Kumamoto J. Math.
Vol.**21** (2008), 29–39

Factorization in certain rings of arithmetical functions

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(Received May 30, 2007)
(Accepted December 19, 2007)

Abstract. In this paper we show that certain subrings of the ring $A_r(F)$ of arithmetical functions in r variables over a given field F are factorial.

1 Introduction

The ring $(A, +, .)$ of complex valued arithmetical functions with Dirichlet convolution consists of all functions $\mathbb{N} \to \mathbb{C}$, where $\mathbb{N}$ is the set of positive integers. Cashwell and Everett [2] proved that $(A, +, .)$ is a unique factorization domain. Yokom [13] investigated the prime factorization of arithmetical functions in a certain subring of the regular convolution ring. He determined a discrete valuation subring of the unitary ring of arithmetical functions. Narkiewicz [4] introduced and studied in [4] the concept of regular convolution. Further work on regular convolutions has been done by Scheid [5], [6], [7], Sitaramaiah [12] and Haukkanen [3]. Schwab and Silberberg [10] constructed an extension of $(A, +, .)$ which is a discrete valuation ring. In [11], they showed that A is a quasi-noetherian ring. Further results have been obtained by Schwab in [8] and [9]. Alkan and the authors [1] studied absolute values and derivations on the ring of arithmetical functions in several variables having values in an integral domain, with the analogue of Dirichlet convolution as multiplication. If R is an integral domain and r is a positive integer, let $A_r(R) = \{f : \mathbb{N}^r \to R\}$. For any $f, g \in A_r(R)$, the convolution $f * g$ of f and g is defined by

$$(f * g)(n_1, ..., n_r) = \sum_{d_1|n_1} ... \sum_{d_r|n_r} f(d_1, ..., d_r) g\left(\frac{n_1}{d_1}, ..., \frac{n_r}{d_r}\right). \tag{1.1}$$

In [14] a natural family of subrings $B_{r,k,p}(R)$ of $A_r(R)$ was considered. For any $k \in \{1, \ldots, r\}$, and any prime number p, $B_{r,k,p}(R)$ consists of all the functions $f \in A_r(R)$ with the property that for all $n_1, \ldots, n_r \in \mathbb{N}$ with p dividing n_k, one has $f(n_1, \ldots, n_r) = 0$. It was shown in [14] that the generating degree of$A_r(R)$

Mathematical Subject Classification (2010): Primary 11K65; Secondary 13F99
Key words: arithmetical functions, factorization, Dirichlet convolution

with respect to each of the subrings $B_{r,k,p}(R)$ is equal to 1. For two commutative topological rings $A \subseteq B$, by the generating degree of B over A, one means the cardinality of the smallest subset M of B for which the ring $A[M]$ is dense in B. In the present paper we complement the results from [14] by showing that if R is a field, then all the subrings of $A_r(R)$ of the form $B_{r,k,p}(R)$ are factorial.

Theorem 1 *For any field F, any integer $r \geq 1$, any $k \in \{1, \ldots, r\}$, and any prime number p, the subring $B_{r,k,p}(F)$ of $A_r(F)$ is factorial.*

2 Valuations

Let r be a positive integer, let R be an integral domain, with identity 1_R, and let $A_r(R) = \{f : \mathbb{N}^r \to R\}$. Then R has a natural embedding in the ring $A_r(R)$, and $A_r(R)$ with addition and convolution defined as in the Introduction naturally becomes an R-algebra. We now recall the construction from [1] of a class of absolute values on $A_r(R)$. Fix $\underline{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1, \ldots, t_r$ linearly independent over $\mathbb{Q}$, and $t_i > 0, (i = 1, 2, \ldots, r)$. Given $n \in \mathbb{N}$, denote by $\Omega(n)$ the total number of prime factors of n, counting multiplicities. Thus, if $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ is the prime factorization of n, then $\Omega(n) = \alpha_1 + \ldots + \alpha_k$. Define also $\Omega_r : \mathbb{N}^r \to \mathbb{N}^r$ by

$$\Omega_r(n_1, \ldots, n_r) = (\Omega(n_1), \ldots, \Omega(n_r)).$$

For any $f \in A_r(R)$ denote $supp(f) = \{\underline{n} \in \mathbb{N}^r | f(\underline{n}) \neq 0\}$, and let

$$V_{\underline{t}}(f) = \min_{\underline{n} \in supp(f)} \underline{t} \cdot \Omega_r(\underline{n}),$$

with the convention $\min(\emptyset) = \infty$. It is shown in [1] that for any $f, g \in A_r(R)$ one has

$$V_{\underline{t}}(f + g) \geq \min(\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}),$$

and

$$V_{\underline{t}}(f * g) = V_{\underline{t}}(f) + V_{\underline{t}}(g).$$

Next, one extends the valuation $V_{\underline{t}}$ to a valuation $\overline{V_{\underline{t}}}$ on the field of fractions $\mathbb{K} = \{\frac{f}{g} | f, g \in A_r(R), g \neq 0\}$ of $A_r(R)$ by letting $V_{\underline{t}}(\frac{f}{g}) = V_{\underline{t}}(f) - V_{\underline{t}}(g)$. The above valuation is nonarchimedean. From now on we will restrict to the case when $R = F$ is a field. One of the important features of this valuation, which will be used in the proof of Theorem 1, is the following. Let f be an element of $A_r(F)$. Then $V_{\underline{t}}(f) = 0$ if and only if f is a unit of $A_r(F)$. Indeed, if f is a unit of $A_r(F)$, then $f(1, 1, \ldots, 1)$ must be a nonzero element of F. Then $(1, 1, \ldots, 1)$ lies in $supp(f)$, and by the definition of $V_{\underline{t}}$ it follows that $V_{\underline{t}}(f) = 0$. Conversely, if $V_{\underline{t}}(f) = 0$, then by the definition of $V_{\underline{t}}$ there exists an $\underline{n} = (n_1, \ldots, n_r)$ in $supp(f)$ for which $(\Omega(n_1), \ldots, \Omega(n_r)) = (0, \ldots, 0)$. This forces each of $n_1, \ldots, n_r$ to equal 1, which in turn implies that $(1, 1, \ldots, 1)$ lies in $supp(f)$, and hence f is a unit of $A_r(F)$. Let us also remark, as another important feature of this valuation, that, although for $r \geq 2$ the image of $\mathbb{K}$ through $\overline{V_{\underline{t}}}$ is dense in $\mathbb{R}$, the image of $A_r(F)$ through $V_{\underline{t}}$ consists of a strictly increasing sequence of nonnegative real numbers with limit infinity. Therefore any strictly decreasing sequence of such values, say $V_{\underline{t}}(\alpha_1) > V_{\underline{t}}(\alpha_2) > \dots$, with $\alpha_1, \alpha_2, \dots$ in $A_r(F)$, must terminate after finitely many steps.

3 Reduction to the full ring $A_r(F)$

Let F be a field. Fix an integer $r \geq 1$, a $k \in \{1, \dots, r\}$, and a prime number p. Consider the subring $B_{r,k,p}(F)$ of $A_r(F)$ consisting of all the functions $f \in A_r(F)$ with the property that for all $n_1, \dots, n_r \in \mathbb{N}$ with p dividing n_k, one has $f(n_1, \dots, n_r) = 0$. As usual, by an irreducible element in a ring R we mean a nonzero, nonunit element a of R with the property that whenever a is written as a product of two elements b and c in the ring, then one of them is a unit.

In proving Theorem 1, our idea is to reduce the satement of the theorem about the subring $B_{r,k,p}(F)$ to the similar statement about the full ring $A_r(F)$, and separately to prove that $A_r(F)$ is factorial. In this section we present the reduction step, that is we show that if $A_r(F)$ is factorial, then $B_{r,k,p}(F)$ is factorial as well.

Lemma 1 *Let $u \in B_{r,k,p}(F)$ be nonzero. Then u is a unit in $B_{r,k,p}(F)$ if and only if u is a unit in $A_r(F)$.*

Proof: It is clear that if u is a unit in $B_{r,k,p}(F)$, then u is a unit in $A_r(F)$. Suppose now that u is a unit in $A_r(F)$. Then there exists $v \in A_r(F)$ such that $u * v = 1$, where 1 is the unity in $A_r(F)$, which as an arithmetical function, is given by

$$1(n_1, \dots, n_r) = \begin{cases} 1 & \text{if } n_1 = \dots = n_r = 1, \\ 0 & \text{else.} \end{cases}$$

for $n_1, \dots, n_r \in \mathbb{N}$. We construct $w \in B_{r,k,p}(F)$ from v as follows. Let $n_1, \dots, n_r \in \mathbb{N}$. Then we let

$$w(n_1, \dots, n_r) = \begin{cases} 0 & \text{if } p | n_k, \\ v(n_1, \dots, n_r) & \text{else.} \end{cases}$$

It follows that $u * w = 1$. To see this, first note that $u * w(1, \dots, 1) = 1$. For $(n_1, \dots, n_r) \neq (1, \dots, 1)$, consider

$$(u * w)(n_1, ..., n_r) = \sum_{d_1 | n_1} ... \sum_{d_r | n_r} w(d_1, ..., d_r) u\left(\frac{n_1}{d_1}, ..., \frac{n_r}{d_r}\right). \tag{3.1}$$

If p does not divide n_k, then p does not divide any divisor d_k of n_k, and the above sum equals

$$\sum_{d_1 | n_1} ... \sum_{d_r | n_r} v(d_1, ..., d_r) u\left(\frac{n_1}{d_1}, ..., \frac{n_r}{d_r}\right) = (u * v)(n_1, ..., n_r) = 0.$$

So $(u * w)(n_1, ..., n_r) = 0$. Also, if p divides n_k, then each term in the sum on the right side of equation (3.1) is zero. This is because p divides either d_k or $\frac{n_k}{d_k}$ in each of the terms. So again $(u * w)(n_1, ..., n_r) = 0$. We conclude that $u * w = 1$ and therefore u is a unit in $B_{r,k,p}(F)$.

Lemma 2 *Let $\pi \in B_{r,k,p}(F)$ be nonzero. Then π is an irreducible element of $B_{r,k,p}(F)$ if and only if π is an irreducible element of $A_r(F)$.*

Proof: Suppose that π is an irreducible element of $A_r(F)$. If π is not an irreducible element of $B_{r,k,p}(F)$, then there exist $a, b \in B_{r,k,p}(F)$ such that both a and b are nonunits in $B_{r,k,p}(F)$, and $\pi = a * b$. By Lemma 1 both a and b are nonunits in $A_r(F)$. This contradicts our assumption that π is irreducible in $A_r(F)$. Hence, π is an irreducible element of $B_{r,k,p}(F)$ as well.

Conversely, suppose that π is an irreducible element of $B_{r,k,p}(F)$. Assume $\pi = a * b$ with $a, b \in A_r(F)$. We construct $a', b' \in B_{r,k,p}(F)$ from a and b in the same way we constructed w from v in the proof of Lemma 1. That is, for $n_1, \ldots, n_r \in \mathbb{N}$ we let

$$a'(n_1, \ldots, n_r) = \begin{cases} 0 & \text{if } p | n_k, \\ a(n_1, \ldots, n_r) & \text{else.} \end{cases}$$

and

$$b'(n_1, \ldots, n_r) = \begin{cases} 0 & \text{if } p | n_k, \\ b(n_1, \ldots, n_r) & \text{else.} \end{cases}$$

Then one easily sees that $\pi = a' * b'$. Since π is irreducible in $B_{r,k,p}(F)$, either a' or b', say a', is a unit in $B_{r,k,p}(F)$. By Lemma 1, a' is also a unit in $A_r(F)$. We now show that a is a unit in $A_r(F)$. Write $a = a' + y$, where y is supported only on the set of all points $(n_1, \ldots, n_r) \in \mathbb{N}^r$ with the property that p divides n_k. Since a' is a unit in $B_{r,k,p}(F)$, there exists an element a'' of $B_{r,k,p}(F)$ such that $a' * a'' = 1$. Thus $a * a'' = a' * a'' + y * a'' = 1 + y * a''$. Define $f \in A_r(F)$ by $f = \sum_{m=0}^{\infty} (-1)^m (y * a'')^m = 1 - y * a'' + (y * a'')^2 - (y * a'')^3 + \ldots$. Note that f is a well defined element of $A_r(F)$ since for each fixed $(n_1, \ldots, n_r) \in \mathbb{N}^r$, there exists $n \in \mathbb{N}$ such that $(y * a'')^m(n_1, \ldots, n_r) = 0$ for all $m > n$. Observe that $a * a'' * f = 1$, and thus a is invertible in $A_r(F)$. This completes the proof of the lemma.

Lemma 3 *If $A_r(F)$ is factorial, then the subring $B_{r,k,p}(F)$ is factorial.*

Proof: First we show that each nonzero element $a \in B_{r,k,p}(F)$ can be expressed as a finite product of irreducible elements of $B_{r,k,p}(F)$. Indeed, if a is not irreducible in $B_{r,k,p}(F)$, then there exist nonunits $b, c \in B_{r,k,p}(F)$ such that $a = b * c$. By Lemma 1, b and c are also non units of $A_r(F)$. If any of the elements b and c is not irreducible in $B_{r,k,p}(F)$, then we may again write that element as a product of two elements in $B_{r,k,p}(F)$ which are nonunits in both $B_{r,k,p}(F)$ and $A_r(F)$. This process must stop after finitely many steps since otherwise we obtain a contradiction with the assumption that $A_r(F)$ is factorial. Hence we conclude that each element $a \in B_{r,k,p}(F)$ can be expressed as a finite product $a = a_1 * a_2 * \cdots * a_m$ of irreducible elements of $B_{r,k,p}(F)$.

Secondly we need to establish the uniqueness of the expressions $a = a_1 * a_2 * \cdots * a_m$ in $B_{r,k,p}(F)$ up to order and units. Suppose that $a = a_1 * a_2 * \cdots * a_m = b_1 * b_2 * \cdots * b_s \quad (m, s \in \mathbb{N})$ where a_i $(i = 1, \ldots, m)$ and b_j $(j = 1, \ldots, s)$ are irreducible elements of $B_{r,k,p}(F)$. By Lemma 2 and the assumption that $A_r(F)$ is factorial we have that $m = s$, and there exist w_i $(i = 1, \ldots, m)$ such that $a_i = b_i * w_i$ $(i = 1, \ldots, m)$ and w_i $(i = 1, \ldots, m)$ are units in $A_r(F)$. Let $n_1, \ldots, n_r \in \mathbb{N}$, and define $v_i \in B_{r,k,p}(F)$ $(i = 1, \ldots, m)$ as follows.

$$v_i(n_1, \ldots, n_r) = \begin{cases} 0 & \text{if } p|n_k, \\ w_i(n_1, \ldots, n_r) & \text{else.} \end{cases}$$

It is easily verified that for each $i \in \{1, \ldots, m\}$, one has $a_i = b_i * w_i = b_i * v_i$, and v_i is a unit in $A_r(F)$. But by Lemma 1, v_i is also a unit in $B_{r,k,p}(F)$. Hence the lemma is proved.

4 Completion of proof of Theorem 1

In order to complete the proof of Theorem 1, it remains to show that $A_r(F)$ is factorial. Let us fix a $\underline{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1, \ldots, t_r$ linearly independent over $\mathbb{Q}$, $t_i > 0, (i = 1, 2, \ldots, r)$, and consider the valuation $V_{\underline{t}}$ defined in Section 2.

Lemma 4 *Every $\alpha \in A_r(F)$ which is not zero and not a unit is expressible as a finite product of irreducible elements of $A_r(F)$.*

Proof: In order to prove the lemma, consider a nonzero element $\alpha \in A_r(F)$ which is not a unit. We need to show that α can be written as a finite product of irreducible elements of $A_r(F)$. If α is itself irreducible, there is nothing to prove. Let us assume that α is not irreducible. Then α can be written as $\alpha = \alpha_1 * \beta_1$, with α_1, β_1 nonunit elements of $A_r(F)$. By the results of Section 2 we know that $V_{\underline{t}}(\alpha_1) > 0$, $V_{\underline{t}}(\beta_1) > 0$, and

$$V_{\underline{t}}(\alpha) = V_{\underline{t}}(\alpha_1) + V_{\underline{t}}(\beta_1).$$

If α_1 and β_1 are both irreducibe then the lemma is proved. If not, we continue the same procedure with α replaced by α_1 or β_1. Note by the properties of the valuation $V_{\underline{t}}$ discussed Section 2, that since each of α_1 and β_1 has a valuation that is strictly smaller than the valuation of α, the above procedure must terminate after finitely many steps. This completes the proof of the lemma.

We now consider the ring of formal r-fold power series, which is defined as follows. Let us list the prime numbers in increasing order: $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. Then every integer n may be written uniquely in the form $n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} \cdots$ and uniquely described by a vector $(\alpha_1(n), \alpha_2(n), \ldots)$ with non-negative integral components where only finitely many of the components are nonzero. All such vectors are realized as n ranges over $\mathbb{N}$. Hence an arithmetic function $a = a(n) \in A_1(F)$

in one variable may be associated with a definite formal power series in a countably infinite number of indeterminates $y_{p_1}, y_{p_2}, y_{p_3}, \dots$, having coefficients in F, by means of the correspondence

$$a \to P(a) = \sum_n a(n) y_{p_1}^{\alpha_1(n)} y_{p_2}^{\alpha_2(n)} \cdots .$$

Here, the summation extends over all $n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} p_3^{\alpha_3(n)} \cdots$ in $\mathbb{N}$. Also, an arithmetic function $a = a(n_1, n_2) \in A_2(F)$ in two variables may be associated with a definite formal power series in a countably infinite number of indeterminates $x_{p_1}, x_{p_2}, x_{p_3}, \dots,\ y_{p_1}, y_{p_2}, y_{p_3}, \dots$ having coefficients in F, by means of the correspondence

$$a \to P(a) = \sum_n \sum_m a(n,m) x_{p_1}^{\alpha_1(n)} y_{p_1}^{\alpha_1(m)} x_{p_2}^{\alpha_2(n)} y_{p_2}^{\alpha_2(m)} \cdots .$$

Here, the summation extends over all

$$n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} p_3^{\alpha_3(n)} \cdots$$

and

$$m = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} p_3^{\alpha_3(n)} \cdots$$

in $\mathbb{N}$. In general, an arithmetic function $a = a(n_1, \dots, n_r) \in A_r(F)$ in r variables may be associated with a definite formal r-fold power series in a countably infinite number of indeterminates $x_{1p_1}, x_{1p_2}, \dots, x_{2p_1}, x_{2p_2}, \dots, \dots, x_{rp_1}, x_{rp_2}, \dots$, having coefficients in F, by means of the correspondence

$$a \to P(a) = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} a(n_1, \dots, n_r) x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \cdots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \cdots x_{rp_2}^{\alpha_2(n_r)} \cdots .$$

Here, the summation extends over all

$$n_1 = p_1^{\alpha_1(n_1)} p_2^{\alpha_2(n_1)} p_3^{\alpha_3(n_1)} \cdots ,$$

$$n_2 = p_1^{\alpha_1(n_2)} p_2^{\alpha_2(n_2)} p_3^{\alpha_3(n_2)} \cdots ,$$

$$\cdots$$

$$n_r = p_1^{\alpha_1(n_r)} p_2^{\alpha_2(n_r)} p_3^{\alpha_3(n_r)} \cdots$$

of $\mathbb{N}$. This correspondence is clearly one to one on $A_r(F)$ to the set

$$F\{\dots, x_{ip_j}, \dots\} = F\{x_{1p_1}, x_{1p_2}, \dots\}\{x_{2p_1}, x_{2p_2}, \dots\} \dots \{x_{rp_1}, x_{rp_2}, \dots\}$$

of all such power series. Moreover, addition is preserved, and as we will see below $P(f * g) = P(f)P(g)$. The latter operation is the usual formal operation on power series involving multiplication and collection of finite number of like terms. Thus the ring $A_r(F)$ is isomorphic to the ring $F\{x_{ip_j}\}$ of all r-fold formal power series. We emphasize that only a finite number of x_{ip_j} actually appear (i.e., have

$\alpha_j(n_i) > 0$) in any term. However, infinitely many x_{ip_j} may well occur in the same series. We now prove that $P(f*g) = P(f)P(g)$. One has

$$
\begin{aligned}
&P(f)P(g)\\
&= \left(\sum_{n_1}\sum_{n_2}\cdots\sum_{n_r} f(n_1,\dots,n_r) x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \cdots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \cdots x_{rp_2}^{\alpha_2(n_r)} \cdots\right)\\
&\left(\sum_{m_1}\sum_{m_2}\cdots\sum_{m_r} g(m_1,\dots,m_r) x_{1p_1}^{\alpha_1(m_1)} x_{2p_1}^{\alpha_1(m_2)} \cdots x_{rp_1}^{\alpha_1(m_r)} x_{1p_2}^{\alpha_2(m_1)} x_{2p_2}^{\alpha_2(m_2)} \cdots x_{rp_2}^{\alpha_2(m_r)} \cdots\right)\\
&= \sum_{\substack{n_1,\dots,n_r\\ m_1,\dots,m_r}} f(n_1,\dots,n_r) g(m_1,\dots,m_r)\\
&x_{1p_1}^{\alpha_1(n_1)+\alpha_1(m_1)} x_{2p_1}^{\alpha_1(n_2)+\alpha_1(m_2)} \cdots x_{rp_1}^{\alpha_1(n_r)+\alpha_1(m_r)} x_{1p_2}^{\alpha_2(n_1)+\alpha_2(m_1)} x_{2p_2}^{\alpha_2(n_2)+\alpha_2(m_2)} \cdots x_{rp_2}^{\alpha_2(n_r)+\alpha_2(m_r)} \cdots
\end{aligned}
$$

This further equals

$$
\begin{aligned}
&\sum_{\substack{n_1,\dots,n_r\\ m_1,\dots,m_r}} f(n_1,\dots,n_r) g(m_1,\dots,m_r)\\
&x_{1p_1}^{\alpha_1(n_1m_1)} x_{2p_1}^{\alpha_1(n_2m_2)} \cdots x_{rp_1}^{\alpha_1(n_rm_r)} x_{1p_2}^{\alpha_2(n_1m_1)} x_{2p_2}^{\alpha_2(n_2m_2)} \cdots x_{rp_2}^{\alpha_2(n_rm_r)} \cdots\\
&= \sum_{k_1,\dots,k_r} \left(\sum_{k_1=m_1n_1} \cdots \sum_{k_r=m_rn_r} f(n_1,\dots,n_r) g(m_1,\dots,m_r)\right)\\
&x_{1p_1}^{\alpha_1(k_1)} x_{2p_1}^{\alpha_1(k_2)} \cdots x_{rp_1}^{\alpha_1(k_r)} x_{1p_2}^{\alpha_2(k_1)} x_{2p_2}^{\alpha_2(k_2)} \cdots x_{rp_2}^{\alpha_2(k_r)} \cdots\\
&= \sum_{k_1,\dots,k_r} (f*g)(k_1,\dots,k_r) x_{1p_1}^{\alpha_1(k_1)} x_{2p_1}^{\alpha_1(k_2)} \cdots x_{rp_1}^{\alpha_1(k_r)} x_{1p_2}^{\alpha_2(k_1)} x_{2p_2}^{\alpha_2(k_2)} \cdots x_{rp_2}^{\alpha_2(k_r)} \cdots\\
&= P(f*g).
\end{aligned}
$$

Next, for any positive integer l, any $k \in \{1,\dots,r\}$, and any power series $Q \in F\{\dots,x_{ip_j},\dots\}$, denote by $deg_{x_{kp_l}}(Q)$ the supremum of the set of exponents of x_{kp_l} that appear in the terms of Q with nonzero coefficients. Also, for a positive integer l, and $k \in \{1,\dots,r\}$, denote by $F\{\dots,x_{ip_j},\dots\}_{(i,p_j)\neq(k,p_l)}$ the subring of $F\{\dots,x_{ip_j},\dots\}$ which consists of all power series Q in $F\{\dots,x_{ip_j},\dots\}$ such that $deg_{x_{kp_l}}(Q)$ is zero. Under the above isomorphism the ring $B_{r,k,p}(F)$ is isomorphic to the subring

$$F\{\dots,x_{ip_j},\dots\}_{(i,p_j)\neq(k,p)}.$$

Cashwell and Everett [2] proved that $(A_1(\mathbb{C}),+,.)$, where $\mathbb{C}$ denotes the field of complex numbers, is a unique factorization domain by showing that the corresponding power series ring $\mathbb{C}\{x_{1p_1},x_{1p_2},\dots\}$ is a unique factorization domain. Next, we show that for any positive integer r, $A_r(F)$ is a unique factorization domain by showing that the corresponding r-fold power series ring

$$F\{\dots,x_{ip_j},\dots\} = F\{x_{1p_1},x_{1p_2},\dots\}\{x_{2p_1},x_{2p_2},\dots\}\dots\{x_{rp_1},x_{rp_2},\dots\}$$

is a unique factorization domain. We have already shown that every element of $(A_r(F),+,.)$ can be written as a finite product of irreducible elements. So we need to establish the uniqueness of the expression of a non-zero, non-unit element of $A_r(F)$ as a product of irreducibles in $A_r(F)$ up to order and units. We first show that the uniqueness holds for the case $r = 2$. We proceed in several steps. Let $S[x_1, \dots, x_l]$ denote the ring of formal power series in l indeterminates with coefficients in S, where S is any ring. Let $H = F\{x_{2p_1}, x_{2p_2}, \dots x_{(2,p_j)}\}$. At the first step, we show that the ring $F\{x_{1p_1}, x_{1p_2}, \dots\}\{x_{2p_1}, x_{2p_2}, \dots\}$ is isomorphic to the ring $H\{x_{1p_1}, x_{1p_2}, \dots\}$ of 1-fold formal power series with coefficents in H. Then we need to show that $H\{x_{1p_1}, x_{1p_2}, \dots\}$ is a a unique factorization domain. By the proof of Cashwell and Everett [2], it is enough to show that for any positive integer l, $H[x_{1p_1}, x_{1p_2}, \dots, x_{1p_l}]$ is a unique factorization domain. At the second step, we show that for any positive integer $l, H[x_{1p_1}, x_{1p_2}, \dots, x_{1p_l}]$ is isomorphic to $F[x_{1p_1}, \dots, x_{1p_l}]\{x_{2p_1}, x_{2p_2}, \dots\}$. Again by the proof of Cashwell and Everett [2], $F[x_{1p_1}, \dots, x_{1p_l}]\{x_{2p_1}, x_{2p_2}, \dots\}$ is a unique factorization domain if for any positive integer l', $F[x_{1p_1}, \dots, x_{1p_l}][x_{2p_1}, x_{2p_2}, \dots, x_{2p'_l}]$ is isomorphic to $F[x_{1p_1}, \dots, x_{1p_l}, x_{2p_1}, x_{2p_2}, \dots, x_{2p'_l}]$. So, at the third step we establish the last isomorphism. We now proceed with the first step. To show that $F\{x_{1p_1}, x_{1p_2}, \dots\}\{x_{2p_1}, x_{2p_2}, \dots\}$ is isomorphic to the ring $H\{x_{1p_1}, x_{1p_2}, \dots\}$, we first replace the set $x_{1p_1}, x_{1p_2}, \dots$ of variables by the set $x_{p_1}, x_{p_2}, \dots$, and the set $x_{2p_1}, x_{2p_2}, \dots$ by $y_{p_1}, y_{p_2}, \dots$. Given

$$f \in F\{x_{p_1}, x_{p_2}, \dots\}\{y_{p_1}, y_{p_2}, \dots\}, f = \sum_n \sum_m f(n,m) x_{p_1}^{\alpha_1(n)} y_{p_1}^{\alpha_1(m)} x_{p_2}^{\alpha_2(n)} y_{p_2}^{\alpha_2(m)} \cdots ,$$

we define $f_H \in H\{x_{p_1}, x_{p_2}, \dots\}$, where $H = F\{y_{p_1}, y_{p_2}, \dots\}$, to be the series

$$f_H = \sum_n f_H(n) x_{p_1}^{\alpha_1(n)} x_{p_2}^{\alpha_2(n)} \cdots ,$$

where

$$f_H(n) = \sum_m f(n,m) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots .$$

for each n. The map $f \to f_H$ is clearly a bijective map.

Also for $f, g \in F\{x_{p_1}, x_{p_2}, \dots\}\{y_{p_1}, y_{p_2}, \dots\}$, we have that

$$\begin{aligned}(fg) &= \sum_{n_1} \sum_{m_1} \sum_{n_2} \sum_{m_2} f(n_1, m_1) g(n_2, m_2) x_{p_1}^{\alpha_1(n_1 n_2)} y_{p_1}^{\alpha_1(m_1 m_2)} x_{p_2}^{\alpha_2(n_1 n_2)} y_{p_2}^{\alpha_2(m_1 m_2)} \cdots \\ &= \sum_n \sum_m \left(\sum_{n = n_1 n_2} \sum_{m = m_1 m_2} f(n_1, m_1) g(n_2, m_2) \right) x_{p_1}^{\alpha_1(n)} y_{p_1}^{\alpha_1(m)} x_{p_2}^{\alpha_2(n)} y_{p_2}^{\alpha_2(m)} \cdots .\end{aligned}$$

So,

$$
\begin{aligned}
(fg)_H &= \sum_n \left(\sum_m \left(\sum_{n=n_1n_2} \sum_{m=m_1m_2} f(n_1,m_1)g(n_2,m_2) \right) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots \right) x_{p_1}^{\alpha_1(n)} x_{p_2}^{\alpha_2(n)} \cdots \\
&= \left(\sum_{n_1} \left(\sum_{m_1} f(n_1,m_1) y_{p_1}^{\alpha_1(m_1)} y_{p_2}^{\alpha_2(m_1)} \cdots \right) x_{p_1}^{\alpha_1(n_1)} x_{p_2}^{\alpha_2(n_1)} \cdots \right) \\
&\quad \left(\sum_{n_2} \left(\sum_{m_2} f(n_2,m_2) y_{p_1}^{\alpha_1(m_2)} y_{p_2}^{\alpha_2(m_2)} \cdots \right) x_{p_1}^{\alpha_1(n_2)} x_{p_2}^{\alpha_2(n_2)} \cdots \right) \\
&= \left(\sum_{n_1} f_H(n_1) x_{p_1}^{\alpha_1(n_1)} x_{p_2}^{\alpha_2(n_1)} \cdots \right) \left(\sum_{n_2} g_H(n_2) x_{p_1}^{\alpha_1(n_2)} x_{p_2}^{\alpha_2(n_2)} \cdots \right) \\
&= f_H g_H.
\end{aligned}
$$

Thus, the map $f \to f_H$ is an isomorphism. This finishes step one.

Next, we prove the second step. To do so it suffices to prove that for any positive integer $l, H[x_{p_1}, x_{p_2}, \ldots x_{p_l}]$ is isomorphic to $F[x_{p_1}, \ldots, x_{p_l}]\{y_{p_1}, y_{p_2}, \ldots\}$, where H is as in the proof of the first step. Let $a \in H[x_{p_1}, x_{p_2}, \ldots, x_{p_l}]$. Let $a = \sum_n a_H(n) x_{p_1}^{\alpha_1(n)} \cdots x_{p_l}^{\alpha_l(n)}$, where for each n, $a_H(n) \in H$, and $a_H(n) = \sum_m a(n,m) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots$. Define $\overline{a} \in F[x_{p_1}, \ldots, x_{p_l}]\{y_{p_1}, y_{p_2}, \ldots\}$ by $\overline{a} = \sum_m \overline{a}(m) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots$, where for each m, $\overline{a}(m) = \sum_n a(n,m) x_{p_1}^{\alpha_1(n)} \cdots x_{p_l}^{\alpha_l(n)}$. The map $a \to \overline{a}(m)$ is a homomorphism. To see this let $a, b \in H[x_{p_1}, x_{p_2}, \ldots, x_{p_l}]$, and let $c = ab$. Then $c_H(n) = \sum_{n=n_1n_2} a_H(n_1) b_H(n_2)$, and

$$
\begin{aligned}
a_H(n_1)b_H(n_2) &= \left(\sum_{m_1} a(n_1,m_1) y_{p_1}^{\alpha_1(m_1)} y_{p_2}^{\alpha_2(m_1)} \cdots \right) \left(\sum_{m_2} b(n_2,m_2) y_{p_1}^{\alpha_1(m_2)} y_{p_2}^{\alpha_2(m_2)} \cdots \right) \\
&= \sum_m \left(\sum_{m=m_1m_2} a(n_1,m_1) b(n_2,m_2) \right) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots .
\end{aligned}
$$

So,

$$
\begin{aligned}
ab &= \left(\sum_{n_1} a_H(n_1) x_{p_1}^{\alpha_1(n_1)} \cdots x_{p_l}^{\alpha_l(n_1)} \right) \left(\sum_{n_2} b_H(n_2) x_{p_1}^{\alpha_1(n_2)} \cdots x_{p_l}^{\alpha_l(n_2)} \right) \\
&= \sum_n \left(\sum_{n=n_1n_2} a_H(n_1) b_H(n_2) \right) x_{p_1}^{\alpha_1(n)} \cdots x_{p_l}^{\alpha_l(n)}
\end{aligned}
$$

Thus,

$$\begin{aligned}
\overline{ab} &= \sum_m \left(\sum_n \left(\sum_{n=n_1 n_2} \sum_{m=m_1 m_2} a(n_1,m_1) b(n_2,m_2) \right) x_{p_1}^{\alpha_1(n)} \cdots x_{p_l}^{\alpha_l(n)} \right) y_{p_1}^{\alpha_1(m)} y_{p_2}^{\alpha_2(m)} \cdots \\
&= \left(\sum_{m_1} \left(\sum_{n_1} a(n_1,m_1) x_{p_1}^{\alpha_1(n_1)} \cdots x_{p_l}^{\alpha_l(n_1)} \right) y_{p_1}^{\alpha_1(m_1)} y_{p_2}^{\alpha_2(m_1)} \cdots \right) \\
&\left(\sum_{m_2} \left(\sum_{n_2} b(n_2,m_2) x_{p_1}^{\alpha_1(n_2)} \cdots x_{p_l}^{\alpha_2(n_2)} \right) y_{p_1}^{\alpha_1(m_2)} y_{p_2}^{\alpha_2(m_2)} \cdots \right) \\
&= \left(\sum_{m_1} \overline{a}(n_1) y_{p_1}^{\alpha_1(m_1)} y_{p_2}^{\alpha_2(m_1)} \cdots \right) \left(\sum_{m_2} \overline{b}(n_2) y_{p_1}^{\alpha_1(m_2)} y_{p_2}^{\alpha_2(m_2)} \cdots \right) \\
&= \overline{a}\overline{b}
\end{aligned}$$

Therefore, the map $a \to \overline{a}$ is a homomorphism. It is clear that this map is also a bijective map. Hence, $H[x_{p_1}, x_{p_2}, \ldots x_{p_l}]$ is isomorphic to $F[x_{p_1}, \ldots, x_{p_l}]\{y_{p_1}, y_{p_2}, \ldots\}$. This finishes step two. As for step three, one can argue similarly as above to conclude that the ring

$$F[x_{1p_1}, \ldots, x_{1p_l}][x_{2p_1}, x_{2p_2}, \ldots, x_{2p'_l}]$$

is isomorphic to the ring

$$F[x_{1p_1}, \ldots, x_{1p_l}, x_{2p_1}, x_{2p_2}, \ldots, x_{2p'_l}],$$

which finishes the proof in this case. The case of a general r is proved similarly, and with this the proof of Theorem 1 is complete.

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