Kumamoto J. Math.
Vol.**21** (2008), 11–20

Logarithmic vector fields along smooth plane cubic curves

Kazushi Ueda and Masahiko Yoshinaga

(Received April 26, 2007)
(Accepted January 30, 2008)

Abstract. We study the sheaves of logarithmic vector fields along smooth cubic curves in the projective plane, and prove a Torelli-type theorem in the sense of Dolgachev–Kapranov [4] for those with non-vanishing j-invariants.

1 Introduction

K. Saito [6] introduced the notion of the sheaf of logarithmic vector fields along a divisor and proved that it is always reflexive. A divisor D in a variety S is said to be *free* if the sheaf of logarithmic vector field along D is a free $\mathcal{O}_S$-module. He proved that the discriminant in the parameter space of the semi-universal deformation of an isolated hypersurface singularity is always free.

When the ambient space is the projective space $\mathbb{P}^\ell$, an $\mathcal{O}_{\mathbb{P}^\ell}$-module is said to be free if it is the direct sum $\bigoplus_i \mathcal{O}_{\mathbb{P}^\ell}(a_i)$ of invertible sheaves. The problem of characterizing free divisors in projective spaces has attracted much attention, especially when the divisor is given as an arrangement of hyperplanes. See e.g. [7]. If a divisor in $\mathbb{P}^\ell$ is free, then the passage from the divisor to the sheaf of logarithmic vector fields causes loss of information; only the sequence $\{a_i\}_{i=1}^{\ell}$ of integers is left, and it is impossible to reconstruct the divisor from this finite amount of information.

In the opposite extreme, Dolgachev and Kapranov [4] asked when the the sheaf $\mathcal{T}(-\log D)$ contains enough information to reconstruct D. A divisor D in $\mathbb{P}^\ell$ is said to be *Torelli* if the isomorphism class of $\mathcal{T}(-\log D)$ as an $\mathcal{O}_{\mathbb{P}^\ell}$-module determines the divisor D. Their main result is the condition for an arrangement of sufficiently many hyperplanes in $\mathbb{P}^\ell$ to be Torelli.

In this paper, we discuss the case when $\ell = 2$ and D is a smooth cubic curve. Our main result asserts that D is Torelli precisely when the j-invariant of D is not zero. The strategy of our proof is the following:

1. The set of jumping lines of the sheaf of logarithmic vector fields along a smooth cubic curve coincides with its Cayleyan curve.

Mathematical Subject Classification (2010): Primary 14J60; Secondary 14F05
Key words: logarithmic vector field, jumping line, Cayleyan curve

2. For a smooth cubic curve with a non-vanishing j-invariant, the Cayleyan curve determines the original curve up to three possibilities.

3. The set of "jumping cubic curves" fixes this left-over ambiguity and the Torelli property holds.

4. When the j-invariant of D is zero, we can construct a family of divisors with isomorphic sheaves of logarithmic vector fields along them.

Smooth cubic curves with vanishing j-invariants provide examples of divisors which are neither free nor Torelli.

Acknowledgment: We thank Igor Dolgachev for a stimulating lecture in Kyoto in winter 2006 and Akira Ishii for valuable discussions and comments. K. U. is supported by Grant-in-Aid for Young Scientists (No.18840029). M. Y. is supported by JSPS Postdoctoral Fellowship for Research Abroad.

2 Preliminaries

2.1 de Rham–Saito's lemma

Let A be a Noetherian ring and $M = \bigoplus_{i=1}^{n} Ae_i$ be a free module over A generated by $e_1, \ldots, e_n$. For $\omega_1, \ldots, \omega_r \in M$, put

$$\omega_1 \wedge \cdots \wedge \omega_r = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1,\ldots,i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}.$$

and define $\mathfrak{a}$ to be the ideal generated by $a_{i_1,\cdots,i_r}$ for $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$. We also define as follows:

$$\begin{aligned} Z^p &= \{\varphi \in \wedge^p M \mid \omega_1 \wedge \cdots \wedge \omega_r \wedge \varphi = 0\}, \\ B^p &= \sum_{k=1}^{r} \omega_k \wedge (\wedge^{p-1} M), \\ H^p &= Z^p / B^p. \end{aligned}$$

Theorem 1 (de Rham–Saito's lemma [3, 5])**.** *(1) There exists an integer $\nu \in \mathbb{Z}_{\ge 0}$ such that $\mathfrak{a}^\nu H^p = 0$ for $0 \le p \le n$.*

(2) For $0 \le p < \operatorname{depth}_{\mathfrak{a}} A$, we have $H^p = 0$.

2.2 Sheaf of logarithmic vector fields

Let $A = \mathbb{C}[z_0, \ldots, z_\ell]$ be a polynomial ring and Der_A be the module of $\mathbb{C}$-derivations of A, which is a free module of rank $\ell + 1$;

$$\mathrm{Der}_A = \sum_{i=0}^{\ell} A \frac{\partial}{\partial z_i}.$$

Definition 2. For a homogeneous polynomial $f \in A$, we define

$$\begin{aligned} D(-\log f) &= \{\delta \in \mathrm{Der}_A \mid \delta f \in (f)\}, \\ D_0(-\log f) &= \{\delta \in \mathrm{Der}_A \mid \delta f = 0\}. \end{aligned}$$

We put $\deg z_i = 1$ and $\deg(\partial/\partial z_i) = -1$ for $i = 0, \ldots, l$. The degree k part of $D_0(-\log f)$ will be denoted by $D_0(-\log f)_k$.

We have the direct sum decomposition

$$D(-\log f) = D_0(-\log f) \oplus A \cdot E,$$

where

$$E = \sum_{i=0}^{\ell} z_i \partial/\partial z_i$$

is the Euler vector field. Let Ω_A be the module of differentials

$$\Omega_A^1 = \bigoplus_{i=0}^{\ell} A dz_i,$$

and Ω_A^k be its k-th exterior power for $k = 0, \ldots, \ell+1$. We have an isomorphism of A-modules

$$D_0(-\log f) \cong \{\omega \in \Omega^\ell \mid df \wedge \omega = 0\}$$

under the identification

$$\begin{array}{ccc} \mathrm{Der}_A & \xrightarrow{\sim} & \Omega^\ell \\ \Uparrow & & \Uparrow \\ \sum_{i=0}^{\ell} f_i \frac{\partial}{\partial z_i} & \longmapsto & \sum_{i=0}^{\ell} (-1)^i f_i dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_\ell. \end{array}$$

Let $D \subset \mathbb{P}^\ell$ be the hypersurface defined by f. If D is smooth, then the origin $0 \in \mathbb{C}^{l+1}$ is the only zero locus of the Jacobi ideal

$$J(f) = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_\ell} \right),$$

and hence we have

$$\mathrm{depth}_{J(f)} A = \ell + 1.$$

Let H^p be the p-th cohomology of the complex

$$0 \longrightarrow \Omega_A^0 \xrightarrow{df\wedge} \Omega_A^1 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega_A^\ell \xrightarrow{df\wedge} \Omega_A^{\ell+1} \longrightarrow 0.$$

If D is smooth, then we have $H^p = 0$ for $p = 0, \ldots, \ell$ by de Rham–Saito's lemma. Since

$$D_0(-\log f) \cong \mathrm{Ker}\left(df\wedge : \Omega^\ell \to \Omega^{\ell+1}\right),$$

the sequence

$$0 \longrightarrow \Omega_A^0 \xrightarrow{df\wedge} \Omega_A^1 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega_A^{\ell-1} \xrightarrow{df\wedge} D_0(-\log f) \longrightarrow 0 \tag{1}$$

gives a free resolution of $D_0(-\log f)$.

The Euler sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(1)^{\ell+1} & \longrightarrow & \mathcal{T}_{\mathbb{P}^\ell} & \longrightarrow & 0, \\ & & \Psi & & \Psi & & & & \\ & & 1 & \mapsto & E & & & & \end{array}$$

shows that the sheafification $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$ of $D_0(-\log f)$ can be considered as a subsheaf of the tangent sheaf $\mathcal{T}_{\mathbb{P}^\ell}$;

$$\mathcal{T}_{\mathbb{P}^\ell}(-\log f) \subset \mathcal{T}_{\mathbb{P}^\ell}.$$

It is the sheaf of holomorphic vector fields tangent to the hypersurface D at smooth points of D. If D is smooth, we have the short exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^\ell}(-\log f) \longrightarrow \mathcal{T}_{\mathbb{P}^\ell} \longrightarrow \mathcal{N}_{D/\mathbb{P}^\ell} \longrightarrow 0,$$

where $\mathcal{N}_{D/\mathbb{P}^\ell}$ is the normal bundle. We have an isomorphism

$$df|_D : \mathcal{N}_{D/\mathbb{P}^\ell} \xrightarrow{\sim} \mathcal{O}_D(d),$$

where

$$d = \deg f.$$

If D is smooth, then the sheaf $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$ has the resolution

$$0 \to \mathcal{O}(1-(d-1)\ell) \to \cdots \to \mathcal{O}(3-2d)^{\oplus\binom{\ell+1}{\ell-2}} \to \mathcal{O}(2-d)^{\oplus\binom{\ell+1}{\ell-1}} \to \mathcal{T}_{\mathbb{P}^\ell}(-\log f) \to 0 \tag{2}$$

obtained by sheafifying the exact sequence (1). We also have

$$\Gamma\left(\mathbb{P}^\ell, \mathcal{T}_{\mathbb{P}^\ell}(-\log f)(k)\right) = D_0(-\log f)_k$$

for $k \in \mathbb{Z}$.

3 Plane curves

Now we set $\ell = 2$ to focus our attention on plane curves. Let $f \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial of degree d and $D \subset \mathbb{P}^2$ be the curve defined by f. Define $\mathcal{F}$ as the cokernel of $df\wedge : \mathcal{O}(3-2d) \to \mathcal{O}(2-d)^{\oplus 3}$ so that we have the exact sequence

$$0 \longrightarrow \mathcal{O}(3-2d) \xrightarrow{df\wedge} \mathcal{O}(2-d)^{\oplus 3} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{3}$$

The Chern polynomial of $\mathcal{F}(k)$ is given by

$$\begin{aligned} c_t(\mathcal{F}(k)) &:= 1 + c_1(\mathcal{F}(k))t + c_2(\mathcal{F}(k))t^2 \\ &= c_t(\mathcal{O}(2-d+k))^3 c_t(\mathcal{O}(3-2d+k))^{-1} \\ &= 1 + (3-d+2k)t + (d^2-3d+3+k^2+(3-d)k)t^2 \end{aligned}$$

for $k \in \mathbb{Z}$. If D is smooth, then we have

$$\begin{aligned}
\mathcal{F} &:= \operatorname{Coker}(df\wedge : \mathcal{O}(3-2d) \to \mathcal{O}(2-d)^{\oplus 3}) \\
&\cong \operatorname{Coim}(df\wedge : \mathcal{O}(2-d)^{\oplus 3} \to \mathcal{O}(1)^{\oplus 3}) \\
&\cong \operatorname{Im}(df\wedge : \mathcal{O}(2-d)^{\oplus 3} \to \mathcal{O}(1)^{\oplus 3}) \\
&\cong \operatorname{Ker}(df\wedge : \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}(d)) \\
&\cong \mathcal{T}_{\mathbb{P}^\ell}(-\log f).
\end{aligned}$$

Lemma 3. *If D is smooth, then $\mathcal{T}_{\mathbb{P}^2}(-\log f)$ is stable.*

Proof. We consider $\mathcal{F}([(d-3)/2])$ instead of $\mathcal{T}_{\mathbb{P}^2}(-\log f)$ whose first Chern number is normalized to either 0 (when d is odd) or -1 (when d is even). Then $\mathcal{F}([(d-3)/2])$ is stable if and only if it has no global section. This follows from the cohomology long exact sequence associated with the short exact sequence (3) tensored with $\mathcal{O}_{\mathbb{P}^2}([(d-3)/2])$.

4 Smooth cubic curves

Let $f \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial of degree three and $D \subset \mathbb{P}(V)$ be a cubic curve defined by f, where $V = \operatorname{Spec} \mathbb{C}[z_0, z_1, z_2]$. We assume that D is smooth.

4.1 Jumping lines

Let L be a point in the dual projective plane $\mathbb{P}(V^*)$ defined by a linear form $\alpha = \alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 \in V^*$. We can think of L as a line in $\mathbb{P}(V)$. Restricting the short exact sequence (3) to L and taking the cohomology long exact sequence, we have

$$0 \longrightarrow H^0(\mathcal{F}|_L) \longrightarrow H^1(\mathcal{O}_L(-3)) \longrightarrow H^1(\mathcal{O}_L(-1))^3 \longrightarrow H^1(\mathcal{F}|_L) \longrightarrow 0.$$

Since

$$H^1(\mathcal{O}_L(-3)) \cong H^0(\mathcal{O}_L(1))^* \cong \mathbb{C}^2$$

and

$$H^1(\mathcal{O}_L(-1)) \cong H^0(\mathcal{O}_L(-1))^* = 0,$$

we have

$$\dim H^0(\mathcal{F}|_L) = 2.$$

Hence $\mathcal{F}|_L$ is either

$$\mathcal{F}|_L = \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L & L \text{ is generic,} \\ \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) & L \text{ is a jumping line.} \end{cases}$$

In particular,

$$L \text{ is a jumping line} \iff H^0(\mathcal{F}(-1)|_L) \neq 0.$$

By tensoring $\mathcal{O}_L(-1)$ with the short exact sequence (3) and taking the cohomology long exact sequence, we have

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{F}(-1)|_L) & \to & H^1(\mathcal{O}_L(-4)) & \xrightarrow{df\wedge} & H^1(\mathcal{O}_L(-2)^{\oplus 3}) & \to & H^0(\mathcal{F}(-1)|_L) & \to & 0 \\ & & & & \wr\| & & \wr\| & & & & \\ & & & & H^0(\mathcal{O}_L(2))^* & & H^0(\mathcal{O}_L^{\oplus 3})^*. & & & & \end{array}$$

Since $H^0(\mathcal{O}(2)|_L) \cong \operatorname{Sym}^2 V^*/(z_0\alpha, z_1\alpha, z_2\alpha)$, the set $S = S(\mathcal{T}_{\mathbb{P}^\ell}(-\log f)) \subset \mathbb{P}(V^*)$ of jumping lines is characterized as follows;

$$\begin{aligned} L \in S &\iff (df\wedge)^* : H^0(\mathcal{O}_L^{\oplus 3} \to H^0(\mathcal{O}_L(2))) \text{ is not an isomorphism} \\ &\iff z_0\alpha, z_1\alpha, z_2\alpha, \partial_0 f, \partial_1 f, \partial_2 f \text{ are linearly dependent in } \operatorname{Sym}^2 V^*. \quad (4) \end{aligned}$$

4.2 Cayleyan curves

Here we prove the following:

Proposition 4. *Let $D \subset \mathbb{P}(V)$ be a smooth cubic curve defined by a polynomial f. Then the set $S = S(\mathcal{T}_{\mathbb{P}^\ell}(-\log f)) \subset \mathbb{P}(V^*)$ of jumping lines of $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$ in the dual projective plane $\mathbb{P}(V^*)$ is the Cayleyan curve of D.*

First we recall the definition of the Cayleyan curve of a plane cubic curve following Artebani and Dolgachev [1]. The *first polar* of a plane curve $D = \{f = 0\}$ with respect to a point $q = [a_0 : a_1 : a_2] \in \mathbb{P}(V)$ is the curve $P_q(D)$ defined by the polynomial

$$g = a_0\partial_0 f + a_1\partial_1 f + a_2\partial_2 f,$$

whose degree is one less than that of f. A point $[x_0 : x_1 : x_2] \in P_q(D)$ is a singularity of the polar curve if

$$\partial_i g(x_0, x_1, x_2) = \sum_{j=0}^{2} a_j \partial_{ij} f(x_0, x_1, x_2) = 0$$

for $i = 0, 1, 2$. Here, ∂_i denotes the partial derivative with respect to x_i and $\partial_{ij} = \partial_i\partial_j$. When f is cubic, one has

$$\sum_{j=0}^{2} a_j \partial_{ij} f(x_0, x_1, x_2) = \sum_{j=0}^{2} x_j \partial_{ij} f(a_0, a_1, a_2)$$

and hence the polar curve $P_q(D)$ has a singularity if and only if q lies on the *Hessian curve* $\operatorname{He}(D) \subset \mathbb{P}(V)$ defined by

$$h = \det \begin{pmatrix} \partial_{00} f & \partial_{01} f & \partial_{02} f \\ \partial_{10} f & \partial_{11} f & \partial_{12} f \\ \partial_{20} f & \partial_{21} f & \partial_{22} f \end{pmatrix}.$$

For $q \in \mathrm{He}(D)$, the polar curve $P_q(D)$ decomposes into the union of two lines. Let $s_q \in \mathbb{P}(V)$ denote the singular point of $P_q(D)$ and $L_q \in \mathbb{P}(V^*)$ be the line connecting q and s_q. Since the equation

$$\sum_{j=0}^{2} a_j \partial_{ij} f(x_0, x_1, x_2) = 0$$

is symmetric with respect to q and s_q, the singularity s_q of $P_q(D)$ always lies on $\mathrm{He}(D)$ and the map

$$\begin{array}{cccccc} s & : & \mathrm{He}(D) & \longrightarrow & \mathrm{He}(D) \\ & & \Psi & & \Psi \\ & & q & \mapsto & s_q \end{array}$$

defines an involution on $\mathrm{He}(D)$. Since $q = s_q$ implies

$$\partial_i f(a_0, a_1, a_2) = 0, \qquad i = 0, 1, 2,$$

so that q is a singular point of D, the involution s has no fixed point. The image of the map

$$\begin{array}{ccc} \mathrm{He}(D) & \longrightarrow & \mathbb{P}(V^*) \\ \Psi & & \Psi \\ q & \mapsto & L_q \end{array}$$

is called the *Cayleyan curve* of D, which is known to be the quotient of $\mathrm{He}(D)$ by the involution s. A linear form $\alpha = \alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 \in V^*$ represents a point in the Cayleyan curve of D if and only if there is a point $[a_0 : a_1 : a_2] \in \mathbb{P}^2$ such that

$$a_0 \partial_0 f + a_1 \partial_1 f + a_1 \partial_1 f \in \alpha \cdot V^*.$$

This is precisely the condition (4) for the line $[\alpha] \in \mathbb{P}(V^*)$ to be a jumping line of $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$.

4.3 The set of jumping lines and j-invariant

Here we prove the following:

Proposition 5. *Let D be the smooth cubic curve defined by a polynomial f. Then the set $S(\mathcal{T}_{\mathbb{P}^\ell}(-\log f))$ of jumping lines is singular if and only if the j-invariant of D is zero.*

Proof. Choose a coordinate of V so that f is a Hesse cubic

$$f_t(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3 - 3t z_0 z_1 z_2, \tag{5}$$

where $t \in \mathbb{C} \setminus \{1, \zeta, \zeta^2\}$ and $\zeta = \exp[2\pi\sqrt{-1}/3]$. Recall that $D = \{f_t = 0\} \subset \mathbb{P}^2$ is smooth if and only if $t^3 \neq 1$. The set $S = S(\mathcal{T}_{\mathbb{P}^\ell}(-\log f))$ of jumping lines, which coincides with the Cayleyan curve of D, is a Hesse cubic

$$t(\alpha_0^3 + \alpha_1^3 + \alpha_2^3) - (t^3 + 2)\alpha_0 \alpha_1 \alpha_2 = 0$$

in the dual projective plane. It is the union of three lines in general position if $t = 0$ or $(3t)^3 = (t^3 + 2)^3$. Since

$$(t^3 + 2)^3 - (3t)^3 = (t^3 - 1)^2(t^3 + 8)$$

and the j-invariant $j(D)$ of D is given by

$$j(D) = \frac{1}{64} t^3 \frac{(t^3 + 8)^3}{(t^3 - 1)^3},$$

the Cayleyan curve of D is smooth if and only if $j(D) \neq 0$, and decomposes into the union of three lines in general position if $j(D) = 0$.

4.4 Restricting $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$ to other cubic curves

Here we consider the restriction of the sheaf $\mathcal{T}_{\mathbb{P}^\ell}(-\log f)$ to another cubic curve E defined by a polynomial g. ¿From the exact sequence (3), we have

$$0 \longrightarrow \mathcal{O}(-3)|_E \longrightarrow \mathcal{O}(-1)^{\oplus 3}|_E \longrightarrow \mathcal{F}|_E \longrightarrow 0.$$

Hence we have

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{F}|_E) & \to & H^1(\mathcal{O}(-3)|_E) & \xrightarrow{df\wedge} & H^1(\mathcal{O}(-1)^{\oplus 3}|_E) & \to & H^0(\mathcal{F}|_E) & \to & 0 \\ & & & & \wr\| & & \wr\| & & & & \\ & & & & H^0(\mathcal{O}(3)|_E)^* & & H^0(\mathcal{O}(1)^{\oplus 3}|_E)^*. & & & & \end{array}$$

Since $H^0(\mathcal{O}(3)|_E) = \operatorname{Sym}^3 V^*/(g)$ and $H^0(\mathcal{O}(1)|_E)^3 = (V^*)^3$, the map $df\wedge$ is dual to the map induced by

$$\begin{array}{ccc} (V^*)^3 & \longrightarrow & \operatorname{Sym}^3 V^* \\ \cup & & \cup \\ (F_0, F_1, F_2) & \longrightarrow & F_0\partial_0 f + F_1\partial_1 f + F_2\partial_2 f. \end{array}$$

This map is injective due to de Rham–Saito's lemma, and the image can be identified with the degree 3 part $J(f)_3$ of the Jacobi ideal. Hence we have

$$H^0(\mathcal{F}|_E) = \begin{cases} \mathbb{C} & g \in J(f)_3, \\ 0 & g \notin J(f)_3. \end{cases}$$

By an explicit calculation, we obtain the following:

Proposition 6. *Let f_t be the Hesse cubic in (5) and put*

$$g = \sum_{0 \le i \le j \le k \le 2} a_{ijk} z_i z_j z_k.$$

Then the hyperplane $J(f_t)_3 \subset \operatorname{Sym}^3 V^*$ is given by

$$J(f_t)_3 = \{g \mid a_{012} + t(a_{000} + a_{111} + a_{222}) = 0\}.$$

5 Torelli theorem

Here we prove our main result:

Theorem 7. *Let C and C' be smooth cubic curves with non-vanishing j-invariants. If $\mathcal{T}(-\log C)$ is isomorphic to $\mathcal{T}(-\log C')$ as an $\mathcal{O}_{\mathbb{P}^2}$-module, then $C = C'$.*

Proof. Take a homogeneous coordinate of the dual projective plane so that the set of jumping lines of $\mathcal{T}(-\log C)$ is a Hesse cubic. Since a smooth cubic whose Cayleyan curve is a smooth Hesse cubic must be a Hesse cubic, C and C' are Hesse cubics. Then Proposition 6 shows that C must coincide with C'.

Remark 8. The Torelli theorem fails for cubic curves with vanishing j-invariants. Indeed, the family

$$az_0^3 + bz_1^3 + cz_0^3 = 0, \qquad a, b, c \in \mathbb{C}^\times$$

consists of cubic curves with identical Cayleyan curves given by

$$\alpha_0\alpha_1\alpha_2 = 0.$$

Since the set of jumping lines determines a unique stable bundle if it consists of three lines in general position by Barth [2], the sheaf of logarithmic vector fields does not depend on a, b, and c.

References

[1] Michela Artebani and Igor Dolgachev. The Hesse pencil of plane cubic curves. arXiv:math.AG/0611590, 2006.

[2] Wolf Barth. Moduli of vector bundles on the projective plane. *Invent. Math.*, *vol.***42**:63–91, 1977.

[3] Georges de Rham. Sur la division de formes et de courants par une forme linéaire. *Comment. Math. Helv.*, *vol.***28**:346–352, 1954.

[4] Igor Dolgachev and Mikhail Kapranov. Arrangements of hyperplanes and vector bundles on $\mathbf{P}^n$. *Duke Math. J.*, *vol.***71**(3):633–664, 1993.

[5] Kyoji Saito. On a generalization of de-Rham lemma. *Ann. Inst. Fourier (Grenoble)*, *vol.***26**(2):vii, 165–170, 1976.

[6] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, *vol.***27**(2):265–291, 1980.

[7] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.*, *vol.***63**(1):159–179, 1981.

Kazushi Ueda
Department of Mathematics,
Graduate School of Science,
Osaka University,
Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.
e-mail: kazushi@math.sci.osaka-u.ac.jp

Masahiko Yoshinaga
Department of Mathematics,
Graduate School of Science,
Kobe University,
1-1, Rokkodai, Nada-ku, Kobe 657-8501, Japan
e-mail: myoshina@math.kobe-u.ac.jp