Kumamoto J. Sci. (Math.)
Vol.9, 75-86 (1973)

ON EQUICONTINUOUS FAMILIES OF HOLOMORPHIC FUNCTIONS

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(Received September 14, 1972)

1. Introduction. The notion of normal family has proved itself to be very important in the theory of functions, since it was introduced by P. Montel. The notion is very qualitative, and this seems to be the reason why the characterization of normality of a family of holomorphic functions has not yet been obtained in a satisfactory way. One of few such examples is the following theorem due to Marty, cf. Ahlfors [1]: *Let D be a domain of the complex plane C. A family $\mathscr{F}$ of the meromorphic functions in D is normal if and only if the family of spherical derivatives* $|f'|(1+|f|^2)^{-1}$, $f \in \mathscr{F}$ *is uniformly bounded on every compact subset of D.*

Since a meromorphic function of several variables is not always a holomorphic mapping, Marty's theorem cited above is not extended in a natural way to the case of several variables. As is well known the condition in Marty's theorem above is also necessary and sufficient for the equicontinuity of $\mathscr{F}$, in other words, normality is equivalent to equicontinuity for $\mathscr{F}$, cf. Wu [3]. Another example not restricted to holomorphic case is Ascoli-Arzelà theorem, also cf. Wu [3]. An example for holomorphic case is famous Montel's theorem, which is a consequence of Ascoli-Arzelà theorem. In these theorems equicontinuity plays an essential rôle.

As is seen in the general theory of tight and taut manifolds, cf. [3], normality and equicontinuity are concerned with each other in a delicate manner.

The purpose of this note is to characterize equicontinuity of a family of holomorphic functions, § 2. In § 3 *equicontinuity implies normality* will be proved for holomorphic functions. In § 4 an analogue of Julia problem will be studied.

2. Equicontinuity of a family of holomorphic functions. Let D be a domain of C^n with coordinates $z_1, z_2, z_3, \cdots, z_n$. We denote the family of functions holomorphic in D by $\mathscr{A}(D)$. For a subfamily $\mathscr{F}$ of $\mathscr{A}(D)$ we denote by $\frac{\partial \mathscr{F}}{\partial z_i}$, $i=1,2,3,\cdots,n$ the families $\left\{\frac{\partial f}{\partial z_i} : f\in\mathscr{F}\right\}$, $i=1,2,3,\cdots,n$.

First we prove:

THEOREM 2.1. *A subfamily $\mathscr{F}$ of $\mathscr{A}(D)$ is equicontinuous in D if and only if n famillies $\frac{\partial \mathscr{F}}{\partial z_i}$, $i=1,2,3,\cdots,n$ are simultaneously uniformly bounded on every compact subset of D.*

It is obvious that the families $\frac{\partial \mathscr{F}}{\partial z_i}$, $i=1,2,3,\cdots,n$ are simultaneously uniformly bounded if and only if the family of real functions $\left\{\left(\left|\sum_{i=1}^{n}\frac{\partial f}{\partial z_i}\right|^2\right)^{\frac{1}{2}}: f\in\mathscr{F}\right\}$ is uniformly bounded on every compact subset of D, and therefore boundedness condition for $\frac{\partial \mathscr{F}}{\partial z_i}$, $i=1,2,3,\cdots,n$ can be replaced by one for $\left\{\left(\sum_{i=1}^{n}\left|\frac{\partial f}{\partial z_i}\right|^2\right)^{\frac{1}{2}}: f\in\mathscr{F}\right\}$.

PROOF OF THEOREM 2.1. Let z_0 be any point of D and B a closed ball around z_0 contained in D. Take any point z of B, let γ be the closed path from z_0 to z given by the line segment. Then we have

$$f(z)-f(z_0)=\int_\gamma df.$$

Hence,

$$\begin{aligned}\left|f(z)-f(z_0)\right| &\leqq \int_\gamma |df| \\ &= \int_\gamma \left|\frac{\partial f}{\partial z_k}\, dz_k\right| \\ &\leqq \int_\gamma \left(\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\right|^2\right)^{\frac{1}{2}}\cdot\left(\left|dz_j\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since B is compact, by assumption there exists a positive constant M such that:

$$\left(\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\right|^2\right)^{\frac{1}{2}}\leqq M$$

for any $z\in B$ and for any $f\in\mathscr{F}$. On the other hand $\left(\sum_{k=1}^{u}\left|dz_k\right|^2\right)^{\frac{1}{2}}$is nothing but the line element. Denoting by δ the radius of B we have

$$\left|f(z)-f(z_0)\right| \leqq M\cdot\|z-z_0\| \leqq M\cdot\delta.$$

Thus, $\mathscr{F}$ is equicontinuous.

Conversely, let us assume that the family $\left\{\left(\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\right|^2\right)^{1/2};\ f\in\mathscr{F}\right\}$ is not uniformly bounded on some compact subset of D. Then, there exists a compact subset K of D such that

$$\operatorname*{Sup}_{f\in\mathscr{F}}\ \operatorname*{Sup}_{z\in K}\left(\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\right|^2\right)^{1/2}=+\infty.$$

Then, we can choose a sequence $\{f_\lambda\}$ from $\mathscr{F}$ such that

$$\operatorname*{Sup}_{z\in K}\left(\sum_{k=1}^{n}\left|\frac{\partial f}{\partial z_k}\right|^2\right)^{1/2}\geqq\lambda.$$

Since every f_λ is holomorphic and K compact, there exists a sequence of points of D, $\{z_\lambda\}$, $\lambda=1,2,3,\cdots$ such that

$$\operatorname*{Sup}_{z\in K}\left(\sum_{k=1}^{n}\left|\frac{\partial f_\lambda}{\partial z_k}(z)\right|^2\right)^{1/2}=\left(\sum_{k=1}^{n}\left|\frac{\partial f_\lambda}{\partial z_k}(z_\lambda)\right|^2\right)^{1/2}.$$

Since K is compact, we can find a subsequence $\{z_\mu\}$ of $\{z_\lambda\}$ which converges to a point z_0 of K: $z\to z_0$ $(\mu\to\infty)$.

By assumption $\mathscr{F}$ is equicontinuous in D and therefore at z_0: for any positive number ε there exists a positive constat δ such that

$$(*)\qquad \left|f(z)-f(z_0)\right|<\varepsilon,\qquad z\in B_\delta,$$

where B_δ is the closed ball around z_0 with radius δ. Corresponding to the subsequence $\{z_\mu\}$ we choose the subsequence $\{f_\mu\}$ from $\{f_\lambda\}$. By restricting $(*)$ to this subsequence we have

$$\left|f_\mu(z)-f_\mu(z_0)\right|<\varepsilon,\qquad z\in B_\delta,$$

that is, the sequence $\{f_\mu(z)-f_\mu(z_0)\}$ is uniformly bounded on B_δ. By well-known Montel's theorem we can find a uniformly convergent subsequence $\{f_\nu(z)-f_\nu(z_0)\}$. Let $\phi(z)$ be the limit function. Then by famous Weierstrass's theorem the sequence of real-valued functions

$$\left\{\left(\sum_{k=1}^{n}\left|\frac{\partial[f_\nu(z)-f_\nu(z_0)]}{\partial z_k}\right|^2\right)^{\frac{1}{2}}\right\}$$

converges uniformly to

$$\left(\sum_{k=1}^{n}\left|\frac{\partial\phi}{\partial z_k}\right|^2\right)^{\frac{1}{2}},$$

which is continuous in B_δ.

Hence,

$$\lim_{\nu\to\infty}\left(\sum_{k=1}^{n}\left|\frac{\partial f_\nu}{\partial z_k}(z_\nu)\right|^2\right)^{\frac{1}{2}}<+\infty.$$

But the subsequence $\{z_\nu\}$ of $\{z_\lambda\}$ was chosen so as

$$\left(\sum_{k=1}^{n}\left|\frac{\partial f_\nu(z_\nu)}{\partial z_k}\right|^2\right)^{\frac{1}{2}}\geqq\nu,$$

which is a contradiction.

As is easily seen Theorem 2.1 is extended to the case of $\mathscr{A}^m(D)$, the family of holomorphic mappings of D to C^m. Let $\mathscr{F}$ be a subfamily of $\mathscr{A}^m(D)$. We denote by $\mathscr{F}_i$ the family consisting of i-th component of the members of $\mathscr{F}$: $\mathscr{F}_i=\{f_i: f=(f_1, f_2, f_3, \cdots, f_i, \cdots, f_m)$ for some $f\in\mathscr{F}\}$.

THEOREM 2.2. *A subfamily $\mathscr{F}$ of $\mathscr{A}^m(D)$ is equicontinuous if and only if $m\cdot n$ families $\frac{\partial\mathscr{F}_i}{\partial z_j}$, $i=1, 2, 3, \cdots, m$, $j=1, 2, 3, \cdots, n$ are all simultaneously uniformly bouded on every compact subset of D.*

PROOF. It suffices to observe that $\mathscr{F}$ is equicontinuous if and only if m families $\mathscr{F}_i$, $i=1, 2, 3, \cdots, m$ are simultaneously equicontinuous in D. Then Theorem 2.1 applies.

The following terminology seems to be convenient. Considering that the condition imposed on $\frac{\partial\mathscr{F}}{\partial z_i}$ in Theorem 2.1 is the same as is found in well-known Montel's theorem, we call a subfamily $\mathscr{G}$ of $\mathscr{A}(D)$ a *Montel family* in D if $\mathscr{G}$ is uniformly bounded on every compact subset of D. Then Theorem 2.1 is stated as follows: *A subfamily $\mathscr{F}$ of $\mathscr{A}(D)$ is equicontinuous in D if and* *only if n families* $\frac{\partial \mathscr{F}}{\partial z_i}$, $i=1,2,3,\cdots,n$ *are all Montel families.*

3. Equicontinuity implies normality. Let D be a domain of C^n and $\mathscr{A}(D)$ again the family of functions holomorphic in D. In the following we shall consider the normality of the subfamily of $\mathscr{A}(D)$. The difinition of normality will be given in somewhat restricted form. For the general one we refer to Wu [3].

In this note we call a sequence of $\mathscr{A}(D)$ *compact-uniformly* convergent in D if it is uniformly convergent on every compact subset of D.

DEFINITION 3.1. *A subfamily* $\mathscr{F}$ *of* $\mathscr{A}(D)$ *is called normal in* D *if any sequence of* $\mathscr{F}$ *contains a compact-uniformly convergent subsequence or a subsequence which diverges compact-uniformly to infinity.*

With this definition we are able to prove a generalization of classical Montel's theorem which asserts that a Montel family of holomorphic functions is normal:

THEOREM 3.2. *An equicontinuous subfamily* $\mathscr{F}$ *of* $\mathscr{A}(D)$ *is normal.*

This is essentially a consequence of Theorem 2.1. But we give here a direct proof.

Before we prove Theorem 3.2 we show

LEMMA 3.3. *Let* D *and* $\mathscr{F}$ *be the same as in* THEOREM 3.2. *Let* z_0 *be any point of* D. *If* $\mathscr{F}$ *is equicontinuous in* D, *then for any connected compact subset* K *of* D, *containing* z_0, *there exists a positive constant* M_K *such that* for every $f \in \mathscr{F}$

$$\left|f(z)-f(z_0)\right| \leqq M_K, \quad z \in K.$$

PROOF. First we want to define a special covering $\mathfrak{V}_\varepsilon$ of D. Take a positive number ε and fix. Since $\mathscr{F}$ is equicontinuous in D, we can associate to every point x of D an open neighborhood $V(x)$ such that $|f(x')-f(x'')|<\varepsilon$ for any two points $x', x'' \in V(x)$. The family $\{V(x)\}$, $x \in D$ thus constructed gives an open covering of D, which we denote by $\mathfrak{V}_\varepsilon$.

Let K be any connected compact subset of D containing z_0. Then K is covered by a finite number of neighborhoods $V(x_1)$, $V(x_2)$, $V(x_3),\cdots,V(x_k)$ of $\mathfrak{V}_\varepsilon$. Let z be any point of K. z and z_0 can be connected by a chain $\mathfrak{C}$ of neighborhoods from $V(x_1), V(x_2), V(x_3),\cdots,V(x_k)$: $\mathfrak{C}\equiv\{V(x_{i_1}),V(x_{i_2}),V(x_{i_3}),\cdots,V(x_{i_s})\}$, $z_0\in V(x_{i_1})$, $V(x_{i_j})\cap V(x_{i_{j+1}})\neq\phi$, $j=1,2,3,\cdots,s-1$, $V(x_{i_s})\ni z$, $i_j\in\{1,2,3,\cdots,k\}$. We can choose $\mathfrak{C}$ so that any member of $\{V(x_1),V(x_2),V(x_3),\cdots,V(x_k)\}$ appears at most once in $\mathfrak{C}$. Hence we may assume that $s\leqq k$ and therefore that the chain $\mathfrak{C}$ is given by $\{V(x_1),V(x_2),V(x_3),\cdots,V(x_s)\}$. Now, take a poin x'_i in $V(x_i)\cap V(x_{i+1})$, $i=1,2,3,\cdots, s-1$, respectively.

Then we have

$$f(z_0)-f(z)=f(z_0)-f(x_1)+\sum_{i=1}^{s-1}\{f(x_i)-f(x'_i)\}$$
$$+\sum_{i=1}^{s-1}\{f(x'_i)-f(x_{i+1})\}+f(x_s)-f(z).$$

Since $z_0, x_1\in V(x_1)$; $x_i, x'_i\in V(x_i)$; $x'_i, x_{i+1}\in V(x_{i+1})$ for $i=1,2,3,\ldots,s-1$ and x_s, $z\in V(x_s)$, by the construction of $\mathfrak{V}_\varepsilon$ we obtain

$$\left|f(z_0)-f(z)\right|\leqq 2s\varepsilon\leqq 2k\varepsilon.$$

Thus, we may take $2k\varepsilon$ as M_K, and Lemma 3.3 is proved.

PROOF OF THEOREM 3.2. Let z_0 be a point of D and $\{K_k\}$ an exaustion by compact sets of D: $K_k\subseteqq K_{k+1}\subseteqq\cdots$, $\bigcup_{k=1}^{\infty}K_k=D$. We assume that $z_0\in K_1$ and every K_k is connected. Since D is a domain of C^n, such exaustion exists.

Now, let $\{f_i\}$ be a sequence of $\mathscr{F}$. Then following two cases occur:

(a) $\{f_i(z_0)\}$ has a convergent subsequence;

(b) $\{f_i(z_0)\}$ tends to infinity.

For the case (a) we shall prove that the sequence $\{f_i\}$ contains a compact-uniformly convergent subsequence. By Lemma 3.3 there exists a positive constant M_1 for K_1 such that $|f(z)-f(z_0)|\leqq M_1$ for any $f\in\mathscr{F}$ and any $z\in K_1$. Specializing this to the sequence $\{f_i\}$ we have $|f_i(z)-f_i(z_0)|\leqq M_1$.

Hence,

$$\left|f_i(z)\right|\leqq\left|f_i(z_0)\right|+M_1.$$

By assumption the sequence $\{f_i(z_0)\}$ has a convergent subsequence $\{f_\lambda(z_0)\}$. Then convergence of $\{f_\lambda(z_0)\}$ implies that there exists a positive constant L_1 such that $|f_\lambda(z_0)|\leqq L_1$. Consequently, we obtain

$$\left| f_\lambda(z) \right| \leqq L_1 + M_1, \ z \in K_1.$$

Then, Montel's theorem applies and there exists a subsequence $\{f_{\lambda_1}\}$ of $\{f_\lambda\}$ which converges uniformly on K_1. Applying Lemma 3.3 again to $\{f_{\lambda_1}\}$ and K_2, we can choose a subsequence $\{f_{\lambda_2}\}$ of $\{f_{\lambda_1}\}$ which converges uniformly on K_2. Thus we obtain inductively a series of subsequences of original $\{f_i\}$: $\{f_{\lambda_1}\}$, $\{f_{\lambda_2}\},\cdots$ where $\{f_{\lambda_k}\}$ is a subsequence of $\{f_{\lambda_{k-1}}\}$ and converges uniformly on K_k. Then, by wellknown diagonal process we can contruct a subsequence of original $\{f_i\}$ which converges compact-uniformly in D.

Now, we shall consider the case (b). By the same arguments as for the case (a) there exists a positive number M_1 such that $|f_i(z)-f_i(z_0)|\leqq M_1$, $z \in K_1$. Hence, we have

$$\left| f_i(z) \right| \geqq \left| f_i(z_0) \right| - M_1.$$

We assumed that the sequence $\{f_i(z_0)\}$ diverges to infinity. So we can find a subsequence $\{f_{\lambda_1}(z_0)\}$ such that

$$\left| f_{\lambda_1}(z_0) \right| \geqq \lambda_1 + M_1.$$

Consequentry, we have

$$\left| f_{\lambda_1}(z) \right| \geqq \lambda_1, \quad z \in K_1.$$

This means that the subsequence $\{f_{\lambda_1}\}$ diverges uniformly to infinity in K_1. In the same way we can choose a subsequence $\{f_{\lambda_2}\}$ of $\{f_{\lambda_1}\}$ which diverges uniformly to infinity in K_2, a subsequence $\{f_{\lambda_3}\}$ of $\{f_{\lambda_2}\}$ which diverges uniformly to infinity in K_3, and so forth. From thus obtained series of subsequences of original $\{ f_i \}$ we can construct a sequence which diverges compact-uniformly to infinity in D. Since in both cases $\{K_k\}$ is a exaustion by the compacts subset of D, the latter part of Theorem 3.2 is easily verified.

In general the converse of Theorem 3.2 can not be proved. In the following we shall show the converse of Theorem 3.2 *under some restriction.*

PROPOSITION 3.4. *If any sequence of $\mathscr{F}$ has a subsequence which converges compact-uniformly in D, then $\mathscr{F}$ is equicontinuous in D.*

PROOF. Assume that $\mathscr{F}$ is not equicontinuous in D. Then, there exists a point z_0 of D such that $\mathscr{F}$ is not equicontinuous at z_0, that is, we can find a positive number ε_0, a sequence $\{z_i\}$ converging to z_0 and a sequence $\{f_i\}$ of $\mathscr{F}$ such that $|f_i(z_i)-f_i(z_0)|\geqq\varepsilon_0$. On the other hand, since $\{f_i\}$ is a sequence of $\mathscr{F}$, by assumption we can choose a subsequence $\{f_\lambda\}$ which converges compact-uniformly in D, the limit of which we shall denote by f.

Let U be a relatively compact neighborhood of z_0. Then, there exists a large number N_1 such that $z_\lambda\in U$ for $\lambda>N_1$. Since f is the uniform limit of $\{f_\lambda\}$, for any positive ε there exists a large number N_2 such that $|f_\lambda(z)-f(z)|<\varepsilon$ for any $z\in U$ and $\lambda>N_2$. Choosing ε and N so as $3\varepsilon<\varepsilon_0$ and $N\geqq\text{Max}\,(N_1, N_2)$, we have

$$\begin{aligned}\varepsilon_0 &\leqq \left| f_\lambda(z_\lambda)-f_\lambda(z_0)\right| \\ &\leqq \left| f_\lambda(z_\lambda)-f(z_\lambda)\right| + \left| f(z_\lambda)-f(z_0)\right| \\ &\quad + \left| f(z_0)-f_\lambda(z_0)\right| \\ &< 2\varepsilon+\left| f(z_\lambda)-f(z_0)\right|, \quad \lambda>N.\end{aligned}$$

f being continuous at z_0, by choosing U so that $\left|f(z)-f(z_0)\right|<\varepsilon$ for every $z\in U$, we have

$$\left| f(z_\lambda)-f(z_0)\right|<\varepsilon \text{ for } \lambda>N.$$

Consequently, we have $\varepsilon_0<3\varepsilon<\varepsilon_0$, which is a contradiction.

From the arguments in the proof of Theorem 3.2 we may state the following:

PROPOSITION 3.5. *Let D be a domain of C^n and $\{f_i\}$ a sequence of $\mathscr{A}(D)$ which is equicontinuous in D.*

Then,

(a) *if the sequence $\{f_i(z_0)\}$ has a converging subsequence for some point $z_0\in D$, then $\{f_i\}$ contains a subsequence which converges compact-uniformly in D;*

(b) *if the sequence $\{f_i(z_0)\}$ has a subsequence which diverges to infinity, for some point $z_0\in D$, then $\{f_i\}$ has a subsequence which diverges compact-uniformly to infinity in D.*

Remark. The restriction imposed upon the converse of Theorem 3.2 is unaboidable because of the fact that the metric of the range of functions in Theorem 3.2 is the usual metric of the complex number plane C. If we adopt the *spherical distance*, then the converse of Theorem 3.2 naturally holds, see §1.

4. **Domain of equicontinuity.** In this section we shall define a domain of equicontinuity for a family of holomorphic functions and consider an analogue of Julia problem raised for a domain of normality.

Let D be a domain of C^n, $\mathscr{F}$ a subfamily of $\mathscr{A}(D)$. We say D is *the domain of existence* of $\mathscr{F}$ if all functions of $\mathscr{F}$ can not continued simultaneously to larger domain, Then, as is well known, D is a domain of holomorphy.

Our results in this section are all carried over to *Riemann domain.* But for the sake of simplicity we restrict ourselves to the domains of the complex number space C^n.

DEFINITION 4.1. *Let D be a domain of C^n and $\mathscr{F}$ a subfamily of $\mathscr{A}$ (D). We assume that D is the domain of existence of $\mathscr{F}$. A subdomain D^* of D is called the domain of equicontinuity of $\mathscr{F}$ if $\mathscr{F}$ is equicontinuous in D and $\mathscr{F}$ is never equicontinuous in any larger domain which contains D^* as its proper subdomain.*

A domain D^* of C^n is simply called a domain of equicontinuity if it is the domain of equicontinuity for some family of holomorphic functions.

DEFINITION 4.2. *Let $\mathscr{F}$ be a subfamily of $\mathscr{A}(D)$ and D be the domain of existence of $\mathscr{F}$. A subdomain D^* of D is called the Montel domain of $\mathscr{F}$ if $\mathscr{F}$ is a Montel family in D^*, but is no longer a Montel family in any larger domain which contains D^* as a proper subdomain. A domain D^* of C^n is called simply a Montel domain if it is the Montel domain of some family of holomorphic functions.*

Remark. Even if D is schlicht, that is, a subdomain of C^n, the domain of existence of a subfamily of $\mathscr{A}(D)$ is not necessarily schlicht.

Let D be a domain of holomorphy in C^n. Then we can construct such a sequence of functions that converges compact-uniformly in D and the limit has D as its domain of existence. Hence, we may state that *a domain of holomorphy is a Montel domain.*

Now, we assert that the converse of this fact also holds:

THEOREM 4.3. *Let D be a domain of C^n and $\mathscr{F}$ a subfamily of $\mathscr{A}(D)$ which has D as its domain of existence. If a subdomain D^* of D is the Montel domain of $\mathscr{F}$, then D^* is a domain of holomorphy.*

PROOF. Our method of proof is due to Cartan and Thullen, cf. [2]. Suppose that D is not a domain of holomorphy. Then, by the fundamental theorem of Cartan and Thullen [2] D is not holomorphically convex. Hence, there exists a compact subset K of D such that the set

$$\tilde{K}=\{ z\in D^*: \left|f(z)\right| \leqq \operatorname{Sup}\left|f(K)\right|, f\in\mathscr{A}(D^*)\}$$

is not compact in D^*. Since the envelope of holomorphy $E(D^*)$ of D^* is a domain of holomorphy, the set

$$\hat{K}=\{ z\in E(D^*): \left|f(z)\right| \leqq \operatorname{Sup}\left|f(K)\right|, f\in\mathscr{A}(E(D^*))\}$$

is compact in $E(D^*)$. It is easily verified that $\tilde{K}=\hat{K}\cap D^*$. In the following we shall reproduce the main consideration in the classical work of Cartan and Thullen [3].

Now, let $S(z_0, r)$ denote the polydisk $\{z\in C^n: |z_i-z_i^0|<r,\ i=1,2,3,\cdots,n\}$. We define a function $d(z)$ by $d(z)=\operatorname{Sup}\{r: S(z,r)\subset D^*\}$ for $z\in D^*$. Then, $d(K)=\operatorname{Inf}_{z\in K} d(z)$ gives the destance between ∂D^* and K. Take positive ρ as $\rho<d(K)$ and fix. Suppose that for a point $z_0\in D^*$ the following inequality holds for every $f\in\mathscr{A}(D^*)$: $|f(z_0)|\leqq \operatorname{Sup}|f(K)|$.

Then, for any differential operator of the form

$$D^{(\nu)}=\frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n}}{\partial z_1^{\nu_1}\,\partial z_2^{\nu_2}\cdots\partial z_n^{\nu}},\quad \nu=(\nu_1,\nu_2,\cdots,\nu_n),$$

We have

$$\left|D^{(\nu)} f(z_0)\right| \leqq \operatorname{Sup}\left|D^{(\nu)}f(K)\right|.$$

Let K_ρ be the closure of the ρ-neighborhood of K. Since $\rho<d(K)$, K_ρ is

compact in D. Let z be any point of K. Then, Cauchy's inequality implies

$$\left| D^{(\nu)}f(z) \right| \leqq \frac{\nu_1!\nu_2!\cdots\nu_n!}{\rho^{\nu_1+\nu_2+\cdots+\nu_n}} \operatorname{Sup} \left| f(S(z,\rho)) \right|$$

$$\leqq \frac{\nu_1!\nu_2!\cdots\nu_n!}{\rho^{\nu_1+\nu_2+\cdots+\nu_n}} \operatorname{Sup} \left| f(K_\rho) \right|.$$

Hence, we obtain

$$\operatorname{Sup} \left| D^{(\nu)}f(K) \right| \leqq \frac{\nu_1!\nu_2!\cdots\nu_n!}{\rho^{\nu_1+\nu_2+\cdots+\nu_n}} \operatorname{Sup} \left| f(K_\rho) \right|.$$

Consequently, we have

$$\left| D^{(\nu)}f(z_0) \right| \leqq \frac{\nu_1!\nu_2!\ldots\nu_n!}{\rho^{\nu_1+\nu_2+\cdots+\nu_n}} \operatorname{Sup} \left| f(K_\rho) \right|.$$

Thus, the Taylor expansion of f at z_0 converges in the polydisk $S(z_0, \rho)$. Since $\hat{K}$ is not compact in D^* there exists a point $z_0 \in \hat{K}$ such that $d(z_0) < \rho$. This means that $z_0 \notin K_\rho$. Then, $S(z_0, \rho) - D^*$ contains non-empty open set. The function f was arbitrary in $\mathscr{A}(D^*)$. So we restrict our arguement to $\mathscr{F} \subset \mathscr{A}(D^*)$. Since by assumption $\mathscr{F}$ is a Montel family and therefore is uniformly bounded in compact subset K, the inequality derived above implies that $\mathscr{F}$ is uniformly bounded in $S(z_0, \rho) - D^*$ which contains non-empty interior Δ. This shows that $\mathscr{F}$ is a Montel family in $D^* \cup \Delta$. This contradicts to the assumption that D^* is the Montel domain of $\mathscr{F}$. Thus, D^* is holomorhically convex, and Theorem 4.3 is proved.

Now, we are ready to solve the following problem, which is the purpose of this section.

Problem. Is a domain of equicontinuity a domain of holomorphy?

Our answer is affirmative:

THEOREM 4.4. *Let D be a domain of C^n, $\mathscr{F}$ a subfamily of $\mathscr{A}(D)$ and D the domain of existence of $\mathscr{F}$. If D^* is the domain of equicontinuity of $\mathscr{F}$, then D^* is a domain of holomorphy.*

PROOF. By definition $\mathscr{F}$ is equicontinuous in D^*, but is no longer equicontinuous in any larger domain. By Theorem 2.1 D^* is equal to the intersection of n Montel domains of n families $\frac{\partial \mathscr{F}}{\partial z_k}$, $k=1, 2, 3, \cdots, n$. Since by Theorem 4.3 a Montel domain is a domain of holomorphy, and D is the intersection of Montel domains, D is a domain of holomorphy. This completes the proof.

The converse of Therem 4.4 is obviously true.

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