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Determination of b-functions of polynomials defining Saito Free Divisors related with simple curve singularities of types E_6, E_7, E_8

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Abstract. We determine b-functions of the seventeen polynomials of three variables classificatied by the second author. The zero set of each of these polynomials defines a Saito free divisor.

1 Introduction

The purpose of this paper is the determination of the b-functions of the polynomials obtained in [10] where the second author classified weighted homogeneous polynomials of three variables with some nice conditions as discriminants of Weyl groups.

Before entering into the main subject of this paper, we explain its background. The seventeen polynomials classified in [10] are decomposed into three families by the weights (p, q, r); (I) $(p, q, r) = (2, 3, 4)$, (II) $(p, q, r) = (1, 2, 3)$, (III) $(p, q, r) = (1, 3, 5)$. These families are related with the reflection groups of rank three. In fact, there are three irreducible finite reflection groups of rank three and their types are A_3, B_3, H_3. The discriminant of the reflection group $W(A_3)$ (resp., $W(B_3)$, $W(H_3)$) is contained in (I) (resp., (II), (III)). Let x, y, z be the variables such that their weights are p, q, r, respectively and let $F(x, y, z)$ be one of the seventeen polynomials. Then we find that not only the hypersurface of $\mathbf{C}^3$ defined by $F(x, y, z) = 0$ is a Saito free divisor but also it is regarded as a special kind of a space of 1-parameter deformations of curves in the yz-space of a simple curve singularity whose type is one of the types E_6, E_7, E_8 (see [10]). Since the complements of discriminants are known to be $K(\pi, 1)$-spaces, it is interesting to determine the fundamental groups of the hypersurfaces defined by $F(x, y, z) = 0$. This is done by T. Ishibe and K. Saito (cf. [4]).

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As a next step to study properties of these polynomials, we focus our attention on the determination of their b-functions and make clear the roles of the roots of the b-functions. Actually this is the main subject of this paper. There are several methods of computing b-functions. In this paper we employ two methods. One is the well-known method developped by M. Kashiwara (cf. [5]) which is quite theoretical. The other is an algorithmic method established by Oaku ([7]); This is improved and implemented to the computer algebra system "Risa/Asir" by Noro ([6]). It is underlined that this method is applicable to arbitary polynomials and that we can easily compute the b-functions of the seventeen polynomials by this method. As an advantage of the former method, it is possible to understand the role of the factors of b-functions. In fact, for each of the seventeen polynomials, its b-function is expected to be the product of three factors associated to a stratification, namely, the first one corresponds to the contribution of the hypersurface, the second does to that of the singular locus of the hypersurface and the third does to that of the origin. Applying these two methods to our case, we actually show that the b-function for each of the seventeen polynomials is decomposed into three factors expected above. In particular, as a consequence of our study, we shall show the following result.

Theorem *If $F(x, y, z)$ is one of the seventeen polynomials, then*

$$b_F(s) = (s+1)\tilde{b}^2_{F,a}(s)\tilde{b}^3_{F,a}(s),$$

where $\tilde{b}^2_{F,a}(s)$ and $\tilde{b}^3_{F,a}(s)$ are factors associated to the singular locus of the hypersurface $F(x, y, z) = 0$ and the origin, respectively.

We are going to explain the contents of this paper briefly. In section 2, we review the results of [10] concerning the seventeen polynomials. In section 3, we explain the definition of "b-function" of an analytic function, some of its basic properties and methods of finding roots of b-functions. In particular, we introduce polynomials $\tilde{b}^k_f(s)$ associated to the b-functions. It is known that $\tilde{b}^1_f(s) = s + 1$. In section 4, we apply the method of section 3 to the seventeen polynomials. In our case, $\tilde{b}^2_F(s)$ corresponds to the contribution of the singular locus of the hypersurface $F(x, y, z) = 0$. Applying the arguments of the previous two sections, we introduce a polynomial $\tilde{b}^2_{F,a}(s)$ which is expected to be a factor of $\tilde{b}^2_F(s)$. As to $\tilde{b}^3_{F,a}(s)$, applying the criterion of finding roots of $\tilde{b}^n_f(s) = 0$ explained in section 3 to our case, we determine some of factors of $\tilde{b}^3_F(s)$. As a result, we introduce a polynomial $\tilde{b}^3_{F,a}(s)$ which is a factor of (but is expected to coincide with) $\tilde{b}^3_F(s)$. The concrete form of $\tilde{b}^3_{F,a}(s)$ is given in Proposition 1. In section 5, we shall show the main result of this paper. We first give b-functions of the seventeen polynomials by using Risa/Asir. Comparing the computation on $\tilde{b}^2_{F,a}(s)$, $\tilde{b}^3_{F,a}(s)$ with that on $b_F(s)$ by the algorithmic method, we obtain Theorem stated above. As a consequence, we find that $\tilde{b}^j_{F,a}(s)$ conincides with $\tilde{b}^j_F(s)$ $(j = 2, 3)$.

2 The seventeen polynomials

In this section, we review the results of [10].

Let x, y, z be variables and consider a weighted homogeneous polynomial $F(x, y, z)$ of x, y, z of weights p, q, r. First we introduce seventeen polynomials $F_{A,1}, F_{A,2}, \ldots, F_{H,8}$ of x, y, z;

(I) The case $(p, q, r) = (2, 3, 4)$. (This case corresponds to the reflection group of Type A_3.)

$F_{A,1} = 16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3$.

$F_{A,2} = 2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3$.

(II) The case $(p, q, r) = (1, 2, 3)$. (This case corresponds to the reflection group of Type B_3.)

$F_{B,1} = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2)$.

$F_{B,2} = z(-2y^3 + 4x^3z + 18xyz + 27z^2)$.

$F_{B,3} = z(-2y^3 + 9xyz + 45z^2)$.

$F_{B,4} = z(9x^2y^2 - 4y^3 + 18xyz + 9z^2)$.

$F_{B,5} = xy^4 + y^3z + z^3$.

$F_{B,6} = 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3$.

$F_{B,7} = \frac{1}{2}xy^4 - 2x^2y^2z - y^3z + 2x^3z^2 + 2xyz^2 + z^3$.

(III) The case $(p, q, r) = (1, 3, 5)$. (This case corresponds to the reflection group of Type H_3.)

$F_{H,1} = -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7y + 60x^4y^2 + 225xy^3)z - \frac{135}{2}y^5 - 115x^3y^4 - 10x^6y^3 - 4x^9y^2$.

$F_{H,2} = 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3$.

$F_{H,3} = 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3$.

$F_{H,4} = y^5 - 2xy^3z + x^2yz^2 + z^3$.

$F_{H,5} = x^3y^4 - y^5 + 3xy^3z + z^3$.

$F_{H,6} = x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3$.

$F_{H,7} = xy^3z + y^5 + z^3$.

$F_{H,8} = x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3$.

Let $F(x, y, z)$ be one of the polynomials defined above. Then $C : \{(y, z); F(0, y, z) = 0\}$ is regarded as a curve having an isolated singularity of type E_6, E_7, E_8 at the origin if $F(x, y, z)$ is one of the polynomials $F_{A,j} (j = 1, 2), F_{B,j} (j = 1, \ldots, 7), F_{H,j} (j = 1, \ldots, 8)$, respectively (cf. [13]). Therefore if we regard x as a parameter, the family of curves $C_x : F(x, y, z) = 0$ on yz-space is a deformation of the curve $C_0 = C$.

It is shown in [10] that, for the polynomial $F(x, y, z)$, there are vector fields V_0, V_1, V_2 defined by

$$\begin{cases} V_0 &= px\dfrac{\partial}{\partial x} + qy\dfrac{\partial}{\partial y} + rz\dfrac{\partial}{\partial z}, \\ V_1 &= qy\dfrac{\partial}{\partial x} + h_{22}(x, y, z)\dfrac{\partial}{\partial y} + h_{23}(x, y, z)\dfrac{\partial}{\partial z}, \\ V_2 &= rz\dfrac{\partial}{\partial x} + h_{32}(x, y, z)\dfrac{\partial}{\partial y} + h_{33}(x, y, z)\dfrac{\partial}{\partial z} \end{cases} \tag{1}$$

having the following properties:

(C0) $h_{ij}(x,y,z)$ are polynomials of x,y,z.
(Ci) $[V_0,V_1]=(q-p)V_1,\ [V_0,V_2]=(r-p)V_2$.
(Cii) There exist polynomials $f_j(x,y,z)$ $(j=0,1,2)$ such that

$$[V_1,V_2]=f_0(x,y,z)V_0+f_1(x,y,z)V_1+f_2(x,y,z)V_2.$$

(Ciii) $\dfrac{\partial h_{22}}{\partial z}$ is a non-zero constant.
(Civ) Define the 3×3 matrix M by

$$M=\begin{pmatrix} px & qy & rz \\ qy & h_{22}(x,y,z) & h_{23}(x,y,z) \\ rz & h_{32}(x,y,z) & h_{33}(x,y,z) \end{pmatrix}. \tag{2}$$

Then $F(x,y,z)$ coincides with $\det(M)$ up to a constant factor.

In particular we have the following facts (cf. [9]):

(D1) There are polynomials $c_j=c_j(x,y,z)$ such that $V_jF=c_jF$ $(j=0,1,2)$. In particular c_0 is a positive constant.

(D2) If V is any vector field logarithmic along $\{F=0\}$ with coefficients in R, then $V\in\sum_{j=0}^{2}RV_j$.

Let Z_F be the hypersurface in $\mathbf{C}^3$ defined by $F=0$ and let S_F be the set of singular points of Z_F. By direct computation, we find that $S_F\cap\{x\neq 0\}$ is smooth. Let S_F^j $(j=1,2,\ldots,k)$ be the totality of irreducible components of S_F. Then it is also easy to see that each S_F^j is a curve. Take a point P of $S_F^j\cap\{x\neq 0\}$. In this section, we review the results on the hypersurface S_F near P given in [10].

Before explaining the result, we recall curves with simple singularities at the origin in $\mathbf{C}^2$;

$$\begin{array}{lcl} A_n & : & u^{n+1}+v^2=0\ (n\geq 1) \\ D_n & : & u(u^{n-2}+v^2)=0\ (n\geq 4) \\ E_6 & : & u^4+v^3=0 \\ E_7 & : & u(u^2+v^3)=0 \\ E_8 & : & u^5+v^3=0 \end{array} \tag{3}$$

If there is a neighbourhood U of P in Z_F which is biholomorphically isomorphic to a neighbourhood T of the origin $(0,0,0)$ in $\{(u,v)\in\mathbf{C}^2 : u^{n+1}+v^2=0\}\times\mathbf{C}$ such that P corresponds to the origin in T_{A_n}, we say that the type of singularity of Z_F along S_F^j is A_n. Similarly, if there is a neighbourhood U of P in Z_F which biholomorphically isomorphic to a neighbourhood T of the origin $(0,0,0)$ in $\{(u,v)\in\mathbf{C}^2 : u(u^{n-2}+v^2)=0\}\times\mathbf{C}$ (resp. $\{(u,v)\in\mathbf{C}^2 : u^4+v^3=0\}\times\mathbf{C}$, $\{(u,v)\in\mathbf{C}^2 : u(u^2+v^3)=0\}\times\mathbf{C}$, $\{(u,v)\in\mathbf{C}^2 : u^5+v^3=0\}\times\mathbf{C}$) such that P corresponds to the origin, we say that the type of singularity of Z_F along S_F^j is D_n (resp. E_6, E_7, E_8).

We now state a theorem in [10].

Theorem 1 *(i) There is a natural bijection between the set of polynomials of (I) and that of corank one subsdiagrams of the Dynkin diagram of type E_6 invariant by its non-trivial symmetry.*

(ii) There is a natural bijection between the set of polynomials of (II) (resp. (III)) and that of corank one subdiagrams of the Dynkin diagram of type E_7 (resp. E_8).

For the seventeen polynomials introduced in this section, the types of singularities are given in the table below.

TABLE 1

F	*Type*
$F_{A,1}$	$A_2 + A_2 + A_1$
$F_{A,2}$	A_5

F	*Type*
$F_{B,1}$	$A_3 + A_2 + A_1$
$F_{B,2}$	$A_5 + A_1$
$F_{B,3}$	D_6
$F_{B,4}$	$D_5 + A_1$
$F_{B,5}$	E_6
$F_{B,6}$	$A_4 + A_2$
$F_{B,7}$	A_6

F	*Type*
$F_{H,1}$	$A_4 + A_2 + A_1$
$F_{H,2}$	$A_4 + A_3$
$F_{H,3}$	$D_5 + A_2$
$F_{H,4}$	D_7
$F_{H,5}$	$E_6 + A_1$
$F_{H,6}$	A_7
$F_{H,7}$	E_7
$F_{H,8}$	$A_6 + A_1$

3 Review on b-functions

We here explain a general method of determining the b-function of an analytic function $f(x)$ (cf. [14]). (Our argument of this section is basically same as a part of [15], §3.)

Let X be a complex manifold of dimension n and let $\mathcal{O}_X$ $(resp.\mathcal{D}_X)$ be the sheaf of germs of holomorphic functions on X (resp. the sheaf of differential operators of finite order with coefficients in $\mathcal{O}_X$). We here restrict our attention to the study of differential equations governing a complex power of a polynomial or a holomorphic function. We put $\mathcal{D}_X[s] = \mathcal{D}_X \otimes \mathbf{C}[s]$ for an indeterminate s that commutes with $\mathcal{D}_X$. We define an Ideal

$$\mathcal{J}_f(s) = \{P(s, x, \partial_x) \in \mathcal{D}_X[s];\ P(s, x, \partial_x)(f(x))^s = 0\}$$

for a holomorphic function $f(x)$ on X and also define the following $\mathcal{D}_X$ (or $\mathcal{D}_X[s]$)-Modules:

$$\begin{array}{lclcl} \mathcal{N} & = & \mathcal{D}_X[s](f(x))^s & \simeq & \mathcal{D}_X[s]/\mathcal{J}_f(s), \\ \mathcal{M} & = & \mathcal{D}_X[s](f(x))^s/\mathcal{D}_X[s](f(x))^{s+1} & \simeq & \mathcal{D}_X[s](f(x))^s/(\mathcal{J}_f(s) + \mathcal{D}_X[s]f(x)), \\ \mathcal{N}_\alpha & = & \mathcal{D}_X/\mathcal{J}_f(\alpha) & & (\alpha \in \mathbf{C}), \end{array}$$

where $\mathcal{J}_f(\alpha) = \{P(\alpha, x, \partial_x) \in \mathcal{D}_X; P(s, x, \partial_x) \in \mathcal{J}_f(s)\}$. Then $\mathcal{N}$, $\mathcal{M}$, $\mathcal{N}_\alpha$ are coherent $\mathcal{D}_X$-Modules.

The b-function of $f(x)$ is, by definition, the monic polynomial $b_f(s)$ of s with the minimal degree such that

$$P(s, x, \partial_x)(f(x))^{s+1} = b_f(s)(f(x))^s$$

for a differential operator $P(s,x,\partial_x) \in \mathcal{D}_X[s]$. It is provable that $\mathcal{N}_\alpha \simeq \mathcal{D}_X(f(x))^\alpha$ if and only if $\alpha \in \mathbf{C}$ satisfies $b_f(\alpha - n) \neq 0$ for all $n \in \mathbf{N}$.

In the sequel, we restrict our attention to the case where $X = \mathbf{C}^n$ with coordinate $x = (x_1, x_2, \ldots, x_n)$ and that $f(x)$ is a weighted homogeneous polynomial of $x_1, x_2, \ldots, x_n$; there are positive rational numbers $r_1, r_2, \ldots, r_n$ $(0 < r_j,\ j = 1, 2, \ldots, n)$ such that $X_0 f = f$, where X_0 is a vector field $X_0 = \sum_{j=1}^n r_j x_j \partial_{x_j}$. Put

$$\tilde{\mathcal{M}} = (s+1)\mathcal{M} \simeq \mathcal{D}_X[s]/(\mathcal{J}_f(s) + \mathcal{D}_X[s](\mathcal{V} + \mathcal{O}_X f)),$$

where $\mathcal{V} = \sum_{i=1}^n \mathcal{O}_X \partial_{x_i} f$. Regarding s an endomorphism of $\tilde{\mathcal{M}}$, we write $\tilde{b}_f(s)$ for the minimal polynomial of s. Then, from the definition, we have

$$b_f(s) = (s+1)\tilde{b}_f(s).$$

¿From the assumption, it follows that $s - X_0 \in \mathcal{J}_f(s)$ and moreover

$$\tilde{\mathcal{M}} = \mathcal{D}_X/(\mathcal{J}_f(0) + \mathcal{D}_X \mathcal{V}),$$

where $\mathcal{J}_f(0) = \mathcal{D}_X \cap \mathcal{J}_f(s)$. We take a regular stratification $X = \bigcup_\alpha X_\alpha$ of H. Whitney such that

$$\tilde{SS}(\tilde{\mathcal{M}}) \subset \bigcup_\alpha T^*_{X_\alpha} X.$$

Then we define $\tilde{b}^k_f(s)$ as the minimal polynomial of the endomorphism s of

$$\bigotimes_{\mathrm{codim} X_\alpha = k} \mathcal{H}om_{\mathcal{D}_X}(\tilde{\mathcal{M}}, \mathcal{B}_{X_\alpha|X})_{x_\alpha} \quad (x_\alpha \in X_\alpha),$$

where $\mathcal{B}_{X_\alpha|X}$ denotes the space of delta functions supported on $X_\alpha \subset X$. We now recall an interdependence between $\tilde{b}_f$ and $\tilde{b}^k_f$, that is,

$$\mathrm{l.c.m}_{2 \le k \le n}(\tilde{b}^k_f) | \tilde{b}_f | \prod_{k=2}^n \tilde{b}^k_f,$$

where l.c.m is an abbreviation of the least common multiple.

There are criterions of finding roots of b-functions. We here explain two of such critrions.

As to $\tilde{b}^n_f(s)$, there is a rather simple criterion of finding roots of $\tilde{b}^n_f(s) = 0$. To be specific, we decompose $\mathcal{F} = \mathcal{H}om_{\mathcal{D}_X}(\tilde{\mathcal{M}}, \mathcal{B}_{pt})_0$ into root subspaces of the endomorphism s, where $\mathcal{B}_{pt}$ is an abbreviation of $\mathcal{B}_{\{0\}|X}$. A homomorphism $1 \mapsto \Delta(x) \in \mathcal{B}_{pt}$ in $\mathcal{F}$ is an eigenvector of s belonging to an eigenvalue β if and only if the following condition holds:

$$\text{(A)} \begin{cases} X_0 \Delta(x) = \beta \Delta(x), \\ Q(x,\partial_x)\Delta(x) = 0 \ \text{for all}\ Q(x,\partial_x) \in \mathcal{J}_f(0), \\ \frac{\partial f}{\partial x_i}\Delta(x) = 0 \ \text{for}\ i = 1, 2, \ldots, n. \end{cases}$$

Then, for a complex number β, $\tilde{b}^n_f(s)$ has the factor $s - \beta$ if and only if there is non-zero $\Delta(x) \in \mathcal{B}_{pt}$ satisfying (A).

On the other hand, if $f(x)$ has an isolated singularity at the origin in addition to weighted homogeneous property, there is a formula of the b-function of $f(x)$. To describe the formula concretely, we write

$$\frac{(t^{r_1}-t)\cdots(t^{r_n}-t)}{(1-t^{r_1})\cdots(1-t^{r_n})}=t^{\alpha_1}+t^{\alpha_2}+\cdots+t^{\alpha_\mu}$$

for some positive rational numbers $\alpha_1,\alpha_2,\ldots,\alpha_\mu$. Then the following theorem holds.

Theorem 2 *(M. Kashiwara, T. Miwa and M. Sato) Under the notation above,*

$$\tilde{b}_f(s)=\prod_j{}'(s+\alpha_j),$$

where $\prod'$ indicates the multiplicity free product.

(This theorem is equivalent to Th. 6.18 in [5].)

As an application of this theorem, it is easy to determine the b-function of the defining polynomial $f(u,v)$ of a curve with a simple singularity at the origin introduced in (3). The result is given in the table below.

TABLE 2

Type	$\tilde{b}_f(s)$
A_n	$\prod_{j=1}^{n}\left(s+\frac{1}{2}+\frac{j}{n+1}\right)$
$D_{2m}(m>1)$	$(s+1)\prod_{j=1}^{m-1}\left(s+\frac{m+j-1}{2m-1}\right)\left(s+\frac{2m+j-1}{2m-1}\right)$
$D_{2m+1}(m>1)$	$(s+1)\prod_{j=1}^{m}\left(s+\frac{2m+2j-1}{4m}\right)\left(s+\frac{4m+2j-1}{4m}\right)$
E_6	$(s+\frac{7}{12})(s+\frac{10}{12})(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{14}{12})(s+\frac{17}{12})$
E_7	$(s+1)(s+\frac{5}{9})(s+\frac{7}{9})(s+\frac{8}{9})(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})$
E_8	$(s+\frac{8}{15})(s+\frac{11}{15})(s+\frac{13}{15})(s+\frac{14}{15})(s+\frac{16}{15})(s+\frac{17}{15})(s+\frac{19}{15})(s+\frac{22}{15})$

In the sequel, $c_Y(s)$ denotes the polynomial $\tilde{b}_f(s)$ for the root system of type Y in the table above.

4 Roots of b-functions of our polynomials

We return to our situation. As before, let $F(x,y,z)$ be a polynomial equal to one of those introduced in §2. Put

$$\begin{aligned}
Z_0 &= \{(x,y,z)\in\mathbf{C}^3;\ F(x,y,z)\neq 0\},\\
Z_1 &= \{(x,y,z)\in\mathbf{C}^3;\ F(x,y,z)=0\}-\{(x,y,z)\in\mathbf{C}^3;\ \partial_xF=0,\partial_yF=0,\partial_zF=0\},\\
Z_2 &= \{(x,y,z)\in\mathbf{C}^3;\ \partial_xF=0,\partial_yF=0,\partial_zF=0,x\neq 0\},\\
Z_3 &= \{(0,0,0)\}.
\end{aligned}$$

Then we easily find that there is a regular stratification $\mathbf{C}^3=\cup_\alpha X_\alpha$ such that the union of strata X_α with $\mathrm{codim}X_\alpha=k$ coincides Z_k. Let $\tilde{b}_F(s)$ and $\tilde{b}_F^k(s)\,(k=2,3)$ be the minimal polynomials for F introduced in the previous section. The purpose of this section is to study the factors of $\tilde{b}_F^k(s)$ $(k=2,3)$.

Remark 1 *In the case of the discriminant of an irreducible finite reflection group, the explicit form of its b-function was conjectured in [15] and solved affirmatively in [8].*

By using the b-function of the defining polynomial of a simple singularity on a curve described in the previous section, we can determine candidates of the roots of $\tilde{b}_F^2(s) = 0$ up to multiplicities. To be specific, let $F = F(x, y, z)$ be one of the seventeen polynomials and let $Y_1, Y_2, \ldots, Y_k$ be the types of irreducible components of singular locus of the hypersurface defined by $F = 0$. Let $\{\beta_1, \beta_2, \ldots, \beta_N\}$ be all the roots of $\prod_{j=1}^k c_{Y_j}(s) = 0$. Assume that $\beta_1, \beta_2, \ldots, \beta_N$ are mutually different. Then we put $\tilde{b}_{F,a}^2(s) = \prod_{i=1}^N (s - \beta_i)$. In virtue of Theorem 1 and TABLE 2, we can easily determine the concrete form of $\tilde{b}_{F,a}^2(s)$. The result is given in the table below:

TABLE 3

F	$\tilde{b}_{F,a}^2(s)$
$F_{A,1}$	$(s+\frac{5}{6})(s+\frac{7}{6})\cdot(s+1)$
$F_{A,2}$	$(s+\frac{2}{3})(s+\frac{5}{6})(s+1)(s+\frac{7}{6})(s+\frac{4}{3})$
$F_{B,1}$	$(s+1)(s+\frac{5}{6})(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{7}{6})$
$F_{B,2}$	$(s+1)(s+\frac{2}{3})(s+\frac{5}{6})(s+\frac{4}{3})(s+\frac{7}{6})$
$F_{B,3}$	$(s+1)(s+\frac{3}{5})(s+\frac{4}{5})(s+\frac{6}{5})(s+\frac{7}{5})$
$F_{B,4}$	$(s+1)(s+\frac{5}{8})(s+\frac{7}{8})(s+\frac{9}{8})(s+\frac{11}{8})$
$F_{B,5}$	$(s+\frac{7}{12})(s+\frac{5}{6})(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{7}{6})(s+\frac{17}{12})$
$F_{B,6}$	$(s+\frac{7}{10})(s+\frac{9}{10})(s+\frac{11}{10})(s+\frac{13}{10})\cdot(s+\frac{5}{6})(s+\frac{7}{6})$
$F_{B,7}$	$(s+\frac{9}{14})(s+\frac{11}{14})(s+\frac{13}{14})(s+\frac{15}{14})(s+\frac{17}{14})(s+\frac{19}{14})$
$F_{H,1}$	$(s+\frac{7}{10})(s+\frac{9}{10})(s+\frac{11}{10})(s+\frac{13}{10})\cdot(s+\frac{5}{6})(s+\frac{7}{6})\cdot(s+1)$
$F_{H,2}$	$(s+\frac{7}{10})(s+\frac{9}{10})(s+\frac{11}{10})(s+\frac{13}{10})\cdot(s+\frac{3}{4})(s+1)(s+\frac{5}{4})$
$F_{H,3}$	$(s+\frac{5}{8})(s+\frac{7}{8})(s+1)(s+\frac{9}{8})(s+\frac{11}{8})\cdot(s+\frac{5}{6})(s+\frac{7}{6})$
$F_{H,4}$	$(s+\frac{7}{12})(s+\frac{3}{4})(s+\frac{11}{12})(s+1)(s+\frac{13}{12})(s+\frac{5}{4})(s+\frac{17}{12})$
$F_{H,5}$	$(s+\frac{7}{12})(s+\frac{5}{6})(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{7}{6})(s+\frac{17}{12})\cdot(s+1)$
$F_{H,6}$	$(s+\frac{5}{8})(s+\frac{3}{4})(s+\frac{7}{8})(s+1)(s+\frac{9}{8})(s+\frac{5}{4})(s+\frac{11}{8})$
$F_{H,7}$	$(s+\frac{5}{9})(s+\frac{7}{9})(s+\frac{8}{9})(s+1)(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})$
$F_{H,8}$	$(s+\frac{9}{14})(s+\frac{11}{14})(s+\frac{13}{14})(s+\frac{15}{14})(s+\frac{17}{14})(s+\frac{19}{14})\cdot(s+1)$

Remark 2 *At this moment, it is not trivial whether $\tilde{b}_{F,a}^2(s)$ is a factor of $\tilde{b}_F^2(s)$ or not. But lalter we observe that they acually coincide.*

In order to determine factors of $\tilde{b}_F^3(s)$, we recall the criterion explained in the previous section, namely what we have to do is to find $\Delta(x, y, z) \in \mathcal{B}_{pt}$ satisfying the condition

$$(\mathrm{A}) \left\{ \begin{array}{l} X_0\Delta(x,y,z) = \beta\Delta(x,y,z), \\ Q(x,y,z,\partial_x,\partial_y,\partial_z)\Delta(x,y,z) = 0 \ \text{ for all } \ Q(x,y,z,\partial_x,\partial_y,\partial_z) \in \mathcal{J}_F(0), \\ \frac{\partial f}{\partial x}\Delta(x) = \frac{\partial f}{\partial y}\Delta(x) = \frac{\partial f}{\partial z}\Delta(x) = 0 \end{array} \right.$$

Here X_0 is the vector field such that $X_0 F = F$.

Unfortunately it is not so easy to determine $\mathcal{J}_F(0)$. For this reason we modify the criterion (A) in the following one:

$$\text{(A')}\ \left\{\begin{array}{l} X_0\Delta(x,y,z) = \beta\Delta(x,y,z), \\ Q(x,y,z,\partial_x,\partial_y,\partial_z)\Delta(x,y,z) = 0 \ \text{for all}\ Q(x,y,z,\partial_x,\partial_y,\partial_z) \in \mathcal{G}_F, \\ \frac{\partial f}{\partial x}\Delta(x) = \frac{\partial f}{\partial y}\Delta(x) = \frac{\partial f}{\partial z}\Delta(x) = 0 \end{array}\right.$$

Here

$$\mathcal{G}_F = \mathcal{D}_{\mathbf{C}^3}(V_1 - \frac{c_1}{c_0}V_0) + \mathcal{D}_{\mathbf{C}^3}(V_2 - \frac{c_2}{c_0}V_0).$$

Since $(V_j - \frac{c_j}{c_0}V_0)F = 0$, it follows that $\mathcal{G}_F \subset \mathcal{J}_F(0)$. As a consequence, if (A) holds for a non-zero $\Delta \in \mathcal{B}_{pt}$ and a rational number β, then (A') also holds for a non-zero $\Delta \in \mathcal{B}_{pt}$ and a rational number β. But it is not trivial whether the converse is true or not. In splite of this fact, we are going to compute the pair (Δ, β) $(\Delta \neq 0)$ such that (A') holds for (Δ, β). Moreover we focus our effort to compute such pairs (Δ, β) that $-2 < \beta < 0$ for simplicity.

Remark 3 *In the first draft of this paper, we used the fact that $\mathcal{G}_F$ coincides with $\mathcal{J}_F$ without proof. The referee pointed out that this is not trivial. For this reason we introduced $\mathcal{G}_F$ and a criterion (A'). The authors thank to the referee for pointing out this error.*

Proposition 1 *Let $\tilde{b}^3_{F,a}(s)$ be the polynomial defined in* **TABLE 4** *below. If β is a root of $\tilde{b}^3_F(s) = 0$ and $-2 < \beta < 0$, so is the root of $\tilde{b}^3_{F,a}(s) = 0$.*

TABLE 4

F	$\tilde{b}^3_{F,a}(s)$
$F_{A,j}$ $(j = 1, 2)$	$(s + \frac{3}{4})(s + \frac{5}{4})$
$F_{B,j}$ $(j = 1, 2, 3, 4)$	$(s + \frac{2}{3})(s + 1)(s + \frac{4}{3})$
$F_{B,j}$ $(j = 5, 6, 7)$	$(s + \frac{2}{3})(s + \frac{4}{3})$
$F_{H,j}$ $(j = 1, 2, \ldots, 8)$	$(s + \frac{3}{5})(s + \frac{2}{3})(s + \frac{4}{5})(s + 1)(s + \frac{6}{5})(s + \frac{4}{3})(s + \frac{7}{5})$

To prove Proposition 1, it suffices to determine pairs (Δ, β) satisfying the conditions that $\Delta(x,y,z) \in \mathcal{B}_{pt}$, $\Delta \neq 0$, $-2 < \beta < 0$ and that the condition (A') holds for (Δ, β). For this purpose, we let $\mathcal{O}_0 = \mathcal{O}_{\mathbf{C}^3,0}$ be the stalk at the orign of the sheaf of holomorphic functions on $\mathbf{C}^3$ with coordinate (x,y,z) and introduce the ring $\mathcal{O}_0[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}]$ and its $\mathcal{O}_0$-ideal $\mathcal{O}_0[\frac{1}{y}, \frac{1}{z}] + \mathcal{O}_0[\frac{1}{x}, \frac{1}{z}] + \mathcal{O}_0[\frac{1}{x}, \frac{1}{y}]$. Then $\mathcal{B}_{pt}$ is identified with the $\mathcal{O}_0$-module

$$\mathcal{O}_0[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}]/(\mathcal{O}_0[\frac{1}{y}, \frac{1}{z}] + \mathcal{O}_0[\frac{1}{x}, \frac{1}{z}] + \mathcal{O}_0[\frac{1}{x}, \frac{1}{y}]).$$

For each positive integers l, m, n, the class in $\mathcal{B}_{pt}$ represented by $\frac{1}{x^l y^m z^n}$ is denote by the same notation for the sake of simplicity. Then it is clear that any element of $\mathcal{B}_{pt}$ is a finite linear combination of $\frac{1}{x^l y^m z^n}$ $(l, m, n > 0)$ with constant coefficients and the operation of partial differential $\partial_x, \partial_y, \partial_z$ and mulitiplication by x, y, z on $\mathcal{B}_{pt}$ are

$$\partial_x\left(\frac{1}{x^l y^m z^n}\right) = \frac{-l}{x^{l+1}y^m z^n}, \partial_y\left(\frac{1}{x^l y^m z^n}\right) = \frac{-m}{x^l y^{m+1} z^n}, \partial_z\left(\frac{1}{x^l y^m z^n}\right) = \frac{-n}{x^l y^m z^{n+1}}$$

$$x\left(\frac{1}{x^l y^m z^n}\right) = \frac{1}{x^{l-1}y^m z^n}, y\left(\frac{1}{x^l y^m z^n}\right) = \frac{1}{x^l y^{m-1} z^n}, z\left(\frac{1}{x^l y^m z^n}\right) = \frac{1}{x^l y^m z^{n-1}}.$$

In particular

$$x\left(\frac{1}{xy^m z^n}\right) = 0, y\left(\frac{1}{x^l y z^n}\right) = 0, z\left(\frac{1}{x^l y z}\right) = 0.$$

By direct computation we find that each of the elements of $\mathcal{B}_{pt}$ below is $\Delta(x, y, z)$ for some β $(-2 < \beta < 0)$.

```
A1
d_{A1,3/4}=1/(x*y*z);
d_{A1,5/4}=1/(8*x^2*y*z^2)+1/(6*x*y^3*z)+1/(x^4*y*z);

A2
d_{A2,3/4}=1/(x*y*z);
d_{A2,3/4}=-5/(2*x^2*y*z^2)+7/(12*x*y^3*z)+1/(x^4*y*z);

B1
d_{B1,2/3}=1/(x*y*z);
d_{B1,1}=1/(4*x^2*y^2*z)+1/(x^4*y*z);
d_{B1,4/3}=-7/(58320*x*y*z^3)+7/(12960*x^2*y^2*z^2)+7/(1080*x^4*y*z^2)+
          7/(1080*x*y^4*z)+7/(144*x^3*y^3*z)+7/(30*x^5*y^2*z)+1/(x^7*y*z);

B2
d_{B2,2/3}=1/(x*y*z);
d_{B2,1}=4/(27*x*y*z^2)-2/(3*x^2*y^2*z)+1/(x^4*y*z);
d_{B2,4/3}=-4/(135*x^2*y^2*z^2)+4/(45*x^4*y*z^2)-8/(45*x*y^4*z)+4/(15*x^3*y^3*z)-
          8/(15*x^5*y^2*z)+1/(x^7*y*z);

B3
d_{B3,2/3}=1/(x*y*z);
d_{B3,1}=3/(50*x*y*z^2)-3/(10*x^2*y^2*z)+1/(x^4*y*z);
d_{B3,4/3}=11/(11250*x*y*z^3)-11/(1500*x^2*y^2*z^2)+11/(375*x^4*y*z^2)+
          11/(250*x^3*y^3*z)-11/(50*x^5*y^2*z)+1/(x^7*y*z);

B4
d_{B4,2/3}=1/(x*y*z);
d_{B4,1}=9/(4*x^2*y^2*z)+1/(x^4*y*z);
d_{B4,4/3}=-187/(720*x*y*z^3)-187/(480*x^2*y^2*z^2)-17/(120*x^4*y*z^2)+
          187/(80*x^3*y^3*z)+17/(10*x^5*y^2*z)+1/(x^7*y*z);

B5
d_{B5,2/3}=1/(x*y*z);
d_{B5,4/3}=-1/(3*x*y*z^3)+1/(6*x^2*y^2*z^2)-1/(45*x^4*y*z^2)+1/(x*y^4*z)-
          1/(9*x^3*y^3*z)+2/(135*x^5*y^2*z)-1/(486*x^7*y*z);

B6
d_{B6,2/3}=1/(x*y*z);
```

```
d_{B6,4/3}=-1729/(10240*x*y*z^3)-1729/(5120*x^2*y^2*z^2)-247/(160*x^4*y*z^2)+
          1729/(10240*x*y^4*z)-247/(640*x^3*y^3*z)-19/(20*x^5*y^2*z)+1/(x^7*y*z);

B7
d_{B7,2/3}=1/(x*y*z);
d_{B7,4/3}=124729/(116640*x*y*z^3)+11339/(38880*x^2*y^2*z^2)-667/(540*x^4*y*z^2)-
          11339/(4320*x*y^4*z)+667/(648*x^3*y^3*z)-29/(45*x^5*y^2*z)+1/(x^7*y*z);

H1
d_{H1,3/5}=1/(x*y*z);
d_{H1,2/3}=1/(x^2*y*z);
d_{H1,4/5}=-(2/(15*x*y^2*z))+1/(x^4*y*z);
d_{H1,1}=1/(x^7*y*z);
d_{H1,6/5}=4/(875*x^2*y^2*z^2)+16/(525*x^5*y*z^2)+8/(2625*x*y^4*z)-4/(525*x^4*y^3*z)+
          2/(105*x^7*y^2*z)+1/(x^10*y*z);
d_{H1,4/3}=32/(19683*x^2*y*z^3)-64/(45927*x*y^3*z^2)+68/(45927*x^4*y^2*z^2)+
          64/(1701*x^7*y*z^2)+8/(5103*x^3*y^4*z)-92/(15309*x^6*y^3*z)+
          2/(81*x^9*y^2*z)+1/(x^12*y*z);
d_{H1,7/5}=1/(625*x^3*y*z^3)-2/(9375*x^2*y^3*z^2)+2/(1875*x^5*y^2*z^2)+
          1/(25*x^8*y*z^2)-8/(5625*x*y^5*z)+2/(1875*x^4*y^4*z)-2/(375*x^7*y^3*z)+
          2/(75*x^10*y^2*z)+1/(x^13*y*z);

H2
d_{H2,3/5}=1/(x*y*z);
d_{H2,2/3}=1/(x^2*y*z);
d_{H2,4/5}=6/(x*y^2*z)+1/(x^4*y*z);
d_{H2,1}=63/(2*x^2*y*z^2)-36/(x*y^3*z)+9/(2*x^4*y^2*z)+1/(x^7*y*z);
d_{H2,6/5}=-(135/(7*x^2*y^2*z^2))+36/(x^5*y*z^2)+594/(7*x*y^4*z)+9/(7*x^4*y^3*z)+
          30/(7*x^7*y^2*z)+1/(x^10*y*z);
d_{H2,4/3}=49894/(729*x^2*y*z^3)-1976/(729*x*y^3*z^2)+247/(9*x^4*y^2*z^2)+
          988/(27*x^7*y*z^2)-1976/(243*x^3*y^4*z)+494/(81*x^6*y^3*z)+
          38/(9*x^9*y^2*z)+1/(x^12*y*z);
d_{H2,7/5}=405/(2*x^3*y*z^3)-243/(4*x^2*y^3*z^2)+216/(5*x^5*y^2*z^2)+
          147/(4*x^8*y*z^2)+1539/(5*x*y^5*z)-189/(20*x^4*y^4*z)+15/(2*x^7*y^3*z)+
          21/(5*x^10*y^2*z)+1/(x^13*y*z);

H3
d_{H3,3/5}=1/(x*y*z);
d_{H3,2/3}=1/(x^2*y*z);
d_{H3,4/5}=7/(300*x*y^2*z)+1/(x^4*y*z);
d_{H3,1}=1/(3200*x^2*y*z^2)+1/(800*x^4*y^2*z)+1/(10*x^7*y*z);
d_{H3,6/5}=299/(28000000*x^2*y^2*z^2)+299/(105000*x^5*y*z^2)-299/(420000000*x*y^4*z)+
          299/(4200000*x^4*y^3*z)+23/(2100*x^7*y^2*z)+1/(x^10*y*z);
d_{H3,4/3}=187/(70543872*x^2*y*z^3)+187/(11757312*x^4*y^2*z^2)+187/(68040*x^7*y*z^2)+
          187/(2449440*x^6*y^3*z)+17/(1620*x^9*y^2*z)+1/(x^12*y*z);
d_{H3,7/5}=7843/(2400000000*x^3*y*z^3)+341/(48000000000*x^2*y^3*z^2)+
          341/(20000000*x^5*y^2*z^2)+217/(80000*x^8*y*z^2)-
          341/(360000000000*x*y^5*z)+341/(2400000000*x^4*y^4*z)+
          31/(400000*x^7*y^3*z)+31/(3000*x^10*y^2*z)+1/(x^13*y*z);

H4
d_{H4,3/5}=1/(x*y*z);
d_{H4,2/3}=1/(x^2*y*z);
d_{H4,4/5}=-26/(75*x*y^2*z)+1/(x^4*y*z);
d_{H4,1}=1/(6*x^2*y*z^2)-1/(6*x^4*y^2*z)+1/(x^7*y*z);
d_{H4,6/5}=-1887/(109375*x^2*y^2*z^2)+444/(4375*x^5*y*z^2)+3774/(546875*x*y^4*z)+
          333/(21875*x^4*y^3*z)-74/(525*x^7*y^2*z)+1/(x^10*y*z);
d_{H4,4/3}=2/(243*x^2*y*z^3)-1/(81*x^4*y^2*z^2)+4/(45*x^7*y*z^2)+2/(135*x^6*y^3*z)-
```

```
          2/(15*x^9*y^2*z)+1/(x^12*y*z);
d_{H4,7/5}=50141/(7031250*x^3*y*z^3)+50141/(70312500*x^2*y^3*z^2)-
          2639/(234375*x^5*y^2*z^2)+637/(7500*x^8*y*z^2)-50141/(87890625*x*y^5*z)-
          2639/(4687500*x^4*y^4*z)+91/(6250*x^7*y^3*z)-49/(375*x^10*y^2*z)+
          1/(x^13*y*z);

H5
d_{H5,3/5}=1/(x*y*z);
d_{H5,2/3}=1/(x^2*y*z);
d_{H5,4/5}=-(52/(25*x*y^2*z))+1/(x^4*y*z);
d_{H5,1}=1/(x^2*y*z^2)+1/(x*y^3*z)-1/(x^4*y^2*z)+1/(x^7*y*z);
d_{H5,6/5}=-6919/(15625*x^2*y^2*z^2)+3256/(4375*x^5*y*z^2)-13838/(78125*x*y^4*z)+
          13838/(21875*x^4*y^3*z)-148/(175*x^7*y^2*z)+1/(x^10*y*z);
d_{H5,4/3}=2/(7*x^2*y*z^3)-3/(7*x^4*y^2*z^2)+24/(35*x^7*y*z^2)-2/(7*x^3*y^4*z)+
          4/(7*x^6*y^3*z)-4/(5*x^9*y^2*z)+1/(x^12*y*z);
d_{H5,7/5}=458983/(1562500*x^3*y*z^3)+458983/(3906250*x^2*y^3*z^2)-
          65569/(156250*x^5*y^2*z^2)+833/(1250*x^8*y*z^2)-
          458983/(19531250*x*y^5*z)-458983/(1562500*x^4*y^4*z)+
          3451/(6250*x^7*y^3*z)-98/(125*x^10*y^2*z)+1/(x^13*y*z);

H6
d_{H6,3/5}=1/(x*y*z);
d_{H6,2/3}=1/(x^2*y*z);
d_{H6,4/5}=-21/(25*x*y^2*z)+1/(x^4*y*z);
d_{H6,1}=-9/(20*x^2*y*z^2)+9/(10*x*y^3*z)-9/(20*x^4*y^2*z)+1/(x^7*y*z);
d_{H6,6/5}=-24219/(218750*x^2*y^2*z^2)-1242/(4375*x^5*y*z^2)-266409/(1093750*x*y^4*z)+
          11799/(43750*x^4*y^3*z)-69/(175*x^7*y^2*z)+1/(x^10*y*z);
d_{H6,4/3}=374/(3645*x^2*y*z^3)+374/(3645*x*y^3*z^2)-34/(135*x^7*y*z^2)-
          187/(1215*x^3*y^4*z)+17/(81*x^6*y^3*z)-17/(45*x^9*y^2*z)+1/(x^12*y*z);
d_{H6,7/5}=32643/(390625*x^3*y*z^3)+293787/(7812500*x^2*y^3*z^2)+
          10881/(625000*x^5*y^2*z^2)-1209/(5000*x^8*y*z^2)+
          1534221/(9765625*x*y^5*z)-402597/(3125000*x^4*y^4*z)+
          1209/(6250*x^7*y^3*z)-93/(250*x^10*y^2*z)+1/(x^13*y*z);

H7
d_{H7,3/5}=1/(x*y*z);
d_{H7,2/3}=1/(x^2*y*z);
d_{H7,4/5}=-11/(25*x*y^2*z)+1/(x^4*y*z);
d_{H7,1}=-1/(15*x^2*y*z^2)+1/(15*x*y^3*z)-1/(5*x^4*y^2*z)+1/(x^7*y*z);
d_{H7,6/5}=551/(62500*x^2*y^2*z^2)-551/(13125*x^5*y*z^2)-551/(78125*x*y^4*z)+
          551/(18750*x^4*y^3*z)-29/(175*x^7*y^2*z)+1/(x^10*y*z);
d_{H7,4/3}=1/(729*x^2*y*z^3)-1/(729*x*y^3*z^2)+1/(162*x^4*y^2*z^2)-
          1/(27*x^7*y*z^2)-1/(243*x^3*y^4*z)+2/(81*x^6*y^3*z)-7/(45*x^9*y^2*z)+
          1/(x^12*y*z);
d_{H7,7/5}=5681/(4687500*x^3*y*z^3)-5681/(5859375*x^2*y^3*z^2)+
          437/(78125*x^5*y^2*z^2)-133/(3750*x^8*y*z^2)+5681/(9765625*x*y^5*z)-
          5681/(1562500*x^4*y^4*z)+437/(18750*x^7*y^3*z)-19/(125*x^10*y^2*z)+
          1/(x^13*y*z);

H8
d_{H8,3/5}=1/(x*y*z);
d_{H8,2/3}=1/(x^2*y*z);
d_{H8,4/5}=-132/(25*x*y^2*z)+1/(x^4*y*z);
d_{H8,1}=9/(x^2*y*z^2)+9/(x*y^3*z)-3/(x^4*y^2*z)+1/(x^7*y*z);
d_{H8,6/5}=-193401/(15625*x^2*y^2*z^2)+27144/(4375*x^5*y*z^2)-
          16632486/(546875*x*y^4*z)+156078/(21875*x^4*y^3*z)-
          468/(175*x^7*y^2*z)+1/(x^10*y*z);
d_{H8,4/3}=249458/(25515*x^2*y*z^3)+997832/(25515*x*y^3*z^2)-
```

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            11339/(945*x^4*y^2*z^2)+5336/(945*x^7*y*z^2)-22678/(1215*x^3*y^4*z)+
            2668/(405*x^6*y^3*z)-116/(45*x^9*y^2*z)+1/(x^12*y*z);
d_{H8,7/5}=140110641/(10937500*x^3*y*z^3)+793960299/(27343750*x^2*y^3*z^2)-
            12860397/(1093750*x^5*y^2*z^2)+6837/(1250*x^8*y*z^2)+
            9200598759/(136718750*x*y^5*z)-187491051/(10937500*x^4*y^4*z)+
            280317/(43750*x^7*y^3*z)-318/(125*x^10*y^2*z)+1/(x^13*y*z);
```

Remark 4 *We now explain the meaning of the classes in $\mathcal{B}_{pt}$ introduced above. Let* `d_{X,r}` *be a class in $\mathcal{B}_{pt}$, where* `X` *is one of* `A3`, `B3`, `H3` *and r is an attached rational number. Then $\mathcal{D}_{\mathbf{C}^3}$*`d_{A1,r}` *is a non-trivial quotient of $\mathcal{D}_{\mathbf{C}^3}/\mathcal{J}_r$, where*

$$\mathcal{J}_r = \mathcal{D}_{\mathbf{C}^3}(V_0 + rc_0) + \mathcal{D}_{\mathbf{C}^3}(V_1 + rc_1) + \mathcal{D}_{\mathbf{C}^3}(V_2 + rc_2).$$

¿From the definition, $\mathcal{D}_{\mathbf{C}^3}F^{-r}$ is also a quotient of $\mathcal{D}_{\mathbf{C}^3}/\mathcal{J}_r$. If $\mathcal{D}_{\mathbf{C}^3}$`d_{A1,r}` *is a quotient of $\mathcal{D}_{\mathbf{C}^3}F^{-r}$, then $-r$ is a root of $\tilde{b}^3_F(s) = 0$.*

5 The Main Theorem

In this section, we first show the b-functions of the seventeen polynomials and then give a relationship between b-functions and $\tilde{b}^2_{F,a}(s), \tilde{b}^3_{F,a}(s)$.

Theorem 3 *The b-functions of the seventeen polynomials are given as follows:*

$$\begin{aligned}
b_{F_{A,1}}(s) &= (s+1)^2(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{5}{6})(s+\tfrac{7}{6})\\
b_{F_{A,2}}(s) &= (s+1)^2(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{5}{6})(s+\tfrac{7}{6})\\
b_{F_{B,1}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{5}{6})(s+\tfrac{7}{6})\\
b_{F_{B,2}}(s) &= (s+1)^3(s+\tfrac{2}{3})^2(s+\tfrac{4}{3})^2(s+\tfrac{5}{6})(s+\tfrac{7}{6})\\
b_{F_{B,3}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})\\
b_{F_{B,4}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{5}{8})(s+\tfrac{7}{8})(s+\tfrac{9}{8})(s+\tfrac{11}{8})\\
b_{F_{B,5}}(s) &= (s+1)(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{7}{12})(s+\tfrac{11}{12})(s+\tfrac{13}{12})(s+\tfrac{17}{12})\\
b_{F_{B,6}}(s) &= (s+1)(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{7}{10})(s+\tfrac{9}{10})(s+\tfrac{11}{10})(s+\tfrac{13}{10})\\
b_{F_{B,7}}(s) &= (s+1)(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{9}{14})(s+\tfrac{11}{14})(s+\tfrac{13}{14})(s+\tfrac{15}{14})(s+\tfrac{17}{14})(s+\tfrac{19}{14})\\
b_{F_{H,1}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{7}{10})\\
&\quad\times(s+\tfrac{9}{10})(s+\tfrac{11}{10})(s+\tfrac{13}{10})\\
b_{F_{H,2}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{7}{10})\\
&\quad\times(s+\tfrac{9}{10})(s+\tfrac{11}{10})(s+\tfrac{13}{10})\\
b_{F_{H,3}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{5}{8})\\
&\quad\times(s+\tfrac{7}{8})(s+\tfrac{9}{8})(s+\tfrac{11}{8})\\
b_{F_{H,4}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{7}{12})\\
&\quad\times(s+\tfrac{11}{12})(s+\tfrac{13}{12})(s+\tfrac{17}{12})\\
b_{F_{H,5}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{5}{6})(s+\tfrac{7}{6})(s+\tfrac{7}{12})\\
&\quad\times(s+\tfrac{11}{12})(s+\tfrac{13}{12})(s+\tfrac{17}{12})\\
b_{F_{H,6}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{4})(s+\tfrac{5}{4})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{5}{8})\\
&\quad\times(s+\tfrac{7}{8})(s+\tfrac{9}{8})(s+\tfrac{11}{8})\\
b_{F_{H,7}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{5}{9})(s+\tfrac{7}{9})(s+\tfrac{8}{9})\\
&\quad\times(s+\tfrac{10}{9})(s+\tfrac{11}{9})(s+\tfrac{13}{9})\\
b_{F_{H,8}}(s) &= (s+1)^3(s+\tfrac{2}{3})(s+\tfrac{4}{3})(s+\tfrac{3}{5})(s+\tfrac{4}{5})(s+\tfrac{6}{5})(s+\tfrac{7}{5})(s+\tfrac{9}{14})(s+\tfrac{11}{14})(s+\tfrac{13}{14})\\
&\quad\times(s+\tfrac{15}{14})(s+\tfrac{17}{14})(s+\tfrac{19}{14})
\end{aligned}$$

This theorem is proved as follows. We first compute $b_F(s)$ by using Oaku's algorithm [7]. (cf. As to its improved version, see [6]). These algorithms use the Groebner basis method in the ring of differential operators. The result is obtained by using the computer algebra system "Risa/Asir".

Theorem 4 *Let $F(x,y,z)$ be one of the seventeen polynomials and let $b_F(s)$ be its b-function. Then $b_F(s) = (s+1)\tilde{b}^2_{F,a}(s)\tilde{b}^3_{F,a}(s)$.*

We can check case by case the claim of the theorem by comparing $b_F(s)$ with $(s+1)\tilde{b}^2_{F,a}(s)\tilde{b}^3_{F,a}(s)$.

Remark 5 *(1) There is a duality for the roots of the b-functions of the seventeen polynomials. Let $F(x,y,z)$ be one of the seventeen polynomials and let $b_F(s)$ be its b-function. Write $b_F(s) = \prod_{i=1}^{m}(s+\alpha_i)$, where $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$. Then $\alpha_i + \alpha_{m-i+1} = 2$ for all i. As a consequence, $\alpha_i < 2$ for all i.*

(2) In Proposition 1, we introduced a polynomial $\tilde{b}^3_{F,a}(s) = 0$ related with $\tilde{b}^3_F(s)$. Noting (1) and Theorem 4, we find that $\tilde{b}^3_F(s) = \tilde{b}^3_{F,a}(s)$. Then we also conclude that $\tilde{b}^2_F(s) = \tilde{b}^2_{F,a}(s)$.

(3) The second author obtained several polynomials which define Saito free divisors and have nice properties as the seventeen polynomials treated in this paper (cf. [11], [12], [2]). They are regarded as spaces of 1-parameter deformations of eight members of the exceptional families of isolated singularities in the sense of V. Arnol'd. It is possible to compute b-functions of such polynomials by Risa/Asir. We will discuss this topic elsewhere.

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