Kumamoto J. Math.
Vol.**39** (2026), 1–24

Some quasi-ergodic limit theorems for purely discontinuous additive functionals under non-local Feynman-Kac transforms

Takara Tagawa

(Received August 28, 2025)
(Accepted November 25, 2025)

Abstract. We establish quasi-ergodic limit theorems for purely discontinuous additive functionals of symmetric stable processes under non-local Feynman-Kac transforms. Some punctual in-time convergence results for non-local Feynman-Kac semigroups induced by discontinuous additive functionals play a crucial role. As an application, we shall demonstrate that our quasi-ergodic limit theorems provide a new approach in establishing a large deviation principle for discontinuous additive functionals by using a rate function with a more direct expression.

1 Introduction

Let $\mathbb{X} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, X_t, \mathbb{P}_x)$ be the symmetric α-stable process in $\mathbb{R}^d$ with $0 < \alpha < 2$ and $d \geq 1$, that is, X is a pure jump conservative Lévy process whose characteristic function is given by $\exp(-t|\xi|^\alpha)$ ($\xi \in \mathbb{R}^d$). Here $(\mathcal{M}_t)_{t\geq 0}$ is the minimal (augumented) filtration. We use m and $\mathbb{E}_x$ to denote the Lebesgue measure in $\mathbb{R}^d$ and the expectation with respect to $\mathbb{P}_x$ for any $x \in \mathbb{R}^d$, respectively.

Let F be a non-trivial symmetric (i.e., $F(y,z) = F(z,y)$ for any $y, z \in \mathbb{R}^d$) bounded Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Consider a summation of Fs induced over all jumps of the process $\mathbb{X}$ up to time $t > 0$ by taking into account both the position before and the position after the jump:

$$A_t^F = \sum_{0<s\leq t} F(X_{s-}, X_s). \tag{1.1}$$

Then, (1.1) forms a purely discontinuous additive functional of $\mathbb{X}$, which often appears when considering pure jump effects in Markov processes, and in certain cases, is thought to represent the number of jumps in various pure jump processes.

Mathematical Subject Classification (2020): Primary 37A30, 60J45; Secondary 60G52, 47D06, 60J46

Key words: discontinuous addtive functionals, non-local Feynman-Kac semigroup, large deviations, quasi-ergodic limits, Kato class

Using (1.1), we consider the following non-local Feynman-Kac transforms, which can be thought of as an extinction time for the biased action: for $x \in \mathbb{R}^d$, $t \geq 0$ and $B \in \mathcal{M}$,

$$\mathbb{P}^F_{x,t}(B) := \mathbb{E}_x\left[e^{-A^F_t}; B\right]. \tag{1.2}$$

With (1.2), define the renormalised probability measure $\mathbb{Q}_{x,t}$ by

$$\mathbb{Q}_{x,t}(B) := \frac{\mathbb{P}^F_{x,t}(B)}{\mathbb{P}^F_{x,t}(\Omega)} = \frac{\mathbb{E}_x[e^{-A^F_t}; B]}{\mathbb{E}_x[e^{-A^F_t}]}.$$

We use $\mathbb{E}^{\mathbb{Q}}_{x,t}$ to denote the expectation with respect to $\mathbb{Q}_{x,t}$ for any $x \in \mathbb{R}^d$ and $t \geq 0$. The purpose of this paper is to study some quasi-ergodic limit theorems for purely discontinuous additive functionals under the probability measure $\mathbb{Q}_{x,t}$. More precisely, let G be a symmetric Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal and A^G_t is the additive functional of the form (1.1) defined similarly for G. Under some condition on the jumping function F on $\mathbb{R}^d \times \mathbb{R}^d$ appearing in the non-local Feynman-Kac transform (1.2), we prove that there exists a jumping measure $\mathcal{J}_{\phi_0}$ on $\mathbb{R}^d \times \mathbb{R}^d$ determined by a bounded continuous function ϕ_0 on $\mathbb{R}^d$ such that the following quasi-ergodic limit theorem holds for any symmetric Borel measurable function G on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on diagonal and satisfying $\iint_{\mathbb{R}^d \times \mathbb{R}^d} |G(y,z)|^k \mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) < \infty$ for $k = 1, 2$:

$$\lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t\right)^k\right] = \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^k. \tag{1.3}$$

for any $x \in \mathbb{R}^d$ (see Theorem 3.4 and Theorem 3.5).

The long-time behavior of the mean-ratio of additive functionals of an almost surely killed Markov process $(X_t)_{t\geq 0}$ conditioned to survive on a state space E (typically, a locally compact separable metric space) has been studied by several authors ([1, 2, 9, 15, 16, 22]). They mainly dealt with (excepting the results in [15, 16] for discontinuous additive functionals) a continuous additive functional of $(X_t)_{t\geq 0}$ of the integral form

$$A^f_t := \int_0^t f(X_s)\mathrm{d}s$$

for any bounded measurable or integrable function f on E. Denote by $\mathfrak{m}$ the underlying measure on E. Breyer and Roberts [1] established the quasi-ergodic limit theorem for A^f_t when $\mathfrak{m}(E) < \infty$, $k = 1$ and $(X_t)_{t\geq 0}$ is positive recurrent. Zhang et al. [22] established a method to derive the quasi-ergodic limit theorem for A^f_t under $\mathfrak{m}(E) < \infty$ and some conditions on the heat kernel of $(X_t)_{t\geq 0}$. See also He et al. [9] for the case that $\mathfrak{m}$ is not necessarily a finite measure on E. However, all their method strongly relies on the additive functional of $(X_t)_{t\geq 0}$ being in integral form, and a different approach is needed to establish (1.3) for the purely discontinuous additive functional under the non-local Feynman-Kac scheme.

It is known that any symmetric Markov process can be transformed into an ergodic process by some multiplicative functional ([8, Chapter 6]). To prove our result (1.3), we will apply this fact to transform a symmetric stable process $\mathbb{X}$ with a non-local Feynman-Kac weight into an ergodic process by a multiplicative functional. Indeed, the Feynman-Kac semigroup p_t^F for any $t > 0$ defined in (2.1) is a compact operator on $L^2(\mathbb{R}^d; m)$ under some condition on F (see Lemma 2.1). From this, there exists the principal eigenfunction ϕ_0 (ground state) of p_t^F, which can be taken to be strictly positive and has a bounded continuous version on $\mathbb{R}^d$. Further, one can see by a similar way of [19, Proposition 3.1 and Lemma 3.4] that ϕ_0 belongs to $L^1(\mathbb{R}^d; m)$. By making use of ϕ_0, we can construct an irreducible and conservative $\phi_0^2 m$-symmetric Markov process $\mathbb{X}^{\phi_0}$ by the multiplicative functional $L_t^{\phi_0}$ defined in (3.3). Then, applying the Fukushima ergodic theorem (see Theorem 3.4) to $\mathbb{X}^{\phi_0}$, with the relation between p_t^F and the semigroup of $\mathbb{X}^{\phi_0}$, we obtain the punctual in-time convergence results for the Feynman-Kac semigroup p_t^F involving the Lévy kernel (Lemma 3.2). Using this, we establish the quasi-ergodic limits for the first and second moments of $\frac{1}{t}A_t^G$ as a suitable semigroup expression via the Lévy system (Theorems 3.4 and 3.5).

Research on large deviation principles for additive functionals of Markov processes is a major topic in probability theory, and it has been studied in various ways (see [5, 6, 14, 20, 21] and references therein). In particular, Chen and Tsuchida [5] established a large deviation principle for pairs of continuous and purely discontinuous additive functionals under a general framework of symmetric Markov processes. Our result (1.3) provides a new approach to derive a large deviation principle for purely discontinuous additive functionals. As an application, we shall show that our quasi-ergodic limit theorem under the non-local Feynman-Kac scheme helps to establish a large deviation principle for the time average of a purely discontinuous additive functional by using a rate function with a more direct expression (see Theorem 4.2).

The remainder of this paper is arranged as follows. In Section 2, we present the setup along with some results on a non-local Feynman-Kac semigroup. We also recall some definitions and known results that will be used in the rest of the paper. In Section 3, we characterize the quasi-ergodic limiting measures of the first and second moments of $\frac{1}{t}A_t^G$ and establish a functional type of weak law of large numbers for $\frac{1}{t}A_t^G$, as a straightforward corollary. The large deviation principle for the time average of the purely discontinuous additive functional A_t^G is studied in Section 4. Throughout the paper, we use c, c', C, C' to denote strictly positive constants and may change from line to line. The symbol ":=" is used to denote a definition, which is read as "is defined to be". We let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for any $a, b \in \mathbb{R}$.

2 Some preliminary results

Let $\mathbb{X} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \theta_t, X_t, \mathbb{P}_x)$ be the symmetric α-stable process in $\mathbb{R}^d$ with $0 < \alpha < 2$ and $d \geq 1$, that is, $\mathbb{X}$ is a pure jump conservative Lévy process whose characteristic function is given by $\exp(-t|\xi|^\alpha)$ ($\xi \in \mathbb{R}^d$). Here $(\mathcal{M}_t)_{t\geq 0}$ is the minimal (augumented) filtration and θ_t is the shift operator satisfying $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geq 0$. Let $\{p_t\}_{t\geq 0}$ and $\{R_\beta\}_{\beta\geq 0}$ be the transition semigroup and resolvent of $\mathbb{X}$, respectively:

$$p_t f(x) := \mathbb{E}_x[f(X_t)], \quad R_\beta f(x) := \int_0^\infty e^{-\beta t} p_t f(x) \mathrm{d}t$$

for any $f \in B_b(\mathbb{R}^d)$, where $B_b(\mathbb{R}^d)$ denotes the space of all bounded Borel functions on $\mathbb{R}^d$. It is known that $\mathbb{X}$ is an irreducible strong Markov process satisfying the strong Feller property (**(SF)** in short), that is, $p_t(B_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$ for any $t > 0$, where $C_b(\mathbb{R}^d)$ denotes the space of all bounded continuous functions on $\mathbb{R}^d$. Further, we note that $\{p_t\}_{t\geq 0}$ is ultracontractive ((UC) in short), $\|p_t\|_{1,\infty} \leq Ct^{-d/\alpha}$.

Denote by $C_0^\infty(\mathbb{R}^d)$ the space of C^∞ functions on $\mathbb{R}^d$ with compact support. Let m be the d-dimensional Lebesgue measure. Let (E, F) be the Dirichlet form on $L^2(\mathbb{R}^d; m)$ generated by $\mathbb{X}$:

$$E(u, u) = \frac{C_{d,\alpha}}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(y) - u(z))^2}{|y - z|^{d+\alpha}} m(\mathrm{d}y) m(\mathrm{d}z), \quad F := \overline{C_0^\infty(\mathbb{R}^d)}^{E_1},$$

where

$$\begin{aligned} C_{d,\alpha} :=& \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}, \\ E_1(u, u) :=& E(u, u) + \int_{\mathbb{R}^d} u^2 \mathrm{d}m \end{aligned}$$

and F is the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $E_1(u, u)^{1/2}$. It is known that $\mathbb{X}$ has a Lévy system $(N(y, \mathrm{d}z), H_t)$ with

$$N(y, \mathrm{d}z) = C_{d,\alpha} |y - z|^{-(d+\alpha)} m(\mathrm{d}z), \quad H_t = t,$$

that is, for any nonnegative Borel function U on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal and any $x \in \mathbb{R}^d$

$$\mathbb{E}_x \left[\sum_{0<s\leq t} U(X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} U(X_s, z) N(X_s, \mathrm{d}z) \mathrm{d}s \right].$$

So the jumping measure J of $\mathbb{X}$ is expressed by

$$J(\mathrm{d}y\mathrm{d}z) = \frac{1}{2} N(y, \mathrm{d}z) m(\mathrm{d}y).$$

To simplify the notation, we shall write

$$N[U](y) := \int_{\mathbb{R}^d} U(y, z) N(y, \mathrm{d}z)$$

for any $y \in \mathbb{R}^d$.

Let $F := F^+ - F^-$ be a non-trivial symmetric (i.e., $F(y,z) = F(z,y)$ for any $y, z \in \mathbb{R}^d$) bounded Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. We say that a function F is in the *Kato class* of $\mathbb{X}$ if

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\int_0^t N[|F|](X_s)\mathrm{d}s\right] = 0,$$

where $|F| := F^+ + F^-$. For example, F is in the Kato class of $\mathbb{X}$ if $N[|F|]$ is a bounded function on $\mathbb{R}^d$. Further, if F is a symmetric bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ with

$$\begin{aligned} &|F(x,y)| \le c|x-y|^\beta \quad \text{for } x, y \in \mathbb{R}^d \text{ and} \\ &F(x,y) = 0 \quad \text{for } (x,y) \in \mathbb{R}^d \times D^c, \end{aligned}$$

where D is a compact subset of $\mathbb{R}^d$, c and β are two positive constants such that $\beta > \alpha$, then F belongs to the Kato class of $\mathbb{X}$ ([3, Lemma 5.1]). For more examples of jumping function Fs belonging to the Kato class, we refer to [3, 4]. Let A_t^F be the discontinuous additive functional of $\mathbb{X}$ given by (1.1). With this additive functional, define the non-local Feynman-Kac semigroup $\{p_t^F\}_{t\ge 0}$ by

$$p_t^F f(x) := \mathbb{E}_x \left[e^{-A_t^F} f(X_t)\right], \quad f \in B_b(\mathbb{R}^d). \tag{2.1}$$

Note that p_t^F also satisfies **(SF)** if F is in the Kato class of $\mathbb{X}$ (cf. [14, Lemma 3.2]). Thus, p_t^F admits a symmetric integral kernel $p_t^F(y,z)$ such that $p_t^F(y,\mathrm{d}z) = p_t^F(y,z)m(\mathrm{d}z)$ for any $y, z \in \mathbb{R}^d$ and $t > 0$.

Let E^F be the symmetric quadratic form on $F \times F$ defined by

$$E^F(u,v) := E(u,v) + \iint_{\mathbb{R}^d\times\mathbb{R}^d} u(y)v(z)\left(1 - e^{-F(y,z)}\right) N(y,\mathrm{d}z)m(\mathrm{d}y).$$

In view of Stollmann-Voigt's inequality, $E^F(u,v)$ is well-defined for F belonging to the Kato class of $\mathbb{X}$ (cf. [13]). It will be clarified that $F \times F$ is associated with $\{p_t^F\}_{t\ge 0}$ (see the proof of Lemma 2.1). Now, let us define the bottom of the spectrum of (E^F, F) by

$$C_F := -\inf\left\{E^F(u,u) : u \in F, \int_{\mathbb{R}^d} u^2 \mathrm{d}m = 1\right\}. \tag{2.2}$$

Lemma 2.1. *Suppose that F is of a Kato class function of $\mathbb{X}$ and satisfies $N[F](x) \to \infty$ as $|x| \to \infty$. Then p_t^F is a compact operator on $L^2(\mathbb{R}^d; m)$ for each $t > 0$.*

Proof. Let

$$F_1 := 1 - e^{-F}, \quad A_t^{F_1} := \sum_{0<s\le t} F_1(X_{s-}, X_s).$$

F_1 is also a function of Kato class of $\mathbb{X}$ and is bounded above away from 1. It is known by using the Lévy system of $\mathbb{X}$ that a positive continuous additive functional

$$A_t^{N[F_1^+]} := \int_0^t N[F_1^+](X_s)\mathrm{d}s \quad (\text{resp. } A_t^{N[F_1^-]} := \int_0^t N[F_1^-](X_s)\mathrm{d}s)$$

is the dual predictable projection of $A_t^{F_1^+}$ (resp. $A_t^{F_1^-}$) (cf. [8, p. 421]), thus $A_t^{N[F_1]} - A_t^{F_1}$ is a local martingale. Its Doléans-Dade exponential is

$$\begin{aligned} Y_t &= e^{A_t^{N[F_1]} - A_t^{F_1}} \prod_{s\le t}(1 - F_1(X_{s-}, X_s))e^{F_1(X_{s-},X_s)} \\ &= e^{A_t^{N[F_1]}} \prod_{s\le t}(1 - F_1(X_{s-}, X_s)) \\ &= e^{-A_t^F + A_t^{N[F_1]}}. \end{aligned}$$

Y_t is a local martingale, so it is a supermartingale multiplicative functional of $\mathbb{X}$ under $\mathbb{P}_x$, called a pure jump Girsanov transform of $\mathbb{X}$. Let $\widetilde{\mathbb{X}}$ be the strong Markov process obtained from $\mathbb{X}$ through Y_t, that is, the transition semigroup $\{\widetilde{p}_t\}_{t\ge 0}$ of $\widetilde{\mathbb{X}}$ is given by

$$\widetilde{p}_t f(x) := \widetilde{\mathbb{E}}_x[f(X_t)] = \mathbb{E}_x[Y_t f(X_t)]$$

for $f \in B_b(\mathbb{R}^d)$. In fact, $\widetilde{\mathbb{X}}$ is known to be a symmetric α-stable-like process on $\mathbb{R}^d$ (cf. [13]). Then, the non-local Feynman-Kac semigroup p_t^F can be expressed as

$$\begin{aligned} p_t^F f(x) =&\mathbb{E}_x\left[e^{-A_t^F} f(X_t)\right] = \mathbb{E}_x\left[Y e^{-A_t^{N[F_1]}} f(X_t)\right] = \widetilde{\mathbb{E}}_x\left[e^{-A_t^{N[F_1]}} f(X_t)\right] \\ :=&\widetilde{p}_t^{N[F_1]} f(x) \end{aligned} \tag{2.3}$$

for $f \in B_b(\mathbb{R}^d)$, that is, the non-local Feynman-Kac semigroup p_t^F of $\mathbb{X}$ can be identified with the local Feynman-Kac semigroup $\widetilde{p}_t^{N[F_1]}$ of $\widetilde{\mathbb{X}}$. It is well known that the local Feynman-Kac semigroup p_t^V of $\widetilde{\mathbb{X}}$ induced by the Kato potential V is to be compact on $L^2(\mathbb{R}^d; m)$ whenever $V(x) \to \infty$ as $|x| \to \infty$ (cf. [10, Lemma 2.1(5)]). Now the conclusion of the present lemma immediately follows from (2.3) and the fact that $CN[F] \le N[F_1]$ for $C := (1 - e^{-\|F\|_\infty})/\|F\|_\infty > 0$. □

From Lemma 2.1, we have the following:

Corollary 2.2. *Suppose that F is of a Kato class function of $\mathbb{X}$ and satisfies $N[F](x) \to \infty$ as $|x| \to \infty$. Then, there is a unique minimizing function (called a ground state) $\phi_0 := \phi_0^F$ in (2.2), that is, there exists $\phi_0 \in F$ with $\int_{\mathbb{R}^d} \phi_0^2 \mathrm{d}m = 1$ such that*

$$C_F = -E^F(\phi_0, \phi_0).$$

Furthermore, ϕ_0 can be taken to be strictly positive and has a bounded continuous version on $\mathbb{R}^d$.

Proof. The existence of a minimizing function ϕ_0 follows from Lemma 2.1. The last assertion is a consequence due to [18, Theorems 3.3 and 5.4]. $\square$

We say that the non-local Feynman-Kac semigroup $\{p_t^F\}_{t\geq 0}$ is intrinsically ultracontractive ((IUC) in short) if there exists a constant $c_t > 0$ such that the integral kernel $p_t^F(y,z)$ of $\{p_t^F\}_{t\geq 0}$ satisfies

$$p_t^F(y,z) \leq c_t\phi_0(y)\phi_0(z), \quad \text{for all } t>0 \text{ and } y,z\in\mathbb{R}^d.$$

Lemma 2.3. *Suppose that F is of a Kato class with respect to $\mathbb{X}$ and satisfies $N[F](x)/\log|x| \to \infty$ as $|x|\to\infty$. Then $\{p_t^F\}_{t\geq 0}$ is (IUC).*

Proof. Let $\widetilde{\mathbb{X}}$ be the transformed process of $\mathbb{X}$ by Y_t as introduced in the proof of Lemma 2.1. Then, the corresponding Dirichlet form $(\widetilde{E},\widetilde{F})$ of $\widetilde{\mathbb{X}}$ is given by

$$\widetilde{E}(u,v) = E^F(u,v) - \int_{\mathbb{R}^d} u(y)v(y)N[F_1](y)m(\mathrm{d}y), \quad u,v\in\widetilde{F}=F$$

(see [13, Theorem 3.2]). Note that the function F used in [13] should be replaced with $-F$ in our setting. Set

$$\widetilde{E}^{N[F_1]}(u,v) := \widetilde{E}(u,v) + \int_{\mathbb{R}^d} u(y)v(y)N[F_1](y)m(\mathrm{d}y).$$

Then, we see that $E^F(u,u) = \widetilde{E}^{N[F_1]}(u,u)$ for $u\in F$ with $\int_{\mathbb{R}^d}u^2\mathrm{d}m = 1$, in particular, $E^F(\phi_0,\phi_0) = \widetilde{E}^{N[F_1]}(\phi_0,\phi_0)$. This means that ϕ_0 is a ground state of the local Feynman-Kac semigroup $\{\widetilde{p}_t^{N[F_1]}\}_{t\geq 0}$ given in (2.3), that is,

$$e^{-C_Ft}p_t^F\phi_0(x) = e^{-C_Ft}\widetilde{p}_t^{N[F_1]}\phi_0(x) = \phi_0(x).$$

It is known in view of [11, Theorem 3] that the local Feynman-Kac semigroup p_t^V of $\widetilde{\mathbb{X}}$ induced by a Kato potential V satisfying $V(x)/\log|x|\to\infty$ as $|x|\to\infty$ is (IUC). From this general fact and the hypothesis $N[F](x)/\log|x|\to\infty$ as $|x|\to\infty$ with $CN[F]\leq N[F_1]$ for some $C>0$, $\{\widetilde{p}_t^{N[F_1]}\}_{t\geq 0}$ is (IUC). Hence $\{p_t^F\}_{t\geq 0}$ is also (IUC). $\square$

The following is known as the Fukushima ergodic theorem (see [7, Corollary to Theorem 1],[19, Theorem 2.2]): Let E be a locally compact separable metric space and $\mathfrak{m}$ a positive Radon measure on E with full topological support.

Theorem 2.4. *Let $\mathbb{X}$ be an $\mathfrak{m}$-symmetric irreducible conservative Markov process on E and $\{p_t\}_{t\geq 0}$ be its transition semigroup. If $\mathfrak{m}(E)<\infty$, then for $f\in L^\infty(E;\mathfrak{m})$,*

$$\lim_{t\to\infty} p_tf(x) = \frac{1}{\mathfrak{m}(E)}\int_E f\,\mathrm{d}\mathfrak{m} \quad \mathfrak{m}\text{-a.e. } x\in E \text{ and in } L^1(E;\mathfrak{m}). \tag{2.4}$$

Remark 2.5. ([19, Corollary 2.1]) In addition to the assumptions of Theorem 2.4, if $\{p_t\}_{t\geq 0}$ is (UC) (that is, $\|p_t\|_{1,\infty}\leq c_t$ for a constant $c_t>0$) and satisfies **(SF)**, then the assertion (2.4) holds for $f\in L^1(E;\mathfrak{m})$ and the phrase $\mathfrak{m}$-a.e. $x\in E$ can be strengthened to all $x\in E$.

3 Quasi-ergodic limit theorems for purely discontinuous additive functionals

In this section, we shall establish the quasi-ergodic limit theorems for purely discontinuous additive functionals under the probability measure $\mathbb{Q}_{x,t}$. For this, we first give some punctual in-time convergence results for the non-local Feynman-Kac semigroup $\widehat{p}_t^F$ defined in (3.2).

In the remainder of this section, we always assume that $F = F^+ - F^-$ is a symmetric bounded Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal such that F is in the Kato class of $\mathbb{X}$ and satisfies

$$N[F](x)/\log|x| \longrightarrow \infty, \quad \text{as} \quad |x| \to \infty. \tag{3.1}$$

3.1 Punctual in-time convergence of Feynman-Kac semigroups

For notational brevity, let

$$\widehat{p}_t^F f(x) := e^{-C_F t} p_t^F f(x) = \mathbb{E}_x \left[e^{-C_F t - A_t^F} f(X_t) \right], \quad f \in B_b(\mathbb{R}^d). \tag{3.2}$$

We exploit the Doob's transform that relies on the definition of a semigroup of the following form:

$$p_t^{\phi_0} f(x) := \frac{1}{\phi_0(x)} \widehat{p}_t^F (\phi_0 f)(x), \quad f \in B_b(\mathbb{R}^d).$$

In view of [8, Lemma 6.3.2], we know that this expression defines the semigroup of a $\phi_0^2 m$-symmetric irreducible and conservative Markov process on $\mathbb{R}^d$, namely $\mathbb{X}^{\phi_0} = (X_t, \mathbb{P}_x^{\phi_0})$ (whose extinction time is thus infinite). In particular, $\{\widehat{p}_t^F\}_{t\geq 0}$ is (IUC) by Lemma 2.3. The fact that ϕ_0 is the ground state of p_t^F implies that the following equality holds:

$$\mathbb{P}_x^{\phi_0}(\Lambda_t) = \int_{\Lambda_t} L_t^{\phi_0}(\omega) \mathbb{P}_x(\mathrm{d}\omega) \quad \Lambda_t \in \mathcal{M}_t,$$

where $L_t^{\phi_0}$ is a multiplicative functional (called a ground state transform) of $\mathbb{X}$ defined by

$$L_t^{\phi_0} = e^{-C_F t - A_t^F} \frac{\phi_0(X_t)}{\phi_0(x)}. \tag{3.3}$$

Note that $\mathbb{X}^{\phi_0}$ also satisfies **(SF)**. Further, $\{p_t^{\phi_0}\}_{t\geq 0}$ is also (UC) because of (IUC) of $\{p_t^F\}_{t\geq 0}$. Thus, by applying Theorem 2.4 (with Remark 2.5) to $\mathbb{X}^{\phi_0}$, we see that for any $f \in L^1(\mathbb{R}^d; \phi_0 m)$,

$$\lim_{t\to\infty} \widehat{p}_t^F f(x) = \lim_{t\to\infty} \phi_0(x) p_t^{\phi_0} \left(\frac{f}{\phi_0} \right)(x) = \phi_0(x) \int_{\mathbb{R}^d} f(y)\phi_0(y) m(\mathrm{d}y) \tag{3.4}$$

for any $x \in \mathbb{R}^d$.

In particular, we obtain

$$\lim_{t\to\infty}\widehat{p}_t^F\mathbf{1}(x)=\lim_{t\to\infty}\phi_0(x)p_t^{\phi_0}\left(\frac{\mathbf{1}}{\phi_0}\right)(x)=\phi_0(x)\int_{\mathbb{R}^d}\phi_0(y)m(\mathrm{d}y) \tag{3.5}$$

for any $x\in\mathbb{R}^d$.

For $N(y,\mathrm{d}z):=C_{d,\alpha}|y-z|^{-(d+\alpha)}m(\mathrm{d}z)$, let

$$N^{\phi_0}(y,\mathrm{d}z):=\frac{\phi_0(z)}{\phi_0(y)}e^{-F(y,z)}N(y,\mathrm{d}z)$$

for $y,z\in\mathbb{R}^d$. Define the measures ν_{ϕ_0} on $\mathbb{R}^d$ and $\mathcal{J}_{\phi_0}$ on $\mathbb{R}^d\times\mathbb{R}^d$ by

$$\nu_{\phi_0}(\mathrm{d}y):=\frac{1}{\int_{\mathbb{R}^d}\phi_0\mathrm{d}m}\,\phi_0(y)m(\mathrm{d}y),$$
$$\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z):=N^{\phi_0}(y,\mathrm{d}z)\phi_0^2(y)m(\mathrm{d}y)=\phi_0(y)\phi_0(z)e^{-F(y,z)}N(y,\mathrm{d}z)m(\mathrm{d}y),$$

respectively.

Lemma 3.1. *$(N^{\phi_0}(y,\mathrm{d}z),t)$ is a Lévy system for $\mathbb{X}^{\phi_0}$. In particular, $\mathcal{J}_{\phi_0}$ is the jumping measure of $\mathbb{X}^{\phi_0}$.*

Proof. Let $W=(W_t)_{t\geq 0}$ be a positive predictable process on Ω and U be a nonnegative Borel measurable function on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on the diagonal. Then, we have by virtue of [17, (62.13)], together with the fact that $\mathbb{X}^{\phi_0}$ is ϕ_0^2m-symmetric,

$$\begin{aligned}
&\mathbb{E}_x^{\phi_0}\left[\sum_{0<s\leq t}(k\wedge W_sU(X_{s-},X_s))\right]\\
&=\sum_{0<s\leq t}\mathbb{E}_x^{\phi_0}\left[(k\wedge W_sU(X_{s-},X_s))\right]\\
&=\sum_{0<s\leq t}\mathbb{E}_x\left[\left(k\wedge L_s^{\phi_0}W_sU(X_{s-},X_s)\right)-\int_0^s(k\wedge W_{s'}U(X_{s'-},X_{s'}))\,\mathrm{d}L_{s'}^{\phi_0}\right]\\
&=\sum_{0<s\leq t}\mathbb{E}_x\left[\left(k\wedge L_s^{\phi_0}W_sU(X_{s-},X_s)\right)\right]\\
&=\mathbb{E}_x\left[\sum_{0<s\leq t}\left(k\wedge L_s^{\phi_0}W_sU(X_{s-},X_s)\right)\right]
\end{aligned}$$

for each fixed $k>0$. Here we used in the third indentity that

$$\mathbb{E}_x[\int_0^s(k\wedge W_{s'}U(X_{s'-},X_{s'}))\mathrm{d}L_{s'}^{\phi_0}]=0$$

by the martingale property of $L_s^{\phi_0}$. Letting $k\to\infty$, it follows that

$$\mathbb{E}_x^{\phi_0}\left[\sum_{0<s\leq t}W_sU(X_{s-},X_s)\right]=\mathbb{E}_x\left[\sum_{0<s\leq t}L_s^{\phi_0}W_sU(X_{s-},X_s)\right].$$

From this and [17, p. 346], together with the fact that

$$\frac{L_s^{\phi_0}}{L_{s-}^{\phi_0}} = \frac{\phi_0(X_s)}{\phi_0(X_{s-})} e^{-F(X_{s-},X_s)},$$

we have

$$\begin{aligned}\mathbb{E}_x^{\phi_0}\left[\sum_{0<s\le t} W_s U(X_{s-},X_s)\right] &= \mathbb{E}_x\left[\sum_{0<s\le t} W_s L_{s-}^{\phi_0} U(X_{s-},X_s)\frac{\phi_0(X_s)}{\phi_0(X_{s-})}e^{-F(X_{s-},X_s)}\right] \\ &= \mathbb{E}_x\left[\int_0^t W_s L_s^{\phi_0}\frac{1}{\phi_0(X_s)} N\left[Ue^{-F}\cdot\phi_0\right](X_s)\mathrm{d}s\right] \\ &= \mathbb{E}_x^{\phi_0}\left[\int_0^t W_s N^{\phi_0}[U](X_s)\mathrm{d}s\right].\end{aligned}$$

This implies the assertion. □

Now, we give some punctual in-time convergence results for the Feynman–Kac semigroup $\widehat{p}_t^F$. To do this, we shall modify a result for killed stable semigroups studied in [16, Lemma 3.1] to $\widehat{p}_t^F$.

Lemma 3.2. (1) *For any nonnegative symmetric Borel measurable function G on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on diagonal such that $N^{\phi_0}[G] \in L^1(\mathbb{R}^d;\phi_0^2 m)$ and $g \in L^1(\mathbb{R}^d;\phi_0 m)$, $x \in \mathbb{R}^d$ and $0<p<1$, we have*

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\widehat{p}_t^F\mathbf{1}(x)}\,\widehat{p}_{pt}^F\left(N\left[Ge^{-F}\cdot\widehat{p}_{(1-p)t}^F g\right]\right)(x) \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\int_{\mathbb{R}^d} g\mathrm{d}\nu_{\phi_0}.\end{aligned}$$

(2) *For any nonnegative symmetric Borel measurable functions G, G' on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on diagonal such that $N^{\phi_0}[G], N^{\phi_0}[G'] \in L^1(\mathbb{R}^d;\phi_0^2 m)$, $g \in L^1(\mathbb{R}^d;\phi_0 m)$, $x \in \mathbb{R}^d$ and $0<p<p'<1$, we have,*

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\widehat{p}_t^F\mathbf{1}(x)}\,\widehat{p}_{pt}^F\Big(N\Big[Ge^{-F}\cdot\widehat{p}_{(p'-p)t}^F\left(N\left[G'e^{-F}\cdot\widehat{p}_{(1-p')t}^F g\right]\right)\Big]\Big)(x) \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\iint_{\mathbb{R}^d\times\mathbb{R}^d} G'(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\int_{\mathbb{R}^d} g\mathrm{d}\nu_{\phi_0}.\end{aligned}$$

Proof. We may and do assume that g is nonnegative. (1) We note by the symmetry of the functions F, G and the measure $N(y,\mathrm{d}z)m(\mathrm{d}y)$ that for any nonnegative measurable function h,

$$\begin{aligned}&\int_{\mathbb{R}^d} N^{\phi_0}[G\cdot h]\phi_0^2\mathrm{d}m \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)h(z)N(y,\mathrm{d}z)e^{-F(y,z)}\phi_0(y)\phi_0(z)m(\mathrm{d}y) \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)h(y)N(y,\mathrm{d}z)e^{-F(y,z)}\phi_0(y)\phi_0(z)m(\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} N^{\phi_0}[G]\cdot h\phi_0^2\mathrm{d}m,\end{aligned} \tag{3.6}$$

For each fixed $r_0 > 0$ and a function $\rho \in L^1(\mathbb{R}^d; \phi_0 m)$, let

$$h_{r_0}^{(\rho)}(x) := \frac{1}{\phi_0(x)} \inf_{r \geq r_0} \widehat{p}_r^F \rho(x) = \inf_{r \geq r_0} p_r^{\phi_0} \left(\frac{\rho}{\phi_0} \right)(x). \tag{3.7}$$

Then, due to (3.6) and (UC) of $\{p_t^{\phi_0}\}_{t \geq 0}$,

$$\begin{aligned} \int_{\mathbb{R}^d} N\left[Ge^{-F} \cdot \phi_0 h_{r_0}^{(g)}\right] \phi_0 \mathrm{d}m &= \int_{\mathbb{R}^d} N^{\phi_0}\left[G \cdot h_{r_0}^{(g)}\right] \phi_0^2 \mathrm{d}m \\ &= \int_{\mathbb{R}^d} N^{\phi_0}[G] \cdot h_{r_0}^{(g)} \phi_0^2 \mathrm{d}m \\ &\leq \int_{\mathbb{R}^d} N^{\phi_0}[G] \cdot p_{r_0}^{\phi_0}\left(\frac{g}{\phi_0}\right) \phi_0^2 \mathrm{d}m \\ &\leq \left\| p_{r_0}^{\phi_0}\left(N^{\phi_0}[G]\right)\right\|_{L^\infty(\mathbb{R}^d;m)} \int_{\mathbb{R}^d} g\phi_0 \mathrm{d}m \\ &< \infty. \end{aligned} \tag{3.8}$$

This implies that the following limit holds for any $x \in \mathbb{R}^d$: for any sufficiently large t such that $(1-p)t \geq r_0$, thanks to (3.4) and (3.5),

$$\begin{aligned} &\liminf_{t \to \infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F g\right]\right)(x) \\ &\geq \liminf_{t \to \infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \phi_0 h_{r_0}^{(g)}\right]\right)(x) \\ &= \frac{1}{\int_{\mathbb{R}^d} \phi_0 \mathrm{d}m} \int_{\mathbb{R}^d} N^{\phi_0}\left[G \cdot h_{r_0}^{(g)}\right] \phi_0^2 \mathrm{d}m. \end{aligned}$$

On the other hand, since $g \in L^1(\mathbb{R}^d; \phi_0 m)$, $h_{r_0}^{(g)}(y) \overset{r_0 \to \infty}{\longrightarrow} \int_{\mathbb{R}^d} g\phi_0 \mathrm{d}m$ for any $y \in \mathbb{R}^d$. Hence, letting $r_0 \to \infty$, by the monotone convergence theorem,

$$\begin{aligned} &\liminf_{t \to \infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F g\right]\right)(x) \\ &\geq \int_{\mathbb{R}^d} N^{\phi_0}[G] \phi_0^2 \mathrm{d}m \cdot \int_{\mathbb{R}^d} g \mathrm{d}\nu_{\phi_0} \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} G(y,z) \mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) \cdot \int_{\mathbb{R}^d} g \mathrm{d}\nu_{\phi_0}. \end{aligned} \tag{3.9}$$

We note that the limit is already known by (3.4) provided $g = \phi_0$, because

$$N[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F \phi_0] = N[Ge^{-F} \cdot \phi_0] \in L^1(\mathbb{R}^d; \phi_0 m)$$

does not depend on t. From this observation, the converse inequality in the limit is deduced as follows: let $g_n := g \wedge (n\phi_0)$. Applying $n\phi_0 - g_n$ to g in the left-hand side of (3.9), one has

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F (n\phi_0 - g_n)\right]\right)(x) \\
&= \liminf_{t\to\infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \left\{ \widehat{p}_{pt}^F \left(N[Ge^{-F} \cdot n\phi_0]\right)(x) - \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F g_n\right]\right)(x)\right\} \\
&= \frac{n}{\int_{\mathbb{R}^d} \phi_0 \mathrm{d}m} \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) \\
&\qquad - \limsup_{t\to\infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F g_n\right]\right)(x),
\end{aligned}$$

while its right-hand side is less than

$$\frac{n}{\int_{\mathbb{R}^d} \phi_0 \mathrm{d}m} \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) \cdot \int_{\mathbb{R}^d} g_n \mathrm{d}\nu_{\phi_0}.$$

So we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{\widehat{p}_t^F \mathbf{1}(x)} \widehat{p}_{pt}^F \left(N\left[Ge^{-F} \cdot \widehat{p}_{(1-p)t}^F g_n\right]\right)(x) \\
&\leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z) \cdot \int_{\mathbb{R}^d} g_n \mathrm{d}\nu_{\phi_0}.
\end{aligned} \tag{3.10}$$

By the monotone convergence theorem, the $L^1(\mathbb{R}^d;\phi_0 m)$-norm of $g - g_n$ tends to 0 as $n \to \infty$. Thus, the $L^1(\mathbb{R}^d;\phi_0^2 m)$-norm of

$$\gamma_{n,t} := N^{\phi_0}[G \cdot p_{(1-p)t}^{\phi_0}(|g-g_n|/\phi_0)]$$

converges to 0 as $n \to \infty$ uniformly for t sufficiently large:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \gamma_{n,t}\phi_0^2 \mathrm{d}m \\
&= \int_{\mathbb{R}^d} N^{\phi_0}\left[G \cdot p_{(1-p)t}^{\phi_0}\left(\frac{|g-g_n|}{\phi_0}\right)\right] \phi_0^2 \mathrm{d}m \\
&\leq \left\| p_{(1-p)t}^{\phi_0}\left(\frac{|g-g_n|}{\phi_0}\right)\right\|_{L^\infty(\mathbb{R}^d;\phi_0^2 m)} \cdot \int_{\mathbb{R}^d} N^{\phi_0}[G]\phi_0^2 \mathrm{d}m \\
&\leq c r_0^{-d/\alpha} \int_{\mathbb{R}^d} |g-g_n|\phi_0 \mathrm{d}m \cdot \int_{\mathbb{R}^d} N^{\phi_0}[G]\phi_0^2 \mathrm{d}m \;\longrightarrow\; 0 \quad \text{as} \;\; n\to\infty.
\end{aligned}$$

Then, we have for large t such that $pt \geq r_1$ for a fixed $r_1 > 0$,

$$
\begin{aligned}
&\left|\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot \widehat{p}^F_{(1-p)t}g\right]\right)(x) - \widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot \widehat{p}^F_{(1-p)t}g_n\right]\right)(x)\right| \\
&\leq \widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot \widehat{p}^F_{(1-p)t}\left(|g-g_n|\right)\right]\right)(x) \\
&= \phi_0(x) p^{\phi_0}_{pt}\left(\frac{1}{\phi_0} N\left[Ge^{-F}\cdot \widehat{p}^F_{(1-p)t}\left(|g-g_n|\right)\right]\right)(x) \\
&\leq \phi_0(x)\left\| p^{\phi_0}_{pt}\left(N^{\phi_0}\left[G\cdot p^{\phi_0}_{(1-p)t}\left(\frac{|g-g_n|}{\phi_0}\right)\right]\right)\right\|_{L^\infty(\mathbb{R}^d;\phi_0^2 m)} \\
&\leq c r_1^{-d/\alpha}\phi_0(x)\int_{\mathbb{R}^d}\gamma_{n,t}\phi_0^2 \mathrm{d}m \;\longrightarrow\; 0 \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

Hence, letting $n \to \infty$ proves that (3.10) extends for g.
(2) For each $r_0 > 0$ such that both $(1-p')t \geq r_0$ and $(p'-p)t \geq r_0$, the inequality below is deduced by virtue of (3.7):

$$
\begin{aligned}
&\frac{1}{\widehat{p}^F_t \mathbf{1}(x)}\widehat{p}^F_{pt}\Big(N\Big[Ge^{-F}\cdot \widehat{p}^F_{(p'-p)t}\left(N\left[G'e^{-F}\cdot \widehat{p}^F_{(1-p')t}g\right]\right)\Big]\Big)(x) \\
&\geq \frac{1}{\widehat{p}^F_t \mathbf{1}(x)}\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot \phi_0 h^{(\rho_1)}_{r_0}\right]\right)(x),
\end{aligned}
$$

where $\rho_1(x) = N[G'e^{-F}\cdot \phi_0 h^{(g)}_{r_0}](x)$. By the definition of $h^{(\rho_1)}_{r_0}$ and the fact that $\{p^{\phi_0}_t\}_{t\geq 0}$ satisfies (UC), one can also check as in (3.8) that $N[Ge^{-F}\cdot \phi_0 h^{(\rho_1)}_{r_0}] \in L^1(\mathbb{R}^d;\phi_0 m)$, with a norm upper-bounded by

$$
\left\|p^{\phi_0}_{r_0}\left(N^{\phi_0}[G]\right)\right\|_{L^\infty(\mathbb{R}^d;m)}\cdot \left\|p^{\phi_0}_{r_0}\left(N^{\phi_0}[G']\right)\right\|_{L^\infty(\mathbb{R}^d;m)}\cdot \int_{\mathbb{R}^d} g\phi_0 \mathrm{d}m < \infty.
$$

Thus, thanks to (3.4), (3.5) and (3.6),

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{\widehat{p}^F_t \mathbf{1}(x)}\widehat{p}^F_{pt}\Big(N\Big[Ge^{-F}\cdot \widehat{p}^F_{(p'-p)t}\left(N\left[G'e^{-F}\cdot \widehat{p}^F_{(1-p')t}g\right]\right)\Big]\Big)(x) \\
&\geq \frac{\int_{\mathbb{R}^d} N\Big[Ge^{-F}\cdot \phi_0 h^{(\rho_1)}_{r_0}\Big]\phi_0 \mathrm{d}m}{\int_{\mathbb{R}^d}\phi_0 \mathrm{d}m} \\
&= \frac{\int_{\mathbb{R}^d} N^{\phi_0}\left[G\cdot h^{(\rho_1)}_{r_0}\right]\phi_0^2 \mathrm{d}m}{\int_{\mathbb{R}^d}\phi_0 \mathrm{d}m} \\
&= \frac{\int_{\mathbb{R}^d} N^{\phi_0}[G] h^{(\rho_1)}_{r_0}\phi_0^2 \mathrm{d}m}{\int_{\mathbb{R}^d}\phi_0 \mathrm{d}m}.
\end{aligned}
$$

Further, by Fatou's lemma,

$$\begin{aligned}\liminf_{r_0\to\infty} h_{r_0}^{(\rho_1)}(y) &\geq \liminf_{r_0\to\infty}\left(\lim_{r\to\infty} p_r^{\phi_0}\left(\frac{\rho_1}{\phi_0}\right)(y)\right)\\ &= \liminf_{r_0\to\infty}\int_{\mathbb{R}^d} N^{\phi_0}\left[G'\cdot h_{r_0}^{(g)}\right]\phi_0^2\mathrm{d}m\\ &\geq \int_{\mathbb{R}^d} N^{\phi_0}[G']\left(\liminf_{r_0\to\infty} h_{r_0}^{(g)}\right)\phi_0^2\mathrm{d}m\\ &= \int_{\mathbb{R}^d} N^{\phi_0}[G']\phi_0^2\mathrm{d}m\cdot\int_{\mathbb{R}^d} g\phi_0\mathrm{d}m\end{aligned}$$

for any $y \in \mathbb{R}^d$. With these two estimates, we can conclude the lower bound by letting $r_0 \to \infty$:

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{\widehat{p}_t^F\mathbf{1}(x)}\widehat{p}_{pt}^F\Big(N\Big[Ge^{-F}\cdot\widehat{p}_{(p'-p)t}^F\Big(N\Big[G'e^{-F}\cdot\widehat{p}_{(1-p')t}^F g\Big]\Big)\Big]\Big)(x)\\ &\geq \int_{\mathbb{R}^d} N^{\phi_0}[G]\phi_0^2\mathrm{d}m\cdot\int_{\mathbb{R}^d} N^{\phi_0}[G']\phi_0^2\mathrm{d}m\cdot\int_{\mathbb{R}^d} g\mathrm{d}\nu_{\phi_0} \qquad (3.11)\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\iint_{\mathbb{R}^d\times\mathbb{R}^d} G'(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\int_{\mathbb{R}^d} g\mathrm{d}\nu_{\phi_0}.\end{aligned}$$

For the converse inequality, we focus again on the specific $g = \phi_0$, in which case the left-hand side of (3.11) takes the following form

$$\liminf_{t\to\infty}\frac{1}{\widehat{p}_t^F\mathbf{1}(x)}\widehat{p}_{pt}^F\left(N\left[Ge^{-F}\cdot\widehat{p}_{(p'-p)t}^F\left(N\left[G'e^{-F}\cdot\phi_0\right]\right)\right]\right)(x).$$

On the other hand, the assertion (1) implies the following convergence, with $g = N[G'e^{-F}\cdot\phi_0] = N^{\phi_0}[G']\phi_0$, $t' = p't$, $q = p/p' \in (0,1)$:

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\widehat{p}_{p't}^F\mathbf{1}(x)}\widehat{p}_{pt}^F\left(N\left[Ge^{-F}\cdot\widehat{p}_{(p'-p)t}^F\left(N\left[G'e^{-F}\cdot\phi_0\right]\right)\right]\right)(x)\\ &= \lim_{t'\to\infty}\frac{1}{\widehat{p}_{t'}^F\mathbf{1}(x)}\widehat{p}_{qt'}^F\left(N\left[Ge^{-F}\cdot\widehat{p}_{(1-q)t'}^F\left(N\left[G'e^{-F}\cdot\phi_0\right]\right)\right]\right)(x)\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\int_{\mathbb{R}^d} N^{\phi_0}[G']\phi_0\mathrm{d}\nu_{\phi_0}\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\iint_{\mathbb{R}^d\times\mathbb{R}^d} G'(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\cdot\int_{\mathbb{R}^d}\phi_0\mathrm{d}\nu_{\phi_0}.\end{aligned}$$

Thanks to (3.5), $\widehat{p}_{p't}^F\mathbf{1}(x)/\widehat{p}_t^F\mathbf{1}(x)$ tends to 1 as $t \to \infty$, which concludes the proof of the assertion (2) in the specific case $g = \phi_0$. From this fact, one can deduce the upper-bound on the supremum limit in the same way as the proof of (1). □

Example 3.3. Let G be a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$|G(x,y)| \leq c|x-y|^\beta,\ (x,y) \in D\times D \ \text{ and } \ G(x,y) = 0,\ \text{otherwise},$$

where D is a compact set of $\mathbb{R}^d$, c and β are two positive constants such that $\beta > \alpha$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |N[G](x)|\phi_0(x)m(\mathrm{d}x)\\
&\le c_1\|\phi_0\|_\infty \int_D\int_D \frac{|G(x,y)|}{|x-y|^{d+\alpha}}m(\mathrm{d}y)m(\mathrm{d}x)\\
&\le c_2\int_D\left\{\int_{D\cap B(x,1)}\frac{1}{|x-y|^{d+\alpha-\beta}}m(\mathrm{d}y)+\int_{D\cap B(x,1)^c}|G(x,y)|m(\mathrm{d}y)\right\}m(\mathrm{d}x)\\
&\le c_2\int_D\left\{\int_{B(x,1)}\frac{1}{|x-y|^{d+\alpha-\beta}}m(\mathrm{d}y)+\|G\|_\infty|D|\right\}m(\mathrm{d}x)<\infty.
\end{aligned}$$

This implies that $N[G]\phi_0$ is integrable on $\mathbb{R}^d$. With our conditions implying that ϕ_0 and F are bounded, we thus deduce that $N^{\phi_0}[G]\in L^1(\mathbb{R}^d;\phi_0^2 m)$.

3.2 Quasi-ergodic limits for purely discontinuous additive functionals

We characterize quasi-ergodic limiting measures for the first and second moments of a discontinuous additive functional under the non-local Feynman-Kac scheme.

Theorem 3.4. *For any symmetric Borel measurable function G on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on diagonal such that $N^{\phi_0}[G]\in L^1(\mathbb{R}^d;\phi_0^2 m)$ and any $x\in\mathbb{R}^d$, we have*

$$\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t}A_t^G\right]=\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z).$$

Proof. First, suppose that G is nonnegative. By the Markov property, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t}A_t^G\right]&=\frac{1}{tp_t^F\mathbf{1}(x)}\mathbb{E}_x\left[e^{-A_t^F}\sum_{0<s\le t}G(X_{s-},X_s)\right]\\
&=\frac{1}{t\widehat{p}_t^F\mathbf{1}(x)}\mathbb{E}_x\left[e^{-A_t^F-C_Ft}\sum_{0<s\le t}G(X_{s-},X_s)\right]\\
&=\frac{1}{t\widehat{p}_t^F\mathbf{1}(x)}\mathbb{E}_x\left[\sum_{0<s\le t}e^{-A_s^F-C_Fs}G(X_{s-},X_s)e^{-C_F(t-s)}\mathbb{E}_{X_s}\left[e^{-A_{t-s}^F}\right]\right]\\
&=\frac{1}{t\widehat{p}_t^F\mathbf{1}(x)}\mathbb{E}_x\left[\sum_{0<s\le t}e^{-A_{s-}^F-C_Fs}G(X_{s-},X_s)e^{-F(X_{s-},X_s)}\widehat{p}_{t-s}^F\mathbf{1}(X_s)\right].
\end{aligned}$$

Then, by applying Lemma 3.2(1) together with [17, p. 346], and using the monotone convergence theorem, it follows that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t}A^G_t\right] &= \lim_{t\to\infty}\frac{1}{t\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\left[\int_0^t e^{-A^F_s - C_F s}N\left[Ge^{-F}\widehat{p}^F_{t-s}\mathbf{1}\right](X_s)\mathrm{d}s\right]\\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_s\left(N\left[Ge^{-F}\widehat{p}^F_{t-s}\mathbf{1}\right]\right)(x)\mathrm{d}s\\ &= \lim_{t\to\infty}\int_0^1\frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\widehat{p}^F_{(1-p)t}\mathbf{1}\right]\right)(x)\mathrm{d}p\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z).\end{aligned}$$

It directly extends to a signed function G by linearity. □

Theorem 3.4 tells us that the quasi-ergodic limits for the number of jumps caused by $\mathbb{X}$ under $\mathbb{Q}_{x,t}$ is characterized by the jumping measure of $\mathbb{X}^{\phi_0}$.

Theorem 3.5. *For any symmetric Borel measurable function G on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on diagonal such that $N^{\phi_0}[G], N^{\phi_0}[G^2] \in L^1(\mathbb{R}^d;\phi_0^2 m)$ and any $x\in\mathbb{R}^d$, we have*

$$\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t\right)^2\right] = \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^2.$$

Proof. Suppose first that G is nonnegative. Note that

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t\right)^2\right]\\ &= \mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t^2}A^{G^2}_t\right] + 2\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t^2}\sum_{0<s<s'\le t}G(X_{s-},X_s)G(X_{s'-},X_{s'})\right].\end{aligned} \tag{3.12}$$

In view of Proposition 3.4, the first term of (3.12) in the right-hand side converges to 0 as $t\to\infty$:

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t^2}A^{G^2}_t\right] &= \lim_{t\to\infty}\frac{1}{t}\cdot\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t}A^{G^2}_t\right]\\ &= \lim_{t\to\infty}\frac{1}{t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}G^2(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\\ &= 0.\end{aligned}$$

What remains is the following sum over the parts of ordered jumps, which we also handle thanks to the Markov property in combination with Lemma 3.2(2):

$$
\begin{aligned}
&2\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t^2}\sum_{0<s<s'\le t} G(X_{s-},X_s)G(X_{s'-},X_{s'})\right] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\left[\sum_{0<s<s'\le t} e^{-A^F_{s'}}e^{-C_F s'}G(X_{s-},X_s)G(X_{s'-},X_{s'})e^{-A^F_{t-s'}\circ\theta_{s'}}e^{-C_F(t-s')}\right] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\left[\sum_{0<s<s'\le t} e^{-A^F_{s'}}e^{-C_F s'}G(X_{s-},X_s)G(X_{s'-},X_{s'})\widehat{p}^F_{t-s'}\mathbf{1}(X_{s'})\right] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\Bigg[\sum_{0<s<t} e^{-A^F_{s}}e^{-C_F s}G(X_{s-},X_s) \\
&\qquad\qquad \cdot\,\mathbb{E}_x\left[\sum_{s<s'\le t} e^{-A^F_{s'-s}\circ\theta_s}e^{-C_F(s'-s)}G(X_{s'-},X_{s'})\widehat{p}^F_{t-s'}\mathbf{1}(X_{s'})\;\Big|\;\mathcal{M}_s\right]\Bigg] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\Bigg[\sum_{0<s<t} e^{-A^F_{s-}}e^{-C_F s}G(X_{s-},X_s)e^{-F(X_{s-},X_s)} \\
&\qquad\qquad \cdot\,\mathbb{E}_x\left[\sum_{s<s'\le t} e^{-A^F_{s'-}+A^F_s}e^{-C_F(s'-s)}G(X_{s'-},X_{s'})e^{-F(X_{s'-},X_{s'})}\widehat{p}^F_{t-s'}\mathbf{1}(X_{s'})\;\Big|\;\mathcal{M}_s\right]\Bigg] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\Bigg[\sum_{0<s<t} e^{-A^F_{s-}}e^{-C_F s}G(X_{s-},X_s)e^{-F(X_{s-},X_s)} \\
&\qquad\qquad \cdot\,\mathbb{E}_x\left[\int_s^t e^{-A^F_{s'}+A^F_s}e^{-C_F(s'-s)}N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right](X_{s'})\mathrm{d}s'\;\Big|\;\mathcal{M}_s\right]\Bigg] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\Bigg[\sum_{0<s<t} e^{-A^F_{s-}}e^{-C_F s}G(X_{s-},X_s)e^{-F(X_{s-},X_s)} \\
&\qquad\qquad \cdot\,\mathbb{E}_{X_s}\left[\int_s^t e^{-A^F_{s'-s}}e^{-C_F(s'-s)}N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right](X_{s'-s})\mathrm{d}s'\right]\Bigg] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\Bigg[\sum_{0<s<t} e^{-A^F_{s-}}e^{-C_F s}G(X_{s-},X_s)e^{-F(X_{s-},X_s)} \\
&\qquad\qquad \cdot\int_s^t\mathbb{E}_{X_s}\left[e^{-A^F_{s'-s}}e^{-C_F(s'-s)}N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right](X_{s'-s})\right]\mathrm{d}s'\Bigg] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\mathbb{E}_x\left[\int_0^t e^{-A^F_s}e^{-C_F s}N\left[Ge^{-F}\cdot\int_s^t\widehat{p}^F_{s'-s}\left(N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right]\right)\mathrm{d}s'\right](X_s)\mathrm{d}s\right] \\
&= \frac{2}{t^2\widehat{p}^F_t\mathbf{1}(x)}\int_0^t\widehat{p}^F_s\left(N\left[Ge^{-F}\cdot\int_s^t\widehat{p}^F_{s'-s}\left(N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right]\right)\mathrm{d}s'\right]\right)(x)\mathrm{d}s \\
&= \frac{2}{t^2}\int_0^t\int_s^t\frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_s\left(N\left[Ge^{-F}\cdot\widehat{p}^F_{s'-s}\left(N\left[Ge^{-F}\widehat{p}^F_{t-s'}\mathbf{1}\right]\right)\right]\right)(x)\mathrm{d}s'\mathrm{d}s \\
&= 2\int_0^1\int_p^1\frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot\widehat{p}^F_{(p'-p)t}\left(N\left[Ge^{-F}\widehat{p}^F_{(1-p')t}\mathbf{1}\right]\right)\right]\right)(x)\mathrm{d}p'\mathrm{d}p \\
&= \int_0^1\int_0^1\frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot\widehat{p}^F_{(p'\vee p-p'\wedge p)t}\left(N\left[Ge^{-F}\widehat{p}^F_{(1-p'\vee p)t}\mathbf{1}\right]\right)\right]\right)(x)\mathrm{d}p'\mathrm{d}p.
\end{aligned}
$$

Hence by applying Lemma 3.2(2) to the cases $G = G'$ and $g = 1$, we have

$$
\begin{aligned}
&2 \lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}_{x,t}\left[\frac{1}{t^2}\sum_{0<s<s'\le t} G(X_{s-},X_s)G(X_{s'-},X_{s'})\right] \\
&= \lim_{t\to\infty}\int_0^1\int_0^1 \frac{1}{\widehat{p}^F_t\mathbf{1}(x)}\widehat{p}^F_{pt}\left(N\left[Ge^{-F}\cdot \widehat{p}^F_{(p'\vee p-p'\wedge p)t}\left(N\left[Ge^{-F}\widehat{p}^F_{(1-p'\vee p)t}\mathbf{1}\right]\right)\right]\right)(x)\mathrm{d}p'\mathrm{d}p \\
&= \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^2 .
\end{aligned}
$$

The desired result is obtained for nonnegative G. It now extends for arbitrary G by linearity. □

As a corollary, we have the following L^2-convergence:

Corollary 3.6. *For any symmetric Borel measurable function G on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on diagonal such that $\iint_{\mathbb{R}^d\times\mathbb{R}^d}|G(y,z)|^k\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)<\infty$ for $k=1,2$, we have*

$$
\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t-\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^2\right]=0
$$

for any $x\in\mathbb{R}^d$.

Proof. The proof is a direct consequence of Theorems 3.4 and 3.5. Indeed,

$$
\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t-\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^2\right] \\
&=\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}_{x,t}\left[\left(\frac{1}{t}A^G_t\right)^2\right]-\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(y,z)\mathcal{J}_{\phi_0}(\mathrm{d}y\mathrm{d}z)\right)^2 \\
&=0
\end{aligned}
$$

for any $x\in\mathbb{R}^d$. □

4 Large deviations for discontinuous additive functionals

The aim of this section is to study the large deviation principle for A^F_t/t. The weak conditional law of large numbers under the special Feynman-Kac functional $e^{-A^F_t}$ helps to establish the large deviation principle for A^F_t/t by using a rate function in a more direct representation via spectral functions.

In the sequel, we assume in this section that F is in the Kato class of $\mathbb{X}$. Thus, θF is also in the Kato class for any $\theta\in\mathbb{R}^1$. Further, we assume that for any $\theta\in\mathbb{R}^1$, the condition (3.1) for θF holds, which is a slightly stronger condition than (3.1) depending on the sign of $\theta\in\mathbb{R}^1$.

For $\theta \in \mathbb{R}^1$, let

$$C_F(\theta) := -\inf\left\{E^{\theta F}(u,u) : u \in F, \int_{\mathbb{R}^d} u^2 \mathrm{d}m = 1\right\}. \tag{4.1}$$

Then, it follows from Lemma 2.1, Corollary 2.2 and Lemma 2.3 that there exists a ground state $\phi_0^{(\theta)} := \phi_0^{\theta F}$ of the quadratic form $(E^{\theta F}, F)$ such that

$$\int_{\mathbb{R}^d} \phi_0^{(\theta)}(x)^2 m(\mathrm{d}x) = 1 \quad \text{and} \quad C_F(\theta) = -E^{\theta F}(\phi_0^{(\theta)}, \phi_0^{(\theta)}). \tag{4.2}$$

Further, by Lemma 2.3, the Feynman-Kac semigroup $\{p_t^{\theta F}\}_{t\geq 0}$ is (IUC), that is, there exist constants $c_t(\theta) > 0$ such that the integral kernel $p_t^{\theta F}(y,z)$ of $\{p_t^{\theta F}\}_{t\geq 0}$ satisfies

$$p_t^{\theta F}(y,z) \leq c_t(\theta)\phi_0^{(\theta)}(y)\phi_0^{(\theta)}(z), \quad \text{for all } t > 0 \text{ and } y,z \in \mathbb{R}^d.$$

The quadratic form $(E^{\theta F}, F)$ corresponding to $\{p_t^{\theta F}\}_{t\geq 0}$ is an analytic function in θ and constitutes a holomorphic family of type (A) in the terminology of [12, p. 395]. Thus, by [12, Chapter VII; Theorems 1.8 and 4.2], the principal L^2-eigenvalue of $(E^{\theta F}, F)$ is differentiable in θ by the analytic perturbation theory and so is $C_F(\theta)$.

For $\vartheta \in \mathbb{R}^1$ and $u \in F$, set $\mathtt{t}(\vartheta)[u] := -E^{\vartheta F}(u,u)$. By the Taylor expansion at $\vartheta = \theta$, we see that $\mathtt{t}(\vartheta)[u]$ can be expressed as

$$\begin{aligned}
\mathtt{t}(\vartheta)[u] =& - E(u,u) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} u(y)u(z)\left(1 - e^{-\vartheta F(y,z)}\right) N(y,\mathrm{d}z)m(\mathrm{d}y) \\
=& - E(u,u) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} u(y)u(z)\left(1 - e^{-\theta F(y,z)}\right) N(y,\mathrm{d}z)m(\mathrm{d}y) \\
&- (\vartheta-\theta)\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(y,z)u(y)u(z)e^{-\theta F(y,z)} N(y,\mathrm{d}z)m(\mathrm{d}y) \\
&+ \sum_{n=2}^{\infty}(\vartheta-\theta)^n \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(-1)^n F^n(y,z)}{n!} u(y)u(z)e^{-\theta F(y,z)} N(y,\mathrm{d}z)m(\mathrm{d}y) \\
:=& \mathtt{t}^{(0)}(\theta)[u] + (\vartheta-\theta)\mathtt{t}^{(1)}(\theta)[u] + \sum_{n=2}^{\infty}(\vartheta-\theta)^n \mathtt{t}^{(n)}(\theta)[u],
\end{aligned}$$

where $\mathtt{t}^{(n)}(\theta)[u]$ denotes the n-th differential coefficient of $\mathtt{t}(\vartheta)[u]$ at $\vartheta = \theta$. By virtue of [12, Chapter VII (4.44)], the first differential coefficient of $C_F(\vartheta)$ at $\vartheta = \theta$ is given as follows:

$$\begin{aligned}
C_F'(\theta) &= \mathtt{t}^{(1)}(\theta)[\phi_0^{(\theta)}] \\
&= -\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(y,z)\phi_0^{(\theta)}(y)\phi_0^{(\theta)}(z)e^{-\theta F(y,z)} N(y,\mathrm{d}z)m(\mathrm{d}y).
\end{aligned} \tag{4.3}$$

Note that $C_F(\theta)$ is a function that is not only convex (cf [5], Lemma 5.5), but actually strictly convex as stated in the next lemma. Thus, $C_F'(\theta)$ is a strictly increasing function in $\mathbb{R}^1$.

Lemma 4.1. *The function* $: \theta \mapsto C_F(\theta)$ *is strictly convex on* $\mathbb{R}^1$.

Proof. Let $\theta_1, \theta_2 \in \mathbb{R}^1$ and $\lambda \in (0,1)$. Denote $\theta_* = \lambda\theta_1 + (1-\lambda)\theta_2$. Since $\phi_0^{(\theta)}$ is the minimizer of (4.1), $E^{\theta F}(\phi_0^{(\theta)}, \phi_0^{(\theta)}) \le E^{\theta F}(\phi_0^{(\theta_*)}, \phi_0^{(\theta_*)})$ for any $\theta \in \mathbb{R}^1$. So we have

$$\lambda C_F(\theta_1) + (1-\lambda)C_F(\theta_2) \ge \lambda \mathtt{t}(\theta_1)\left[\phi_0^{(\theta_*)}\right] + (1-\lambda)\mathtt{t}(\theta_2)\left[\phi_0^{(\theta_*)}\right].$$

The right-hand side above is equal to

$$\begin{aligned}&- E\left(\phi_0^{(\theta_*)}, \phi_0^{(\theta_*)}\right)\\ &+ \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi_0^{(\theta_*)}(y)\phi_0^{(\theta_*)}(z)\left(\lambda e^{-\theta_1 F(y,z)} + (1-\lambda)e^{-\theta_2 F(y,z)} - 1\right) N(y,\mathrm{d}z)m(\mathrm{d}y).\end{aligned}$$

Since $\theta \mapsto e^{-\theta F}$ is strictly convex in θ, it holds that

$$e^{-\theta_* F} - 1 < \lambda e^{-\theta_1 F} + (1-\lambda)e^{-\theta_2 F} - 1.$$

Hence, we have, together with the fact $\phi_0^{(\theta_*)} > 0$, that

$$\begin{aligned}&\lambda C_F(\theta_1) + (1-\lambda)C_F(\theta_2)\\ &> -E\left(\phi_0^{(\theta_*)}, \phi_0^{(\theta_*)}\right)\\ &\qquad + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi_0^{(\theta_*)}(y)\phi_0^{(\theta_*)}(z)\left(e^{-\theta_* F(y,z)} - 1\right) N(y,\mathrm{d}z)m(\mathrm{d}y)\\ &= -E^{\theta_* F}\left(\phi_0^{(\theta_*)}, \phi_0^{(\theta_*)}\right) = C_F(\lambda_1\theta_1 + (1-\lambda)\theta_2).\end{aligned}$$

It concludes the proof of the present lemma. □

Let $\Psi_F(\theta)$ be the function given by

$$\Psi_F(\theta) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} F(y,z)\mathcal{J}_{\phi_0^{(\theta)}}(\mathrm{d}y\mathrm{d}z).$$

Then we have $\Psi_F(\theta) = -C_F'(\theta)$ by (4.3). Hence $\Psi_F(\theta)$ is a strictly decreasing and continuous function on $\mathbb{R}^1$. Denote by Ψ_F^{-1} the inverse function of Ψ_F. Put $\mathbb{R}^1_- := (-\infty, 0)$. Set $\Psi_F(\mathbb{R}^1_-) := \{\Psi_F(\theta) : \theta \in \mathbb{R}^1_-\}$ and denote by $\Psi_F(\mathbb{R}^1_-)^o$ the interior of $\Psi_F(\mathbb{R}^1_-)$. We note that $\Psi_F(\mathbb{R}^1_-)^o \neq \emptyset$ because of the strict convexity of $C_F(\theta)$. Now, we give the following large deviation principle for A_t^F/t.

Theorem 4.2. *For any* $\gamma \in \Psi_F(\mathbb{R}^1_-)^o$ *and any* $x \in \mathbb{R}^d$, *we have*

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}_x\left(\frac{A_t^F}{t} \in [\gamma, \infty)\right) = C_F(\theta_\gamma) - \theta_\gamma C_F'(\theta_\gamma), \tag{4.4}$$

where θ_γ *is the nonpositive real number given by* $\theta_\gamma = \Psi_F^{-1}(\gamma)$.

Proof. First, we prove the upper bound. By the convergence of the Feynman-Kac heat content (3.5), the logarithmic moment generating function of $-\theta A_t^F$ has the limit $C_F(\theta)$:

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\left[e^{-\theta A_t^F}\right]=C_F(\theta),\quad \theta\in\mathbb{R}^1.$$

Since the function $\Psi_F(\theta)$ is strictly decreasing and continuous on $\mathbb{R}^1_-$, there exists $\theta_\gamma\in\mathbb{R}^1_-$ such that $\gamma=\Psi_F(\theta_\gamma)$ for any $\gamma\in\Psi_F(\mathbb{R}^1_-)^o$. Then, by the celebrated Gärtner-Ellis theorem (cf. [6]), it holds that for any $x\in\mathbb{R}^d$,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_x\left(\frac{A_t^F}{t}\in[\gamma,\infty)\right)&=\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_x\left(\frac{-A_t^F}{t}\in(-\infty,-\gamma]\right)\\&\le-\inf_{\lambda\in(-\infty,-\gamma]}\sup_{\theta\in\mathbb{R}^1}\{\lambda\theta-C_F(\theta)\}\\&\le\sup_{\lambda\in(-\infty,-\Psi_F(\theta_\gamma)]}\{C_F(\theta_\gamma)-\lambda\theta_\gamma\}\\&\le C_F(\theta_\gamma)+\theta_\gamma\Psi_F(\theta_\gamma)\\&=C_F(\theta_\gamma)-\theta_\gamma C_F'(\theta_\gamma).\end{aligned}$$

Next, we turn to the proof of the lower bound. Let $\{\theta_n\}\subset\mathbb{R}^1_-$ be a sequence such that $\theta_n\uparrow\theta_\gamma$ as $n\to\infty$. Put $\varepsilon_n:=\Psi_F(\theta_n)-\Psi_F(\theta_\gamma)=\Psi_F(\theta_n)-\gamma>0$. Then, we have for large enough $t>0$,

$$\begin{aligned}&\mathbb{P}_x\left(\frac{A_t^F}{t}\in[\gamma,\infty)\right)\\&\ge\mathbb{P}_x\left(\frac{A_t^F}{t}\in[\gamma,\gamma+2\varepsilon_n]\right)\\&=e^{C_F(\theta_n)t}e^{-C_F(\theta_n)t}\mathbb{E}_x\left[e^{-\theta_nA_t^F}e^{\theta_nA_t^F};\ \frac{A_t^F}{t}\in[\gamma,\gamma+2\varepsilon_n]\right]\\&\ge e^{(C_F(\theta_n)+(\gamma+2\varepsilon_n)\theta_n)t}e^{-C_F(\theta_n)t}\\&\quad\times\mathbb{E}_x\left[e^{-\theta_nA_t^F};\ \frac{A_t^F}{t}\in[\Psi_F(\theta_n)-\varepsilon_n,\Psi_F(\theta_n)+\varepsilon_n]\right].\end{aligned}$$

By applying Corollary 3.6 with (θ_nF,F) instead of (F,G), and using the fact due to (3.5) that

$$e^{-C_F(\theta_n)t}\mathbb{E}_x\left[e^{-\theta_nA_t^F}\right]=\widehat{p}_t^{\theta_nF}\mathbf{1}(x)\quad\overset{t\to\infty}{\longrightarrow}\quad\phi_0^{(\theta_n)}(x)\int_{\mathbb{R}^d}\phi_0^{(\theta_n)}\mathrm{d}m,$$

one can see

$$\begin{aligned}&e^{-C_F(\theta_n)t}\mathbb{E}_x\left[e^{-\theta_nA_t^F};\frac{A_t^F}{t}\in[\Psi_F(\theta_n)-\varepsilon_n,\Psi_F(\theta_n)+\varepsilon_n]\right]\\&\overset{t\to\infty}{\longrightarrow}\quad\phi_0^{(\theta_n)}(x)\int_{\mathbb{R}^d}\phi_0^{(\theta_n)}\mathrm{d}m.\end{aligned}$$

Hence we have for $\{\theta_n\} \subset \mathbb{R}^1_-$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A^F_t}{t} \in [\gamma, \infty) \right) \geq C_F(\theta_n) + (\gamma + 2\varepsilon_n)\theta_n.$$

Letting $n \to \infty$, the right-hand side of the above converges to $C_F(\theta_\gamma)+\gamma\theta_\gamma$ because $\varepsilon_n = \Psi_F(\theta_n) - \gamma = \Psi_F(\theta_n) - \Psi_F(\theta_\gamma) \to 0$ and $C_F(\theta_n) \to C_F(\theta_\gamma)$ as $n \to \infty$ by the continuities of $\Psi_F(\theta)$ and $C_F(\theta)$, respectively. This leads us that the lower bound is given by $C_F(\theta_\gamma) + \theta_\gamma\Psi_F(\theta_\gamma) = C_F(\theta_\gamma) - \theta_\gamma C'_F(\theta_\gamma)$. □

We can represent the rate $C_F(\theta_\gamma) - \theta_\gamma C'_F(\theta_\gamma)$ given by (4.4) in a more direct way via quadratic form:

Corollary 4.3. *For any $\gamma \in \Psi_F(\mathbb{R}^1_-)^o$ and any $x \in \mathbb{R}^d$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A^F_t}{t} \in [\gamma, \infty) \right) = -A\left(\phi_0^{(\theta_\gamma)}, \phi_0^{(\theta_\gamma)}\right),$$

where θ_γ is the nonpositive real number given by $\theta_\gamma = \Psi_F^{-1}(\gamma)$ and

$$\begin{aligned} &A\left(\phi_0^{(\theta_\gamma)}, \phi_0^{(\theta_\gamma)}\right) \\ &= E\left(\phi_0^{(\theta_\gamma)}, \phi_0^{(\theta_\gamma)}\right) \\ &\quad + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi_0^{(\theta_\gamma)}(y)\phi_0^{(\theta_\gamma)}(z)\left(1 - e^{-\theta_\gamma F(y,z)} - \theta_\gamma F(y,z)e^{-\theta_\gamma F(y,z)}\right) N(y, \mathrm{d}z)m(\mathrm{d}y). \end{aligned}$$

Acknowledgements

The author would like to thank the reviewer for his/her careful reading and comments. The author also wishes to express gratitude to Professor Daehong Kim for his valuable suggestions and encouragement.

References

[1] L. A. Breyer and G. O. Roberts, A quasi-ergodic theorem for evanescent processes, Stochastic Process. Appl. **84** (1999), no. 2, 177–186.

[2] J. Chen and S. Jian, A deviation inequality and quasi-ergodicity for absorbing Markov processes, Ann. Mat. Pura. Appl. (4) **197** (2018), no. 2, 641–650.

[3] Z.-Q. Chen, L^p-independence of spectral bounds of generalized non-local Feynman–Kac semigroups, J. Funct. Anal. **226** (2012), no. 6, 4120–4139.

[4] Z.-Q. Chen and R. Song, Drift transforms and Green function estimates for discontinuous processes, J. Funct. Anal. **201** (2003), no. 1, 262–281.

[5] Z.-Q. Chen and T. Tsuchida, Large deviation for additive functionals of symmetric Markov processes, Trans. Amer. Math. Soc. **373** (2020), no. 4, 2981–3005.

[6] A. Dembo and O. Zeitouni, Large Deviation Techniques and Applications, in: Applications of Mathematics, **38**, second ed. Springer-Verlag, New York, 1998.

[7] M. Fukushima, A note on irreducibility and ergodicity of symmetric Markov processes, Lecture Notes in Physics, **173** (1982), 200–207.

[8] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes, Second revised and extended edition. de Gruyter Studies in Mathematics, **19**. Walter de Gruyter & Co., Berlin, 2011.

[9] G. He, G. Yang and Y. Zhu, Some conditional limiting theorems for symmetric Markov processes with tightness property, Electron. Commun. Probab. **24** (2019), no. 60, 1–11.

[10] K. Kaleta and J. Lörinczi, Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman–Kac semigroups for a class of Lévy processes, Ann. Probab. **43** (2015), no. 3, 1350–1398.

[11] K. Kaleta and T. Kulczycki, Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians, Potential Anal. **33** (2010), no. 3, 313–339.

[12] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[13] D. Kim and K. Kuwae, Analytic characterizations of gaugeability for generalized Feynman–Kac functionals, Trans. Amer. Math. Soc. **369** (2017), no. 7, 4545–4596.

[14] D. Kim, K. Kuwae and Y. Tawara, Large deviation principle for generalized Feynman–Kac functionals and its applications, Tohoku Math. J. **68** (2016), no. 2, 161–197.

[15] D. Kim, T. Tagawa and A. Velleret, Quasi-ergodic theorems for Feynman-Kac semigroups and large deviation for additive functionals, to appear in Ann. Inst. Henri Poincaré (B) Probabilité et Statistiques.

[16] D. Kim and T. Tagawa, Quasi-ergodic limits for moments of jumps under absorbing stable processes, J. Theor. Probab. **38** (2025), no. 1, Paper No. 10.

[17] M. Sharpe, General Theory of Markov Processes, Academic Press, Boston, 1988.

[18] M. Takeda, Compactness of symmetric Markov semigroups and boundedness of eigenfunctions, Trans. Amer. Math. Soc. **372** (2019), no. 6, 3905–3920.

[19] M. Takeda, Existence and uniqueness of quasi-stationary distributions for symmetric Markov processes with tightness property, J. Theor. Probab. **32** (2019), no. 4, 2006–2019.

[20] M. Takeda and Y. Tawara, A large deviation principle for symmetric Markov processes normalized by Feynman–Kac functionals, Osaka J. Math. **50** (2013), no. 2, 287–307.

[21] M. Takeda and K. Tsuchida, Large deviations for discontinuous additive functionals of symmetric stable processes, Math. Nachr. **284** (2011), no. 8-9, 1148–1171.

[22] J. Zhang, S. Li and R. Song, Quasi-stationarity and quasi-ergodicity of general Markov processes, Sci China Math. **57** (2014), no. 10, 2013–2024.

Takara Tagawa
Department of Mathematics and Engineering
Graduate School of Science and Technology
Kumamoto University
Kumamoto, 860-8555, Japan
e-mail: 236d9321@st.kumamoto-u.ac.jp