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ON THE NUMBER OF BLOCKS OF IRREDUCIBLE CHARACTERS OF A FINITE GROUP WITH A GIVEN DEFECT GROUP

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Introduction

Let $\mathfrak{G}$ be a group of finite order g and p a fixed rational prime; $g=p^a g'$, $(p, g')=1$. Let K be the number field of g-th roots of unity and $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers of K, where $\mathfrak{p}$ is a prime ideal divisor of p in K. Denote by Z_0 the center of the group ring of $\mathfrak{G}$ over $\mathfrak{o}$. The natural homomorphism of $\mathfrak{o}$ onto the residue class field $K^*=\mathfrak{o}/\mathfrak{p}$ induces a ring homomorphism of Z_0 onto the center Z^* of the group ring of $\mathfrak{G}$ over K^*, which has the kernel $\mathfrak{p}Z_0$. For every element x of Z_0, we shall denote by x^* the image of x by the ring homomorphism. Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_n$ be the classes of conjugate elements in $\mathfrak{G}$ and $G_1, G_2, \cdots, G_n$ a complete system of representatives for the classes. Denote by K_ν the sum of all elements of $\mathfrak{K}_\nu$ and $|\mathfrak{K}_\nu|$ the cardinality of $\mathfrak{K}_\nu$, $\nu=1, 2, \cdots, n$. $\mathfrak{G}$ has n distinct absolutely irreducible ordinary characters $\chi_1, \chi_2, \cdots, \chi_n$. For each χ_i, there corresponds one and only one primitive idempotent of the center Z of the group ring of $\mathfrak{G}$ over K which is given by

$$(1)\qquad e_i = \frac{1}{g}\sum_{\nu=1}^{n}\chi_i(1)\chi_i(G_\nu^{-1})K_\nu :$$

$$(2)\qquad 1 = e_1+e_2+\cdots+e_n,$$

$$(3)\qquad e_i^2 = e_i,\ e_i e_j = 0\ (i\neq j).$$

It is well known that Z has n distinct linear characters $\omega_1, \omega_2, \cdots, \omega_n$;

$$(4)\qquad \omega_i(K_\mu) = \frac{|\mathfrak{K}_\mu|\chi_i(G_\mu)}{\chi_i(1)},$$

$$(5)\qquad K_\mu e_i = \omega_i(K_\mu)e_i = \frac{1}{g}\sum_{\nu=1}^{n}|\mathfrak{K}_\mu|\chi_i(G_\mu)\chi_i(G_\nu^{-1})K_\nu .$$

Let $B_1, B_2, \cdots, B_t$ be the p-blocks of absolutely irreducible ordinary characters χ_i and $\eta_1, \eta_2, \cdots, \eta_t$ the corresponding block idempotents of Z_0;

(6) $$\eta_\tau = \sum_{\chi_i \in B_\tau} e_i = \frac{1}{g} \sum_{\nu=1}^{n} \sum_{\chi_i \in B_\tau} \chi_i(1)\chi_i(G_\nu^{-1})K_\nu,$$

(7) $$1 = \eta_1 + \eta_2 + \ldots + \eta_t,\ \eta_\tau^2 = \eta_\tau,\ \eta_\tau\eta_\rho = 0\ (\tau \neq \rho),$$

(8) $$K_\mu \eta_\nu = \frac{|\mathfrak{K}_\mu|}{g} \sum_{\nu=1}^{n} \sum_{\chi_i \in B_\tau} \chi_i(G_\mu)\chi_i(G_\nu^{-1})K_\nu.$$

For each class $\mathfrak{K}_\nu$, we denote by $\mathfrak{D}(\mathfrak{K}_\nu)$ a defect group of $\mathfrak{K}_\nu$ and by $d(\mathfrak{K}_\nu)$ the defect of $\mathfrak{K}_\nu$, i.e., $\mathfrak{D}(\mathfrak{K}_\nu)$ is a Sylow p-subgroup of the normalizer $\mathfrak{N}_{\mathfrak{G}}(G_\nu)$ in $\mathfrak{G}$ and $p^{d(\mathfrak{K}_\nu)}$ the order of $\mathfrak{D}(\mathfrak{K}_\nu)$.

The following results were given in Osima [9] (Cf. Curtis-Reiner [6]).

(A) We may set

(9) $$\eta_\tau = \sum_\nu a_\nu^\tau K_\nu\ (a_\nu^\tau \in \mathfrak{o}),$$

where $\mathfrak{K}_\nu$ ranges only over p-regular classes.

(B) For each η_τ, there exists a p-regular class $\mathfrak{K}_\mu$ satisfying the following conditions:

(a) $a_\mu^\tau \not\equiv 0 \pmod{\mathfrak{p}}$.

(b) For $\chi_i \in B_\tau$, $\omega_i(K_\mu) \not\equiv 0 \pmod{\mathfrak{p}}$ and, if $d(\mathfrak{K}_\nu) < d(\mathfrak{K}_\mu)$, $\omega_i(K_\nu) \equiv 0 \pmod{\mathfrak{p}}$.

(c) If $\mathfrak{D}(\mathfrak{K}_\nu)$ is not conjugate to any subgroup of $\mathfrak{D}(\mathfrak{K}_\mu)$, then $a_\nu^\tau \equiv 0 \pmod{\mathfrak{p}}$.

In this case, $\mathfrak{D}(\mathfrak{K}_\mu)$ is called a defect group of B_τ and $d(\mathfrak{K}_\mu)$ the defect of B_τ;

$$d(\mathfrak{K}_\mu) = \min\{\, m \mid \chi_i(1) \equiv 0 \pmod{p^{a-m}} \text{ for all } \chi_i \in B_\tau \,\}.$$

R. Brauer, in his papers [1, 2], gave the following theorem: *The class sums* K_ν^* *with* $d(\mathfrak{K}_\nu) = 0$ *form a basis of a subalgebra* M *of the center* Z^*. *The number of blocks of defect* 0 *is equal to the rank of* M^n *for sufficiently large* n.

In the present paper, in § 1, we shall show that we can take $n = 2$ in Brauer's theorem. In § 2, we shall prepare some lemmas for the following section. In § 3, we shall count the number of blocks B_τ with a given defect group $\mathfrak{D}$.

§ 1

In the following, as is discribed above, we shall give an improvement of Brauer's theorem.

It is well known that the space M, in Brauer's theorem, is an ideal of Z^* and that the primitive idempotents of M are identical with those idempotents η_τ^* such that B_τ have defect 0; we denote by E_0 the sum of all those idempotents. We have

(10) $$M = Z^* E_0 \oplus M(1-E_0).$$

LEMMA 1. *We have*

$$Z^* E_0 = \sum_{d(B_\tau)=0} \oplus K^* \eta_\tau^*,$$

where $d(B_\tau)$ denotes the defect of B_τ.

PROOF. If $d(B_\tau)=0$, then B_τ consists of exactly one absolutely irreducible character χ_i hence we have

$$Z^* \eta_\tau^* = K^* \eta_\tau^*.$$

LEMMA 2. *We have*

$$M^2 = Z^* E_0.$$

PROOF. We have to show that for any two classes $\mathfrak{K}_\lambda$, $\mathfrak{K}_\mu$ with defect 0 and for any block B_ρ with $d(B_\rho) > 0$,

$$K_\lambda K_\mu \eta_\rho \equiv 0 \pmod{\mathfrak{p} Z_0}$$

holds. In the expression

$$K_\lambda K_\mu \eta_\rho = \sum_{\nu=1}^{n} \sum_{\chi_i \in B_\rho} \frac{|\mathfrak{K}_\lambda| \, |\mathfrak{K}_\mu| \, \chi_i(G_\lambda) \chi_i(G_\mu) \chi_i(G_\nu^{-1})}{g \chi_i(1)} K_\nu,$$

hold $\chi_i(G_\lambda)\chi_i(G_\mu)\chi_i(G_\nu^{-1}) \in \mathfrak{o}$, $|\mathfrak{K}_\lambda| \, |\mathfrak{K}_\mu| \equiv 0 \pmod{p^{2a}}$ and $g\chi_i(1) \not\equiv 0 \pmod{p^{2a}}$. This proves Lemma 2.

From Lemma 1, 2, we obtain the following:

THEOREM 1. *If M is the linear subspace of Z^* spanned by those K_ν^* with $d(\mathfrak{K}_\nu)=0$, then the number of blocks B_τ with defect 0 is equal to* $\mathrm{rank}_{K^*} M^2$.

§ 2

In this section, we prepare for § 3.

Let $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_r$ be the p-regular sections (p'-sections) of $\mathfrak{G}$ and denote by S_β the sum of all K_ν such that $\mathfrak{K}_\nu \subseteq \mathfrak{S}_\beta$. It is easy to see that two elements G, H of $\mathfrak{G}$ belong to the same p'-section $\mathfrak{S}_\beta$ if and only if G^{p^a} and H^{p^a} are conjugate to each other in $\mathfrak{G}$. Let N be the radical of Z^* and assume that the unit element 1 belongs to $\mathfrak{S}_1$. We have the following:

LEMMA 3. *For each $\mathfrak{S}_\alpha$, we have*

(11) $$S_\alpha^* N = \{0^*\}$$

and

(12) $$N = \{x^* \in Z^* \mid S_1^* x^* = 0^*\}.$$

PROOF. First, we show (11). Let x be any element of Z_0 such that $x^* \in N$. By (1), (2), we have

$$S_\alpha x = \sum_{\nu=1}^{n} \sum_{i} \frac{\chi_i(S_\alpha)\omega_i(x)\chi_i(G_\nu^{-1})}{g} K_\nu.$$

It follows from Frobenius's theorem (Cf. Curtis-Reiner [6, Corollary (41.9)]) that

$$\chi_i(S_\alpha) \equiv 0 \pmod{p^a \mathfrak{o}}$$

holds, hence $\chi_i(S_\alpha)\chi_i(G_\nu^{-1})/g \in \mathfrak{o}$. Since $x^* \in N$, we have

$$\omega_i(x) \equiv 0 \pmod{\mathfrak{p}}.$$

Therefore, we get

$$S_\alpha x \equiv 0 \pmod{\mathfrak{p} Z_0}$$

which implies (11). Next, we show (12). Assume that $S_1^* x^* = 0^*$ $(x \in Z_0)$. By (5), (7), we have

$$S_1^* x^* \eta_\tau^* = \omega_\tau^*(x^*)\{\sum_{\nu} \sum_{\chi_i \in B_\tau} \frac{\chi_i(S_1)\chi_i(G_\nu^{-1})}{g} K_\nu\}^* = \omega_\tau^*(x^*) S_1^* \eta_\tau^* = 0^*,$$

where ω_τ^* is a linear character of Z^* associated with B_τ. It follows from Lemma 4 that

$$\omega_\tau^*(x^*) = 0$$

holds for $\tau = 1, 2, \cdots, t$. Hence, x^* belongs to N.

LEMMA 4. *We have*

(13) $$S_1 \eta_\tau \not\equiv 0 \pmod{\mathfrak{p} Z_0} \qquad (\tau = 1, 2, \cdots, t),$$

(14) $$S_1 \eta_\tau = \eta_\tau \qquad (\textit{for } B_\tau \textit{ with defect } 0).$$

PROOF. Assume $S_1 = \{1\} \cup \mathfrak{K}_1 \cup \cdots \cup \mathfrak{K}_l$ (disjoint union). For each μ $(1 \leq \mu \leq l)$, $\mathfrak{K}_\mu$ is a p-singular class and, by Osima [9] or Brauer-Nesbitt [3] with (8) (Cf. also Iizuka [7; 8, Theorem 2]), $K_\mu \eta_\tau$ is a linear combination of only those K_ν such that $\mathfrak{K}_\nu$ are p-singular classes. This fact combined with (A) in Introduction yields

$$S_1 \eta_\tau = \eta_\tau + \sum_{\mu} K_\mu \eta_\tau \not\equiv 0 \pmod{\mathfrak{p} Z_0}.$$

If $d(B_\tau) = 0$, then $K_\mu \eta_\tau = 0$ and $S_1 \eta_\tau = \eta_\tau$.

§ 3

In this section, for every p-subgroup $\mathfrak{D}$ of $\mathfrak{G}$ or every number d $(0 \leq d \leq a)$, we shall give a certain submodule of Z^*, the K^*-rank of which is equal to the number of blocks B_τ with $\mathfrak{D}(B_\tau) = \mathfrak{D}$ or with $d(B_\tau) = d$, $\mathfrak{D}(B_\tau)$ being the defect group of B_τ.

First, we consider the number of blocks of a given defect d. Let $V^{(d)}$ be a linear subspace of Z^* which is spanned by those K_ν^* such that $d(\mathfrak{K}_\nu) \leq d$. As is well known,

$$V^{(-1)} = \{0^*\} \subseteq V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(a)}$$

is a chain of ideals in Z^* (Cf. Osima [9]). We have

$$V^{(d)} = \sum_{d(B_\tau)<d} \oplus Z^* \eta_\tau^* \oplus \sum_{d(B_\rho)=d} \oplus Z^* \eta_\rho^* \oplus \sum_{d(B_\sigma)>d} \oplus V^{(d)} \eta_\sigma^*,$$

hence,

$$V^{(d)} s = \sum_{d(B_\tau)<d} \oplus Z^* \eta_\tau^* s \oplus \sum_{d(B_\rho)=d} \oplus Z^* \eta_\rho^* s \oplus \sum_{d(B_\sigma)>d} \oplus V^{(d)} \eta_\sigma^* s,$$

where $s = S_1^*$ (Cf. Tsushima [10, 11]). If $d(B_\sigma) > d$ then, by Lemma 3, we see that $V^{(d)} \eta_\sigma^* s = \{0^*\}$ holds, because $V^{(d)} \eta_\sigma^* \subseteq N$. Therefore, we get

$$V^{(d)} s = V^{(d-1)} s \oplus \sum_{d(B_\rho)=d} \oplus Z^* \eta_\rho^* s.$$

If $d(B_\rho) = d$ then, by Lemma 3, we have

$$K^* \cong Z^* \eta_\rho^* / N^* \eta_\rho^* \cong Z^* \eta_\rho^* s \qquad \text{(as } K^*\text{-modules)}.$$

Hence we obtain the following:

PROPOSITION 1. *The number of blocks B_τ of defect d is equal to* $\mathrm{rank}_{K^*} V^{(d)} s - \mathrm{rank}_{K^*} V^{(d-1)} s$.

Next, in the analogous way as above, we consider the number of blocks B_τ with a given defect group $\mathfrak{D}$. For two subgroups $\mathfrak{H}_1$, $\mathfrak{H}_2$ of $\mathfrak{G}$, the notation $\mathfrak{H}_1 \leq \mathfrak{H}_2$ $(\mathfrak{H}_1 < \mathfrak{H}_2)$ will mean that $\mathfrak{H}_1$ is conjugate to a subgroup (a proper subgroup) of $\mathfrak{H}_2$, and the notation $\mathfrak{H}_1 \approx \mathfrak{H}_2$ will mean that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjugate to each other in $\mathfrak{G}$. Let $\mathfrak{D}$ be a p-subgroup of $\mathfrak{G}$ and $V(\mathfrak{D})$ the linear subspace of Z^* which is spanned by those K_ν^* such that $\mathfrak{D}(\mathfrak{K}_\nu) \leq \mathfrak{D}$. It is also well known that $V(\mathfrak{D})$ is an ideal of Z^*. We have

$$V(\mathfrak{D}) = \sum_{\mathfrak{D}(B_\tau)<\mathfrak{D}}\oplus V(\mathfrak{D})\eta_\tau^* \oplus \sum_{\mathfrak{D}(B\rho)\approx\mathfrak{D}}\oplus Z^*\eta_\rho^* \oplus \sum_{\mathfrak{D}(B_\sigma)\not\leq\mathfrak{D}}\oplus V(\mathfrak{D})\eta_\sigma^*,$$

hence,

$$V(\mathfrak{D})s = \sum_{\mathfrak{D}(B_\tau)<\mathfrak{D}}\oplus V(\mathfrak{D})\eta_\tau^* s \oplus \sum_{\mathfrak{D}(B\rho)\approx\mathfrak{D}}\oplus Z^*\eta_\rho^* s \oplus \sum_{\mathfrak{D}(B_\sigma)\not\leq\mathfrak{D}}\oplus V(\mathfrak{D})\eta_\sigma^* s.$$

For B_σ with $\mathfrak{D}(B_\sigma)\not\leq\mathfrak{D}$, we have $V(\mathfrak{D})\eta_\sigma^* \subseteq N$ hence, by Lemma 3,

$$V(\mathfrak{D})\eta_\sigma^* s = \{0^*\}.$$

For B_τ with $\mathfrak{D}(B_\tau)<\mathfrak{D}$, we have

$$V(\mathfrak{D})\eta_\tau^* = V(\mathfrak{D}(B_\tau))\eta_\tau^*.$$

Therefore we see that

$$V(\mathfrak{D})s = \sum_{\mathfrak{D}(B_\tau)<\mathfrak{D}}\oplus V(\mathfrak{D}(B_\tau))\eta_\tau^* s \oplus \sum_{\mathfrak{D}(B\rho)\approx\mathfrak{D}}\oplus Z^*\eta_\rho^* s$$

$$= \sum_{\mathfrak{H}<\mathfrak{D}} V(\mathfrak{H})s \oplus \sum_{\mathfrak{D}(B\rho)\approx\mathfrak{D}}\oplus Z^*\eta_\rho^* s$$

holds. Thus we obtain the following:

PROPOSITION 2. *The number of blocks B_τ with a given defect group $\mathfrak{D}$ is equal to* $\text{rank}_{K^*}V(\mathfrak{D})s - \text{rank}_{K^*}\sum_{\mathfrak{H}<\mathfrak{D}} V(\mathfrak{H})s$.

Finally, by making use of the first main theorem on blocks (Brauer [2, (10B)]), we consider the number of blocks B_τ with a given defect group $\mathfrak{D}$. Let $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{D})$ be the normalizer of $\mathfrak{D}$ in $\mathfrak{G}$. By the first main theorem on blocks, the number of blocks of $\mathfrak{G}$ with defect group $\mathfrak{D}$ is equal to that of blocks of $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{D})$ which have defect group $\mathfrak{D}$. Hence we may reduce our problem to the case in which $\mathfrak{D}$ is a normal p-subgroup of $\mathfrak{G}$. By Proposition 2, we obtain the following:

THEOREM 2. *If $\mathfrak{D}$ is a normal p-subgroup of $\mathfrak{G}$, then the number of blocks of $\mathfrak{G}$ with defect group $\mathfrak{D}$ is equal to* $\text{rank}_{K^*} U_0(\mathfrak{D})s$, *where $U_0(\mathfrak{D})$ is the linear subspace of Z^* spanned by those K_ν^* such that $\mathfrak{K}_\nu$ are p-regular classes with defect group $\mathfrak{D}$.*

REMARK 1. Let $U(\mathfrak{D})$ $(U^{(d)})$ be the linear subspace of Z^* spanned by those K_μ^* such that $\mathfrak{K}_\mu$ are p-regular classes with $\mathfrak{D}(\mathfrak{K}_\mu)\leq\mathfrak{D}$ $(d(\mathfrak{K}_\mu)\leq d)$. It is easy to see that

$$V(\mathfrak{D})s = U(\mathfrak{D})s \quad (V^{(d)}s = U^{(d)}s)$$

holds. Then, in Proposition 2 (Proposition 1), we may replace $V(\mathfrak{D})$ $(V^{(d)})$ by $U(\mathfrak{D})$ $(U^{(d)})$.

REMARK 2. If the group $\mathfrak{G}$ has a normal Sylow p-subroup $\mathfrak{P}$ and $\mathfrak{K}_{\nu_1}$, $\mathfrak{K}_{\nu_2}, \cdots, \mathfrak{K}_{\nu_m}$ are the p-regular classes with defect group $\mathfrak{P}$ then, for each ν_j $(1 \leq j \leq m)$, there corresponds a p'-section $\mathfrak{S}_\beta$ such that $K_{\nu_j}^* s = S_\beta^*$. Considering this fact, it is easy to see from Thorem 2 that *the number of blocks with maximum defect is equal to the numer of p-regular classes with maximum defect* (Brauer [4, Theorem 2; 2, 6(D)]).

REMARK 3. From Theorem 2 follows some of the results in Bovdi [5] also.

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