)$, and in the sequel we shall write such a boundary condition as $\beta_1 u_1 - \beta_2 u_2 = \varphi_3$ for brevity.

Let $\tilde{C}^2(\bar{\Omega})$ be a set of functions $u(x, y)$ which are defined on $\bar{\Omega}$, belong to $C^2(\Omega) \cap C^0(\bar{\Omega})$, have the directional derivatives $u_\tau$ on $\sigma \cup \overline{DA} \cup \overline{BE}$ and have the characteristic derivatives $u_\eta$ which are continuous up to $\gamma_1$ inclusive.

DEFINITION 2. The linear problem is to seek a function $u(x, y) \in \tilde{C}^2(\bar{\Omega})$ which satisfies the equation

$$Lu = f(x, y)$$

in $\Omega$ and the boundary condition

$$u \in B_1(A, B, \Phi),$$

where $L$ is supposed to satisfy Assumption I and $f(x, y)$ is an arbitrary continuous and bounded function on $\Omega$.

ASSUMPTION II. Let the functions $K(y)$ and $g(x, y, z, p, q) \in C^2(\Omega \times R^3)$ satisfy the following conditions:

$$\begin{cases} g_z \geqslant 0 \text{ in } \Omega \times R^3, \\ g_p + g_q \sqrt{-K} - (\sqrt{-K})_y > 0 \text{ in } \Omega_2 \times R^3, \\ 4(-K)\, g_z + [-g_p + g_q\sqrt{-K} + 3(\sqrt{-K})_y] \cdot [g_p + g_q \sqrt{-K} - (\sqrt{-K})_y] \\ \qquad - 2\sqrt{-K}[g_p + g_q\sqrt{-K} - (\sqrt{-K})_y]_\xi \leqslant 0 \text{ in } \Omega_2 \times R^3. \end{cases}$$

DEFINITION 3. Consider the following six functions defined on each domain:

$\varphi_1(x, y, z, s)$ on $(\overline{DA} \cup \overline{BE}) \times R^2$; $\varphi_2(x, y, z, s)$ on $\sigma \times R^2$;

$\beta_1(x, y)$, $\beta_2(x, y)$, $\varphi_3(x, y)$ on $\bar{\gamma}_1$; $\varphi_4(x, y)$ on $\gamma_1$,

where $\varphi_1$, $\varphi_2$ are continuously differentiable and bounded with the requirements $\partial\varphi_i/\partial z \leqslant 0$, $\partial\varphi_i/\partial s \geqslant 0$ and $-\partial\varphi_i/\partial z + \partial\varphi_i/\partial s \geqslant t_i > 0$ $(i = 1, 2)$ in the respective domains, and $\beta_1$, $\beta_2$, $\varphi_3$ and $\varphi_4$ have the same requirements as in Definition 1. Then we shall say a function $u(x, y)$ defined on $\bar{\Omega}$ satisfies the boundary condition $u \in B_2(B, \Phi)$ where $B = (\beta_1, \beta_2)$ and $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, if it satisfies

$$\begin{aligned} &\varphi_1(x, y, u, u_\tau) = 0 \text{ on } \overline{DA} \cup \overline{BE} \\ &\varphi_2(x, y, u, u_\tau) = 0 \text{ on } \sigma \\ &\beta_1 u(x, y) - \beta_2 u(X, Y) = \varphi_3 \\ &u_\eta = \varphi_4 \text{ on } \gamma_1, \end{aligned}$$

where in the third relation the points $(x, y)$ and $(X, Y)$ are taken as in Definition 1.

DEFINITION 4. The nonlinear problem is to seek a function $u(x, y) \in \tilde{C}^2(\bar{\Omega})$ which satisfies the equation

$$Tu = g(x, y, u, u_x, u_y)$$

in $\Omega$ and the boundary condition

$$u \in B_2(B, \Phi),$$

where the functions $K(y)$ and $g(x, y, z, p, q)$ are supposed to satisfy Assumption II.

## 3. Maximum principle.

THEOREM 1. *Let a function $u(x, y) \in \tilde{C}^2(\bar{\Omega})$ satisfy the inequalities $Lu \geqslant 0$ in $\Omega$, $\alpha_{11} u + \alpha_{12} u_\tau \geqslant 0$ on $\sigma_0$, $\beta_1 u_1 - \beta_2 u_2 \geqslant 0$ and $u_\eta \geqslant 0$ on $\gamma_1$, then the positive maximum*[2] *of $u$ in $\bar{\Omega}$ cannot be attained except on $\overline{DA} \cup \overline{BE} \cup \overline{\sigma \setminus \sigma_0}$.*

To prove this fact, we require the following lemmas, which may be proved in a similar manner as in Agmon, Nirenberg and Protter [1], Oleĭnik [10] and Protter and Weinberger [11].

LEMMA 1. *Let $L$ satisfy Assumption I. Consider a function $u(x, y) \in C^2(\Omega_2) \cap C^0(\bar{\Omega}_2)$ having the characteristic derivative $u_\eta$ which is continuous up to $\gamma_1$ inclusive. If the function $u(x, y)$ satisfies the inequalities $Lu \geqslant 0$ in $\Omega_2$ and $u_\eta \geqslant 0$ on $\gamma_1$, then the positive maximum of $u$ in $\bar{\Omega}_2$ cannot be attained except on $\bar{\gamma}_1 \cup \overline{AB}$. Moreover, if the maximum is attained at some point $(x_0, 0) \in AB$, then we have*

$$\liminf_{y \to -0} \frac{u(x_0, y) - u(x_0, 0)}{y} > 0.$$

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2) When $c \equiv 0$, the assumption of positivity of the maximum may be omitted and the same holds for Lemmas 1 and 2.

LEMMA 2. *Consider a function* $u(x, y) \in C^2(\Omega_1) \cap C^0(\bar{\Omega}_1)$. *Assume* $c \leqslant 0$ *in* $\Omega_1$. *If the function* $u(x, y)$ *satisfies the inequality* $Lu \geqslant 0$ *in* $\Omega_1$, *then the positive maximum of* $u$ *in* $\bar{\Omega}_1$ *cannot be attained in the interior of* $\Omega_1$. *Moreover, if the maximum is attained at some point* $(x_0, y_0) \in \sigma \cup DE$, *then we have*

$$\limsup_{t \to +0} \frac{u(x_0 + t\kappa_1, y_0 + t\kappa_2) - u(x_0, y_0)}{t \mid \kappa \mid} < 0,$$

*where the inner product of the vector* $\kappa = (\kappa_1, \kappa_2)$ *and the inner normal vector to* $\sigma$ *or* $DE$ *at* $(x_0, y_0)$ *is positive*[3)].

PROOF OF THEOREM 1. Lemma 1 and Lemma 2 show that the positive maximum of $u$ is not attained at an interior point of $\Omega_1$ and $\Omega_2$ and on $\gamma_2$. Thus it is attained at a point of the boundary of $\Omega$ except $\gamma_2$ or at a point of $\overline{AB}$. But the points of $AB$ cannot be the maximum point by Lemmas 1 and 2. If the maximum point lies on $\gamma_1$, the maximum must be attained at a corresponding point on $\sigma_0$ from the assumption, and then $u_\tau < 0$ owing to Lemma 2. This contradicts with the assumption, then the theorem is proved.

THEOREM 2. *If the function* $f(x, y)$ *and the boundary functions* $\varphi_1, \varphi_2, \varphi_3$ *and* $\varphi_4$ *are nonnegative, then the solution of the linear problem* $Lu = f$ *with* $u \in B_1(A, B, \Phi)$ *is nonpositive.*

PROOF. By virtue of Theorem 1, the positive maximum in $\bar{\Omega}$, if it exists, is attained at some point of $\overline{DA} \cup \overline{BE} \cup \overline{\sigma \setminus \sigma_0}$. But using Lemma 2 the value of $\alpha_{i1} u + \alpha_{i2} u_\tau$ $(i = 1$ or $2)$ at the point is negative. Therefore the contradiction concludes the proof.

## 4. Linear problem.

THEOREM 3. *The solution of the linear problem is unique.*

PROOF. This is evident from Theorem 2.

THEOREM 4. *Assume* $c \leqslant -k < 0$ *with a positive constant* $k$. *If a function* $u(x, y)$ *is the solution of the linear problem, there holds the estimation*

$$|u| \leqslant C_1\left(\frac{\sup|\varphi_1|}{a_1} + \frac{\sup|\varphi_2|}{a_2} + \frac{\sup|\varphi_3|}{a_3} + \sup|\varphi_4|\right)$$
$$\times \{1 + \frac{C_2}{k} \sup[\,|K| + |a| + |b| + |c| + 1\,]\} + \frac{\sup|f|}{k},$$

3) If the boundary curve of $\Omega$ is smooth at the point $D$ or $E$, then the point can be taken as $(x_0, y_0)$.

*where the constants* $C_1$ *and* $C_2$ *are independent of the coefficients in* $L$ *and the boundary functions. Here it is required that for arbitrary continuous functions* $\Psi$, *the problem* $Lu=0$ *with* $u \in B_1(A, B, \Psi)$ *has a solution in* $\tilde{C}^2(\bar{\Omega})$.

PROOF. We shall divide the solution $u$ into solutions of two problems such as

$$u = u_1 + u_2\ ;\ u_1 : Lu_1 = f \text{ with } u_1 \in B_1(A, B, \mathbf{0})$$
$$u_2 : Lu_2 = 0 \text{ with } u_2 \in B_1(A, B, \Phi).$$

(I) *estimation of* $u_1$. Let $D_1 = \sup|f|/k$ and let $v_\pm = \pm u_1 - D_1$. Then $v_\pm$ satisfies $Lv_\pm = \pm f - cD_1 \geqslant 0$ with $v_\pm \in B_1(A, B, -\alpha_{11}D_1, -\alpha_{21}D_1, -D_1(\beta_1-\beta_2), 0)$. Since $\alpha_{i1} \leqslant 0$ $(i=1, 2)$ and $\beta_1 - \beta_2 \leqslant 0$, it follows by Theorem 2 that $v_\pm \leqslant 0$ in $\bar{\Omega}$, and thus $|u_1| \leqslant D_1$.
(II) *estimation of* $u_2$. Let a function $U(x, y) \in \tilde{C}^2(\bar{\Omega})$ be such that

$$\alpha_{11}U + \alpha_{12}U_\tau \equiv \lambda_1 \geqslant c_1 > 0 \text{ on } \overline{DA} \cup \overline{BE}$$
$$\alpha_{21}U + \alpha_{22}U_\tau \equiv \lambda_2 \geqslant c_2 > 0 \text{ on } \sigma$$
$$\beta_1 U_1 - \beta_2 U_2 \equiv \lambda_3 \geqslant c_3 > 0$$
$$U_\eta \equiv \lambda_4 \geqslant c_4 > 0 \text{ on } \gamma_1,$$

for some positive constants $c_i (i=1, 2, 3, 4)$ and also satisfy $|U| \leqslant 1$. Further, if a function $v(x, y) \in \tilde{C}^2(\bar{\Omega})$ is a solution of the problem

$$Lv = 0$$
$$v \in B_1(A, B, \Lambda)$$

for $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, then from Theorem 2 we have $v \leqslant 0$. Let $D_2 = \Sigma_{i=1}^4 \sup|\varphi_i|/c_i$. Since $w_\pm = \pm u_2 + D_2 v$ is a solution of the problem

$$Lw_\pm = 0$$
$$w_\pm \in B_1(A, B, \pm\Phi + D_2\Lambda),$$

then again by Theorem 2 we have $w_\pm \leqslant 0$. Thus $|u_2| \leqslant D_2|v|$. In order to estimate $|v|$, we shall take $\tilde{v} = U - v$, then since

$$L\tilde{v} = LU$$
$$\tilde{v} \in B_1(A, B, \mathbf{0}),$$

we have from (I)

$$|v| \leqslant 1 + \frac{C_2}{k} \sup[|K| + |a| + |b| + |c| + 1].$$

Consequently

$$| u_2 | \leqslant C_1 \Big( \frac{\sup | \varphi_1 |}{a_1} + \frac{\sup | \varphi_2 |}{a_2} + \frac{\sup | \varphi_3 |}{a_3} + \sup | \varphi_4 | \Big)$$

$$\times \Big\{ 1 + \frac{C_2}{k} \sup [ | K | + | a | + | b | + | c | + 1 ] \Big\}.$$

(III) From the estimations (I) and (II) we have the estimation for $u$, which completes the proof.

THEOREM 5. *Let the boundary functions* $\alpha_{11}$ *and* $\alpha_{21}$ *have the properties* $\alpha_{i1} \leqslant - d_i < 0$ *for some positive constants* $d_i$ $(i = 1, 2)$. *If a function* $u(x, y)$ *is the solution of the linear problem, then there holds the estimation*

$$| u | \leqslant \frac{\sup | \varphi_1 |}{d_1} + \frac{\sup | \varphi_2 |}{d_2}$$

$$+ C_3 \Big\{ \frac{\sup | \varphi_3 |}{a_3} + \sup | \varphi_4 | + C_4 \sup | f | \cdot [ \sup | \beta_1 | + \sup | \beta_2 | + 1 ] \Big\}$$

$$\times \Big\{ \frac{\sup | \alpha_{11} | + \sup | \alpha_{12} |}{d_1} + \frac{\sup | \alpha_{21} | + \sup | \alpha_{22} |}{d_2} + 1 \Big\},$$

*where the constants* $C_3$ *and* $C_4$ *are independent of the coefficients in* $L$ *and the boundary functions. Here it is required that for arbitrary continuous function* $g$ *in* $\bar{\Omega}$ *and boundary functions* $\psi_3$ *on* $\bar{\gamma}_1$, $\psi_4$ on $\gamma_1$, *there exists a solution in* $\tilde{C}^2(\bar{\Omega})$ *of the problem* $Lu = g$ *with* $u \in B_1(A, B, 0, 0, \psi_3, \psi_4)$.

PROOF. Suppose that the solution $u$ is composed of solutions of three problems such as

$$u = v_1 + v_2 + v_3 ;$$

$$v_1 : Lv_1 = 0 \text{ with } v_1 \in B_1(A, B, \varphi_1, \varphi_2, 0, 0)$$

$$v_2 : Lv_2 = 0 \text{ with } v_2 \in B_1(A, B, 0, 0, \varphi_3, \varphi_4)$$

$$v_3 : Lv_3 = f \text{ with } v_3 \in B_1(A, B, 0, 0, 0, 0).$$

(I) *estimation of* $v_1$. Let $E_1 = \sup | \varphi_1 | / d_1 + \sup | \varphi_2 | / d_2$ and let $v_\pm = \pm v_1 - E_1$. Then $v_\pm$ satisfies

$$Lv_\pm = - cE_1$$

$$v_\pm \in B_1(A, B, \pm \varphi_1 - \alpha_{11} E_1, \pm \varphi_2 - \alpha_{21} E_1, E_1(\beta_2 - \beta_1), 0).$$

Since $\pm \varphi_i - \alpha_{i1} E_1 \geqslant 0$ $(i = 1, 2)$ and $\beta_2 \geqslant \beta_1$, by Theorem 2 we have $v_\pm \leqslant 0$ in $\bar{\Omega}$, and then $| v_1 | \leqslant E_1$.

(II) *estimation of* $v_2$. Let a function $V(x, y) \in \tilde{C}^2(\bar{\Omega})$ be such that

$$LV \geqslant 0$$

$$\beta_1 V_1 - \beta_2 V_2 \equiv \chi_3 \geqslant e_1 > 0$$

$$V_\eta \equiv \chi_4 \geqslant e_2 > 0$$

$$| V | \leqslant 1,$$

where $e_i$ are positive constants $(i=1, 2)$. Further, if a function $\tilde{u}(x, y) \in \tilde{C}^2(\bar{\Omega})$ is a solution of the problem

$$L\tilde{u} = LV$$
$$\tilde{u} \in B_1(A, B, 0, 0, \chi_3, \chi_4),$$

then by Theorem 2 $\tilde{u} \leqslant 0$. Let $E_2 = \sup|\varphi_3|/e_1 + \sup|\varphi_4|/e_2$. Since $\tilde{w}_\pm = \pm v_2 + E_2\tilde{u}$ is a solution of the problem

$$L\tilde{w}_\pm = E_2 LV$$
$$\tilde{w}_\pm \in B_1(A, B, 0, 0, \pm\varphi_3 + E_2\chi_3, \pm\varphi_4 + E_2\chi_4),$$

then by Theorem 2 and the conditions for $V$ and $E_2$, $\tilde{w}_\pm \leqslant 0$. Thus $|v_2| \leqslant E_2|\tilde{u}|$. Next we shall estimate $|\tilde{u}|$. Taking $w_1 = V - \tilde{u}$, $w_1$ is a solution of the problem

$$Lw_1 = 0$$
$$w_1 \in B_1(A, B, \chi_1, \chi_2, 0, 0),$$

where $\chi_i = \alpha_{i1}V + \alpha_{i2}V_\tau \quad (i=1, 2)$. Then from (I)

$$|w_1| \leqslant \frac{\sup|\chi_1|}{d_1} + \frac{\sup|\chi_2|}{d_2}.$$

Thus

$$|\tilde{u}| \leqslant 1 + \frac{\sup|\chi_1|}{d_1} + \frac{\sup|\chi_2|}{d_2}.$$

Consequently

$$|v_2| \leqslant C_3'\left(\frac{\sup|\varphi_3|}{e_1} + \frac{\sup|\varphi_4|}{e_2}\right)$$
$$\times\left\{1 + \frac{\sup|\alpha_{11}| + \sup|\alpha_{12}|}{d_1} + \frac{\sup|\alpha_{21}| + \sup|\alpha_{22}|}{d_2}\right\}.$$

(III) From (I) and (II) we have for a solution $v_0$ of the problem

$$Lv_0 = 0$$
$$v_0 \in B_1(A, B, \Phi),$$

the estimation

$$|v_0| \leqslant \frac{\sup|\varphi_1|}{d_1} + \frac{\sup|\varphi_2|}{d_2} + C_3''\left(\frac{\sup|\varphi_3|}{a_3} + \sup|\varphi_4|\right)$$
$$\times\left\{1 + \frac{\sup|\alpha_{11}| + \sup|\alpha_{12}|}{d_1} + \frac{\sup|\alpha_{21}| + \sup|\alpha_{22}|}{d_2}\right\}.$$

(IV) *estimation of* $v_3$. Suppose a function $W(x, y) \in \tilde{C}^2(\Omega)$ is such that

$LW \geqslant m > 0$ in $\Omega$, where $m$ is some positive constant. Let a function $\hat{w}(x, y) \in \tilde{C}^2(\bar{\Omega})$ be a solution of the problem

$$L\hat{w} = LW$$
$$\hat{w} \in B_1(A, B, \mathbf{0}),$$

then by Theorem 2 we have $\hat{w} \leqslant 0$. Let $E_3 = \sup |f|/m$. Since $\hat{v}_\pm = \pm v_3 + E_3 \hat{w}$ is a solution of the problem

$$L\hat{v}_\pm = \pm f + E_3 LW$$
$$\hat{v}_\pm \in B_1(A, B, \mathbf{0}),$$

then again by Theorem 2 we have $\hat{v}_\pm \leqslant 0$. Thus $|v_3| \leqslant E_3 |\hat{w}|$. In order to estimate $|\hat{w}|$ we set $\tilde{v} = W - \hat{w}$, then $\tilde{v}$ is a solution of the problem

$$L\tilde{v} = 0$$
$$\tilde{v} \in B_1(A, B, \Delta),$$

where $\Delta = (\delta_1, \delta_2, \delta_3, \delta_4)$, $\delta_i = \alpha_{i1} W + \alpha_{i2} W_\tau$ $(i = 1, 2)$, $\delta_3 = \beta_1 W_1 - \beta_2 W_2$ and $\delta_4 = W_\eta$. Then from (III) we have

$$|\tilde{v}| \leqslant C_4' (\sup|\beta_1| + \sup|\beta_2| + 1)$$
$$\times \left\{ \frac{\sup|\alpha_{11}| + \sup|\alpha_{12}|}{d_1} + \frac{\sup|\alpha_{21}| + \sup|\alpha_{22}|}{d_2} + 1 \right\}.$$

Therefore

$$|v_3| \leqslant \frac{C_4'}{m} \sup|f| (\sup|\beta_1| + \sup|\beta_2| + 1)$$
$$\times \left\{ \frac{\sup|\alpha_{11}| + \sup|\alpha_{12}|}{d_1} + \frac{\sup|\alpha_{21}| + \sup|\alpha_{22}|}{d_2} + 1 \right\}.$$

(V) Combining (III) and (IV), we have the estimation for $u$, which completes the proof.

REMARK. Let the boundary of the domain $\Omega$ be supposed that $\gamma_1$ and $\sigma_0$ are parallel with $y$-axis whose end points are $A(a, 0)$, $C(a, -l)$, $D(d, 0) = F$ and $G(d, l)$, and let the other parts of the boundary satisfy the assumptions in Section **2**. Then if $\alpha_{12} = 0$ on $\overline{DA} \cup \overline{BE}$, $\alpha_{21} = 0$ on $\sigma_0$, $\alpha_{22} = 0$ on $\sigma \setminus \sigma_0$ and $\tau = (1, 0)$ on $\sigma_0$, the function $U$ in Theorem 4 may be set, for instance, $U = \mu[\exp\{\nu(y + l)\} - (x - a/3 - 2d/3)^2]$, where $\nu$ is a sufficiently large number and $\mu$ is a constant chosen to be $|U| \leqslant 1$. Concerning $V$ and $W$ in Theorem 5 for the same domain, we may set as previous $U$, but in this case no other condition is taken on $\alpha_{ij}$ except the assumptions in the theorem.

THEOREM 6. *Let the assumptions of Theorem 4 be satisfied. Then the solution $u$ of the linear problem which has the bounded second derivative $u_{xx}$ and the bounded first derivatives $u_x$, $u_y$ on $\bar{\Omega}$ depends continuously on the coefficients of $L$, the function $f$ and the boundary functions.*

PROOF. Let the functions $u$ and $\tilde{u}$ be the solutions of the linear problems

$$Lu = Ku_{xx} + u_{yy} + au_x + bu_y + cu = f$$
$$u \in B_1(A, B, \Phi)$$

and

$$\tilde{L}\tilde{u} = \tilde{K}\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{a}\tilde{u}_x + \tilde{b}\tilde{u}_y + \tilde{c}\tilde{u} = \tilde{f}$$
$$\tilde{u} \in B_1(\tilde{A}, \tilde{B}, \tilde{\Phi}),$$

respectively. Then the difference function $v = \tilde{u} - u$ satisfies

$$\tilde{L}v = \tilde{f} - f + [(K - \tilde{K})u_{xx} + (a - \tilde{a})u_x + (b - \tilde{b})u_y + (c - \tilde{c})u]$$
$$v \in B_1(\tilde{A}, \tilde{B}, \Psi),$$

where $\psi_i = \tilde{\varphi}_i - \varphi_i + (\alpha_{i1} - \tilde{\alpha}_{i1})u + (\alpha_{i2} - \tilde{\alpha}_{i2})u_\tau$ $(i = 1, 2)$, $\psi_3 = \tilde{\varphi}_3 - \varphi_3 + (\beta_1 - \tilde{\beta}_1)u_1 - (\beta_2 - \tilde{\beta}_2)u_2$ and $\psi_4 = \tilde{\varphi}_4 - \varphi_4$. Therefore by virtue of Theorem 4 there holds the inequality

$$|\tilde{u} - u| \leqslant C_1\Big(\sum_{i=1}^{2} \frac{\sup(|\tilde{\varphi}_i - \varphi_i| + |\tilde{\alpha}_{i1} - \alpha_{i1}| + |\tilde{\alpha}_{i2} - \alpha_{i2}|)}{\inf(|\tilde{\alpha}_{i1}| + |\tilde{\alpha}_{i2}|)}$$
$$+ \frac{\sup(|\tilde{\varphi}_3 - \varphi_3| + |\tilde{\beta}_1 - \beta_1| + |\tilde{\beta}_2 - \beta_2|)}{\inf(|\tilde{\beta}_1| + |\tilde{\beta}_2|)} + \sup|\tilde{\varphi}_4 - \varphi_4|\Big)$$
$$\times \{1 + \frac{C_2}{\inf|\tilde{f}|}\sup[|\tilde{K}| + |\tilde{a}| + |\tilde{b}| + |\tilde{c}| + 1]\}$$
$$+ \frac{C_5}{\inf|\tilde{f}|}\sup\{|\tilde{f} - f| + |\tilde{K} - K| + |\tilde{a} - a| + |\tilde{b} - b| + |\tilde{c} - c|\},$$

from which the required results may be deduced.

## 5. Nonlinear problem.

THEOREM 7. *The solution of the nonlinear problem is unique.*

PROOF. Suppose that there are two solutions $u_1$ and $u_2$ for the problem. Then the difference function $v = u_1 - u_2$ satisfies

$$Tv + \bar{a}v_x + \bar{b}v_y + \bar{c}v = 0$$
$$v \in B_1(\bar{A}, B, \mathbf{0}),$$

where

$$\bar{a} = -\int_0^1 \frac{\partial g}{\partial p}(x, y, tu_1 + (1-t)u_2, tu_{1_x} + (1-t)u_{2_x}, tu_{1_y} + (1-t)u_{2_y})\,dt,$$

$$\bar{b} = -\int_0^1 \frac{\partial g}{\partial q}(\cdots\cdots\cdots)\,dt,\ \bar{c} = -\int_0^1 \frac{\partial g}{\partial z}(\cdots\cdots)\,dt,$$

$$\bar{\alpha}_{i1} = \int_0^1 \frac{\partial \varphi_i}{\partial z}(x, y, tu_1 + (1-t)u_2, tu_{1_\tau} + (1-t)u_{2_\tau})\,dt \quad \text{and}$$

$$\bar{\alpha}_{i2} = \int_0^1 \frac{\partial \varphi_i}{\partial s}(\cdots\cdots)\,dt,\ (i = 1, 2).$$

By virtue of Assumption II, Definitions 3 and 4, $v \equiv 0$ in $\bar{\Omega}$ is guaranteed by Theorem 2.

THEOREM 8. *Consider the nonlinear problem having the property that* $|\partial g/\partial z|$, $|\partial g/\partial p|$, $|\partial g/\partial q|$ *are bounded and* $\partial g/\partial z \geqslant k > 0$ *on* $\Omega \times R^3$ *for some positive constant k. Then if a function* $u(x, y)$ *is the solution of the problem, there holds the estimation*

$$|u| \leqslant C_6\Big(\frac{\sup|\varphi_1(x, y, 0, 0)|}{b_1} + \frac{\sup|\varphi_2(x, y, 0, 0)|}{b_2}$$
$$+ \frac{\sup|\varphi_3|}{a_3} + \sup|\varphi_4|\Big)\cdot\{1 + \frac{C_7}{k}\sup[|K| + 1]\} + \frac{\sup|g(x, y, 0, 0, 0)|}{k},$$

*where the constant* $C_6$ *is independent of K, g and the boundary functions, and* $C_7$ *depends only on the bounds of the derivatives of g. Here it is required that there exists a solution in* $\tilde{C}^2(\bar{\Omega})$ *of the problem mentioned in Theorem 4 for the equation* $Lu = 0$ *whose coefficients a, b and c are replaced by*

$$\hat{a} = -\int_0^1 \frac{\partial g}{\partial p}(x, y, tu, tu_x. tu_y)\,dt,$$

$$\hat{b} = -\int_0^1 \frac{\partial g}{\partial q}(\cdots\cdots)\,dt \quad \textit{and}$$

$$\hat{c} = -\int_0^1 \frac{\partial g}{\partial z}(\cdots\cdots)\,dt, \quad \textit{respectively,}$$

*with the boundary condition* $u \in B(A, B, \Phi)$ *where* $\alpha_{ij}$ $(i, j = 1, 2)$ *are replaced by*

$$\hat{\alpha}_{i1} = \int_0^1 \frac{\partial \varphi_i}{\partial z}(x, y, tu, tu_\tau)\,dt,$$

$$\hat{\alpha}_{i2} = \int_0^1 \frac{\partial \varphi_i}{\partial s}(\cdots\cdots)\,dt\ (i = 1, 2).$$

THEOREM 9. *Consider the nonlinear problem having the property that* $|\partial g/\partial z|$, $|\partial g/\partial p|$, $|\partial g/\partial q|$ *are bounded on* $\Omega \times R^3$. *Further, assume that* $\partial\varphi_i/\partial z \leqslant -n_i < 0$ *for some positive constants* $n_i$ $(i=1, 2)$. *Then, if a function* $u(x, y)$ *is the solution of the problem, there holds the estimation*

$$|u| \leqslant \frac{\sup|\varphi_1(x, y, 0, 0)|}{n_1} + \frac{\sup|\varphi_2(x, y, 0, 0)|}{n_2}$$
$$+ C_8\{\sup|\varphi_3| + \sup|\varphi_4| + C_9 \sup|g(x, y, 0, 0, 0)|[\sup|\beta_1| + \sup|\beta_2| + 1]\},$$

*where the constant* $C_8$ *depends only on* $\partial\varphi_i/\partial z$, $\partial\varphi_i/\partial s$ $(i=1, 2)$ *and* $C_9$ *is independent of* $K$, $g$ *and the boundary functions. Here it is required that there exists a solution in* $\tilde{C}^2(\bar{\Omega})$ *of the problem mentioned in Theorem* 5 *for* $L$ *with the coefficients* $\hat{a}$, $\hat{b}$ *and* $\hat{c}$ *and for* $\hat{\alpha}_{ij}$ $(i, j=1, 2)$ *described in Theorem* 8.

PROOF of THEOREMS 8 AND 9. There hold the relations

$$g(x, y, u, u_x, u_y) = g(x, y, 0, 0, 0) - \hat{a}u_x - \hat{b}u_y - \hat{c}u,$$
$$\varphi_i(x, y, u, u_\tau) = \varphi_i(x, y, 0, 0) + \hat{\alpha}_{i1}u + \hat{\alpha}_{i2}u_\tau \quad (i=1, 2).$$

Then we can produce from Theorems 4 and 5 the respective estimations.

THEOREM 10. *Let the assumptions of Theorem* 8 *be satisfied. Then the solution* $u$ *of the nonlinear problem which has the bounded second derivative* $u_{xx}$ *on* $\bar{\Omega}$ *depends continuously on the coefficient* $K$, *the function* $g$ *and the boundary functions.*

PROOF. Let $u$ and $\tilde{u}$ be the solutions of the nonlinear problems

$$Ku_{xx} + u_{yy} = g(x, y, u, u_x, u_y)$$
$$u \in B_2(B, \Phi)$$

and

$$\tilde{K}\tilde{u}_{xx} + \tilde{u}_{yy} = \tilde{g}(x, y, \tilde{u}, \tilde{u}_x, \tilde{u}_y)$$
$$\tilde{u} \in B_2(\tilde{B}, \tilde{\Phi}),$$

respectively. Then the difference function $w = \tilde{u} - u$ satisfies

$$\tilde{K}w_{xx} + w_{yy} + \bar{a}w_x + \bar{b}w_y + \bar{c}w$$
$$= \tilde{g}(x, y, \tilde{u}, \tilde{u}_x, \tilde{u}_y) - g(x, y, \tilde{u}, \tilde{u}_x, \tilde{u}_y) - (\tilde{K} - K)u_{xx}$$
$$w \in B_1(\tilde{A}, \tilde{B}, \Psi),$$

where

$$\bar{a} = -\int_0^1 \frac{\partial g}{\partial p}(x, y, t\tilde{u} + (1-t)u, t\tilde{u}_x + (1-t)u_x, t\tilde{u}_y + (1-t)u_y)\,dt$$

$$\tilde{b} = -\int_0^1 \frac{\partial g}{\partial q}(\cdots\cdots)\,dt,\ \tilde{c} = -\int_0^1 \frac{\partial g}{\partial z}(\cdots\cdots)\,dt,$$

$$\tilde{\alpha}_{i1} = -\int_0^1 \frac{\partial \varphi_i}{\partial z}(x,\ y,\ t\tilde{u} + (1-t)\,u,\ t\tilde{u}_\tau + (1-t)\,u_\tau)\,dt,$$

$$\tilde{\alpha}_{i2} = -\int_0^1 \frac{\partial \varphi_i}{\partial s}(\cdots\cdots)\,dt,\quad \psi_i = \tilde{\varphi}_i(x,\ y,\ \tilde{u},\ \tilde{u}_\tau) - \varphi_i(x,\ y,\ \tilde{u},\ \tilde{u}_\tau)\ (i=1,\ 2),$$

$$\psi_3 = \tilde{\varphi}_3 - \varphi_3 + (\beta_1 - \tilde{\beta}_1)\,u_1 - (\beta_2 - \tilde{\beta}_2)\,u_2 \quad\text{and}\quad \psi_4 = \tilde{\varphi}_4 - \varphi_4.$$

Then from Theorem 4 we have the inequality

$$|\tilde{u} - u| \leqslant C_{10}\Big(\frac{\sup|\tilde{\varphi}_1 - \varphi_1|}{b_1} + \frac{\sup|\tilde{\varphi}_2 - \varphi_2|}{b_2}$$

$$+ \frac{\sup|\tilde{\varphi}_3 - \varphi_3| + \sup|\tilde{\beta}_1 - \beta_1| + \sup|\tilde{\beta}_2 - \beta_2|}{a_3} + \sup|\tilde{\varphi}_4 - \varphi_4|\Big)$$

$$\times \{1 + \frac{C_{11}}{k}\sup[|\tilde{K}| + 1]\}$$

$$+ \frac{C_{12}}{k}\sup(|\tilde{g} - g| + |\tilde{K} - K|),$$

from which the result of the theorem follows.

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