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TWO-FOLD ASSIGNING FAMILIES OF HOLOMORPHIC MAPPINGS

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§ 1. Preliminaries.

The main purpose of this note is to give a generalization of a theorem proved by H. Wu in his paper [5], which states:

If (N, d) is a tight complex manifold, then there is no non-constant holomorphic map from C^n into N.

The space C^n in the theorem will be replaced by a complex manifold satisfying more general condition, stated in § 4.

The spaces considered here will be assumed connected and second countable. Let M, N be complex manifolds. We denote by $\mathscr{C}(M, N)$ the set of all continuous mappings from M to N and by $\mathscr{H}(M, N)$ the set of all holomorphic mappings from M to N, respectively. To $\mathscr{C}(M, N)$ we introduce the compact-open topology. Then, as is easily verified by the Cauchy integral formula $\mathscr{H}(M, N)$ is closed in $\mathscr{C}(M, N)$. This implies that a subset of $\mathscr{H}(M, N)$ is compact or closed in $\mathscr{C}(M, N)$ if and only if it is so in $\mathscr{H}(M, N)$.

By the previous assumption the complex manifolds under our consideration are metrizable, cf. Kelley [2]. So we choose on complex manifold N a metric d_N, which converts N into a metric space. With metric d_N a sequence $\{f_i\} \subset \mathscr{C}(M, N)$ converges to an $f \in \mathscr{C}(M, N)$ if and only if $\{f_i\}$ converges to f uniformly on every compact subset. In the following we shall say a sequence $\{f_i\} \subset \mathscr{C}(M, N)$ converges *compact-uniformly* in M if it converges uniformly on every compact subsets of M. The compact-uniform limit f of a sequence $\{f_i\} \subset \mathscr{C}(M, N)$ belongs to $\mathscr{H}(M, N)$ if every $f_i \in \mathscr{H}(M, N)$.

A sequence $\{f_i\} \subset \mathscr{C}(M, N)$ is said to be *compactly divergent* if and only if for any compact subset K in M and compact subset L in N there exists a number i_0 such that $f_i(K) \cap L = \phi$ for all $i \geqq i_0$.

DEFINITION 1.1 *A subset $\mathscr{F}$ of $\mathscr{C}(M, N)$ is called normal if and only if every sequence of $\mathscr{F}$ contains a subsequence which is either relatively compact in*

$\mathscr{C}(M, N)$ or compactly divergent.

Since we assumed M and N second countable, $\mathscr{C}(M, N)$ is also second countable, and hence the compactness of a subset of $\mathscr{C}(M, N)$ is checked by its sequential compactness.

Let d_N be a metric on N.

DEFINITION 1.2. *A subset $\mathscr{F}\subset\mathscr{C}(M, N)$ is called equicontinuous if and only if for any positive number ε and any point $x\in M$ there exists a neighborhood U of x such that $x'\in U$ implies $d_N(f(x), f(x'))<\varepsilon$ for all $f\in\mathscr{F}$.*

DEFINITION 1.3. *A complex manifold N is called taut if and only if for every complex manifold M the set of holomorphic mappings $\mathscr{H}(M, N)$ is normal.*

A complex manifold N with a metric d_N which metrizes N is called tight if and only if the set of holomorphic mappings $\mathscr{H}(M, N)$ is equicontinuous.

For the detailed informations about taut spaces and tight spaces we refer to Wu [5] and Barth [1].

§ 2. Two-fold assigning family of mappings.

We begin with

DEFINITION 2.1. *Let M, N be complex manifolds. A subset $\mathscr{F}$ of $\mathscr{C}(M, N)$ is called two-fold assigning if and only if given any two different points p, $q\in M$ and any two different points P, $Q\in N$ there exists an $f\in\mathscr{F}$ such that $f(p)=P$ and $f(q)=Q$.*

We have

THEOREM 2.2. *Let M, N be complex manifolds. A subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ can not be normal if it is two-fold assigning.*

PROOF. Let us assume that $\mathscr{F}$ is normal. Take a point $x_0\in M$, a sequence $\{x_i\}$ converging to x_0 and two different points $P, Q\in N$. By assumption there exists a sequence $\{f_i\}$ of $\mathscr{F}$ such that $f_i(x_i)=P$ and $f_i(x_0)=Q$. Since $\mathscr{F}$ is normal, the sequence $\{f_i\}$ should contain a subsequence which is either compactly divergent or compact-uniformly convergent.

We assert that the sequence $\{f_i\}$ does not contain any compactly divergent subsequence. For this purpose let K be a compact neighborhood of x_0. Then there exists an i_0 such that $x_i \in K$ for all $i \geqq i_0$. Put $L=\{P, Q\}$. Then, L is compact in N and $f_i(K) \cap L \supset L \neq \phi$ for all $i \geqq i_0$. Thus, $\{f_i\}$ can not contain any compactly divergent subsequence. So $\{f_i\}$ should contain a compact-uniformly convergent subsequence, say $\{f_j\}$. Let f_0 be the limit of $\{f_j\}$. We choose subsequence $\{x_j\}$ corresponding to $\{f_j\}$. Since $f_j \to f_0$ and $x_j \to x_0$ as $j \to \infty$, we have $f_j(x_j) \to f_0(x_0)$ as $j \to \infty$. On the other hand by the choice of $\{f_i\}$ we have $f_j(x_j) = P$ and $f_j(x_0) = Q$. Thus we have $P = Q$. This contradicts to the assumption that P and Q are different.

COROLLARY 2.3. *Let N be a complex manifold. If there exists a complex manifold M such that a subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ is two-fold assigning, then N can not be taut.*

In connection with equicontinuity we have a quite analogous result:

THEOREM 2.4. *Let M, N be complex manifolds. If a subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ is two-fold assigning, then $\mathscr{F}$ can not be equicontinuous.*

PROOF. As in the proof of Theorem 2.2 we choose a point $x_0 \in M$, a sequence of points $\{x_i\}$ of M converging to x_0, two different points P, $Q \in N$ and a sequence $\{f_i\}$ of $\mathscr{F}$ such that $f_i(x_i)=P$ and $f_i(x_0)=Q$. Assume that $\mathscr{F}$ is equicontinuous. Then, given any positive number ε there exists a neighborhood V of x_0 such that $x \in V$ implies that $d(f(x), f(x_0)) < \varepsilon$ for any $f \in \mathscr{F}$, where d is the metric inducing the topology of N. Since $x_i \to x_0$ as $i \to \infty$, there exists an i_0 such that $x_i \in V$ for all $i \geqq i_0$. Then, by the fact that $\{f_i\} \subset \mathscr{F}$ $d(f_i(x_i), f_i(x_0)) < \varepsilon$ for all $i \geqq i_0$. Let us choose ε so that $\varepsilon < d(P, Q)$. By the choice of $\{f_i\}$ that $f_i(x_i)=P$ and $f_i(x_0)=Q$, we obtain $d(f_i(x_i), f_i(x_0)) = d(P, Q) < \varepsilon < d(P, Q)$. This is a contradiction. Thus $\mathscr{F}$ can not be equicontinuous.

COROLLARY 2.5. *Let N be a complex manifold. If there exists a complex manifold M such that a subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ is two-fold assigning, then N can not be tight.*

PROOF. Assume that N is tight with respect to some metric d, then any subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ is equicontinuous with respect to d. Suppose that $\mathscr{F}$ is two-fold assigning. Then, Theorem 2.4 applies.

§ 3. Manifolds admitting two-fold transitive group of automorphisms.

Let M be a complex manifold and $Aut\ (M)$ the group of biholomorphic mappings of M onto M itself. $Aut(M)$ is a subset of $\mathscr{H}(M, M)$ and therefore $\mathscr{H}(M, M)$ is two-fold assigning if $Aut(M)$ is two-fold assigning. On the other hand $Aut\ (M)$ is two-fold assigning if and only if it is two-fold transitive. Suppose M is either taut or tight. Then, $\mathscr{H}(M, M)$ should be normal or equicontinuous, respectively. This, by Theorem 2.2 and Theorem 2.4, contradicts to the assumption that $\mathscr{H}(M, M)$ is two-fold assigning. Hence $Aut\ (M)$ can not be two-fold assigning. Thus in an obvious way we proved the following.

THEOREM 3.1 *Let M be a complex manifold. If its group of automorphisms $Aut(M)$ acts two-fold transitively on M, then M can neither be taut nor tight.*

COROLLARY 3.2. *The spaces P^n, C^n are neither taut nor tight.*

COROLLARY 3.3. *If a complex manifold M is either taut or tight, then $Aut(M)$ can not be two-fold transitive.*

Since any bounded domain of C^n is tight, we have the following

PROPOSITION 3.4. *The group of automorphisms of a bounded domain of C^n is not two-fold transitive.*

This fact is implicitly known, see Wu [5]. We only notice that $Aut(M)$ is two-fold transitive if and only if its isotropy subroup G_x, $x \in M$ acts transitively on $M-\{x\}$.

Equicontinuity and normality are considerably rigid conditions. In this connection we have

PROPOSITION 3.5. *Let M be a complex manifold. If a family $\mathscr{F}$ of holomorphic functions on M is a vector space over the complex number field C containing C and seperates the points of M, then $\mathscr{F}$ is two-fold assingning and therefore can neither be normal nor equicontinuous.*

PROOF. Let p, q and c_1, c_2 be two different points of M and C, respectively. Since $\mathscr{F}$ seperates the points of M, there exists an $f \in \mathscr{F}$ such that $f(p) \neq f(q)$. We define a function $g(x)$ by

$$g(x)=c_1+(c_2-c_1)\frac{f(x)-f(p)}{f(q)-f(p)}.$$

Then, g belongs to $\mathscr{F}$ and satisfies $g(p)=c_1$ and $g(q)=c_2$. Thus, $\mathscr{F}$ is two-fold assigning. The latter part of the proposition is the direct consequence of Theorem 2.2 and Theorem 2.4.

§ 4. Some theorems of Liouville-Wu type.

It is easily checked that the condition "*two-fold assigning*" in Theorem 2.2 and Theorem 2.4 can be relaxed to weaker one: there exist a sequence $\{x_i\}$ of M converging to an inner point x_0 of M, two different points P, $Q \in N$ and a sequence $\{f_i\}$ of $\mathscr{H}(M, N)$ such that $f_i(x_i)=P$ and $f_i(x_0)=Q$. Though this assumption is almost equivalent to assume "*not equicontinuous*", the results are very important as the following several theorems show.

LEMMA 4.1. *Let M, N be complex manifolds and $Aut(M)$ act two-fold transitively on M. Then, for any non-constant mapping $f \in \mathscr{H}(M, N)$ the family $f \circ Aut(M)=\{f \circ \sigma : \sigma \in Aut(M)\}$ is neither normal nor equicontinuous with respect to any metric on N.*

PROOF. Since f is not constant, there exist two points p, $q \in M$ such that $f(p) \neq f(q)$. Put $f(p)=P$ and $f(q)=Q$. The assumption that $Aut(M)$ is two-fold transitive implies that for any two different points a, $b \in M$ there exists a $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(a)=p$ and $\sigma(b)=q$. Hence, $f \circ \sigma(a)=P$ and $f \circ \sigma(b)=Q$. Thus, the family $f \circ Aut(M)$ satisfies the weakened condition cited above. Our Lemma is proved.

Now, we are able to prove a generalization of Wu's theorem cited in § 1.

THEOREM 4.2. *Let M be a complex manifold and $Aut(M)$ act two-fold transitively on M. Then there is no non-constant holomorphic mapping from M to either a taut space or a tight space.*

PROOF. A direct consequence of Lemma 4.1.

COROLLARY 4.3. *A holomorphic mapping from P^n* (or C^n) *to a taut space or a tight space is constant.*

PROOF. A two-fold transitive group of automorphisms acts on P^n, and on C^n also.

The proof of Lemma 4.1 suggests an idea to extend Theorem 4.2 to more general case. We begin with the complex number space C^n. The space C^n is most popular as the range of holomorphic mappings. It may be said that the space C^n is sufficiently wide as the range of holomorphic mappings and that classical Liouville's theorem and Wu's theorem characterize this fact to some extent. From this point of view we define

DEFINITION 4.4. *A complex manifold N is called an exact range of complex manifold M if and only if for any two different points P, Q of N there exists an $f \in \mathscr{H}(M, N)$ such that $f(M)$ contains both P and Q.*

For example the complex number plane C is the exact range of C^n, but a bounded domain of C^m can not be the exact range of C^n. On the other hand C can not be the exact range of compact spaces.

Our result concerning exact range is

THEOREM 4.5. *Let M be a complex manifold and $Aut(M)$ act two-fold transitively on M. If a complex manifold N is the exact range of M, then there is no non-constant holomorphic mapping from N to either a taut space or a tight space.*

PROOF. Assume that there exist a taut space L and a non-constant holomorphic mapping f of N to L. It is easily verified that the family $\mathscr{H}(M,$

N) is two-fold assigning. Then, the family $f\circ\mathscr{H}(M, N)$ satisfies the weakened condition remarked at the beginning of this section. On the other hand, as a subset of $\mathscr{H}(M, L)$ the family $f\circ\mathscr{H}(M, N)$ must be normal. This is a contradiction. Hence f must be constant. The proof for the case of tight L is quite analogous.

COROLLARY 4.6. *If M is a complex manifold on which the complex number plane C as a complex Lie group acts transitively, then there is neither non-constant holomorphic mapping from M to a taut space nor a tight space.*

PROOF. It suffices to notice that the condition "*C acts transitively on M*" implies that M is the exact range of C.

It will be interesting to compare Corollary 4.6 with Theorem 1.1 of Kobayashi [3].

§ 5. A property of C^n.

As is remarked at the beginning of § 4 the condition "*two-fold assigning*" can be replaced by weaker one in most of the theorems in § 2, § 3. Along this line the theorems in § 4 were proved. Through these sections we have in mind that the properties "*two-fold assigning*" and "*two-fold transitive*" are the most remarkable characters of the set of all holomorphic functions and of the group of automorphisms of C^n, respectively.

Now, we consider another property of C^n which is not necessarilly equivalent to "*two-fold assigning*" property. Let x be a point of C^n. Then x can be identified with the constant mapping with value x, which we denote again by x. Then, there exists a sequence $\{f_i\}$ of $Aut(C^n)$ which converges compact-uniformly to x.

Abstracting this fact we define

DEFINITION 5.1. *Let M and N be complex manifolds. We call a subset $\mathscr{F}$ of $\mathscr{H}(M, N)$ compactly assigning if and only if given any compact subset K of M and any open subset U of N there exists an $f\in\mathscr{F}$ such that $f(K)\subset U$.*

We have

THEOREM 5.2. *Let M be a complex manifold such that $Aut(M)$ is compactly assigning as a subset of $\mathscr{H}(M, N)$. Then M can neither be taut nor tight.*

PROOF. Let P, Q be any two different points of M and α any point of M different from P and Q. Let $\{U_n\}$, $\{V_n\}$, $n=1, 2, 3, \cdots$ be monotonely decreasing sequences of open neighborhoods of α such that $\bigcap_{n=1}^{\infty} \overline{U}_n = \bigcap_{=1n}^{\infty} \overline{V}_n = \{\alpha\}$, $\overline{V}_n \subset U_n$ and $U_{n+1} \subsetneqq V_n$. Then $\{U_n - \overline{V}_u\}$, $n=1, 2, 3, \cdots$ is a sequence of open sets. By assumption there exists an $f_n \in Ant(M)$ such that $f_n(\{P, Q\}) \subset U_n - \overline{V}_n$. By $U_{n+1} \subsetneqq V_n$ the sequence $\{f_n\}$ consist of different elements. By putting $f_n(P) = x_n$, $f_n(Q) = y_n$ and $\phi_n = f_n^{-1}$ we obtain:

(1) two sequences $\{x_n\}$, $\{y_n\}$ converging to the point α such that $x_n \neq y_n$, and

(2) a sequence $\phi_n \in Aut(M)$ such that $\phi_n(x_n) = P$ and $\phi_n(y_n) = Q$.

This is the same circumstances as in § 4. Hence Theorem 5.2 is proved.

It is obvious that $Aut(M)$ is compactly assigning if and only if for any constant mapping $\alpha \in \mathscr{H}(M, M)$, which is identified with the point α of M, there exists $\{f_n\} \subset Aut(M)$ which converges compact-uniformly to α. So we have another expression of Theorem 5.2:

PROPOSITION 5.3. *Let M be a complex manifold. If for any constant mapping $\alpha \in \mathscr{H}(M, M)$, which is identified with the point $\alpha \in M$, there exists a sequence $\{f_n\} \subset Aut(M)$ which converges compact-uniformly to α, then M can neither be taut nor tight.*

§ 6. Holomorphic functions in two-fold homogeneous manifold.

By a *two-fold homogeneous manifold* we understand a complex manifold the group of automorphisms of which is two-fold transitive. In this section we shall investigate the value-distribution of holomorphic functions in a two-fold homogeneous domain of C^n, which will lead us to a Picard-type theorem.

Oeljeklaus proved in his article [4] that a compact two-fold homogeneous manifold is projective-algebraic. This was worked out from the view point of the theory of *almost homogeneous spaces*. In connection with the result of Oeljeklaus and our results in preceding sections there naturally arises the following question: *what is the non compact two-fold homogeneous manifold*?

Especially, does there exist a proper subdomain of C^n which is two-fold homogeneous?

At present we can not give complete answer to this question. Here we only give several properties of such manifolds.

First we prove

PROPOSITION 6.1. *Let M be a two-fold homogeneous manifold. Then for any non-constant holomorphic function f on M the image $f(M)$ of M by f is dence in C, that is, $\overline{f(M)}=C$.*

PROOF. Assume that the image $f(M)$ is not dence in C. Then there exist a point $\alpha \in C$ and a closed disk Δ containing α such that $\Delta \cap f(M)=\phi$. Now, let σ be an automorphism of Riemann sphere which maps the complement of Δ into the unit disk $D=\{\zeta : |\zeta|<1\}$. The composite $\sigma \circ f$ maps M into D. Thus $\sigma \circ f$ is a bounded holomorphic function on M. By Theorem 4.5 $\sigma \circ f$ should be constant. Since σ is an automorphism, f itself should be constant. This constradicts to the assumption that f is non-constant.

Concerning the two-fold homogeneous subdomain of C^n we have

PROPOSITION 6.2. *If D is a two-fold homogeneous subdomain of C^n, then $D \cap L \neq \phi$ for any real hyperplane L.*

PROOF. Assume that there exists a real hyperplane L such that $D \cap L=\phi$. Let $z_1, z_2, z_3, \cdots, z_n$; $z_j=x_j+\sqrt{-1}\, y_j$, $j=1,2,3,\cdots,n$ be the coordinates of C^n and L be given by $\{z : \phi(z)=0\}$ where

$$\phi(z)=\sum_{j=1}^{n}(a_j x_j+b_j y_j)+d;\quad a_j, b_j, d \in R.$$

Since D is connected and $D \cap L=\phi$, we may assume that $D \subset \{z : \phi(z)<0\}$. Now, let us define a complex linear function $\Phi(z)$ by

$$\Phi(z)=\sum_{j=1}^{n}(a_j-\sqrt{-1}\, b_j)z_j+d.$$

Then, we have $\phi(z)=\mathrm{Re}\,\Phi(z)$ and therefore we see that $|\exp[\Phi(z)]|<1$ if and only if $\phi(z)<0$. By the construction of $\Phi(z)$ the function $\Psi(z)=\exp[\Phi(z)]$ is holomorphic and $|\Psi(z)|<1$ for every $z \in D$, that is, $\Psi(z)$ is bounded holomorphic function in D. This contradicts again to Theorem 4.5, because

$\Phi(z)$ is non-constant and therefore $\Psi(z)$ is non-constant.

Now, we want to improve Proposition 6.1, 6.2 and prove a Picard-type theorem for two-fold homogeneous manifolds.

We start from

PROPOSITION 6.3. $C-\{a, b\}$ is taut.

There are several proofs. The simplest is one making use of classical Schottky's theorem and Barth's recent result, cf. [1].

LEMMA 6.4. (Schottky, Bohr and Landau) *Let*

$$f(z)=a_0+a_1z+\cdots$$

be holomophic and $f(z)\neq 0, \neq 1$ *in the unit disk* $\{z:|z|<R\}$. *If* $|a_0|\leqq k$, *then there exists a constant* $M(k, \theta)$ *such that*

$$|f(z)|\leqq M(k, \theta)$$

for every z satisfying $|z|\leqq\theta R$, $0<\theta<1$, *where* $M(k, \theta)$ *depends only upon k and* θ.

LEMMA 6.5. (Barth) *Let M be a complex manifold. If the set of all holomorphic mappings from the unit disk to M is normal, then M is taut.*

The proof of Proposition 6.3 is direct from Lemma 6.4 and Lemma 6.5.

Our result is as follows.

THEOREM 6.6. *Let M be a two-fold homogeneous complex manifold. If a function* $f(z)$ *holomorphic in M is non-constant, then there is at most one value a such that the equation* $f(z)=a$ *has no solution.*

PROOF. Assume that there exists a holomorphic function $f(z)$ which misses two values $a, b\ (\neq\infty)$ in M. By Proposition 6.3 the family $f\circ Aut(M)=\{f\circ\sigma:\sigma\in Aut(M)\}$ is normal. On the other hand $f\circ Aut(M)$ can not be normal except when $f(z)$ is a constant, because $Aut(M)$ acts two-fold transitively on M. Hence, if $f(z)$ is non-constant, $f(z)$ misses at most one value.

As an application of Theorem 6.6 we obtain the following statement which is a refinement of Proposition 6.2.

COROLLARY 6.7. *If D is a two-fold homogeneous subdomain of C^n, then for any non-constant entire function $f(z)$ there is at most one value a such that the niveau set $L_a=\{z\in C^n: f(z)=a\}$ does not meet D: $L_a\cap D=\phi$.*

PROOF. If there exist two values a, b such that $L_a\cap D=\phi$ and $L_b\cap D=\phi$. Then the restriction of $f(z)$ to D defines a holomorphic mapping from D to $C-\{a,b\}$. Since D is two-fold homogeneous, $f(z)$ must be constant by Theorem 4.2. This contradicts to the assumption that $f(z)$ is non-constant.

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