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Groups that can be the union of n proper normal subgroups [1]

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Abstract

In this paper groups that can be the union of n proper normal subgroups are investigated and the following results are obtained:

(1) Let p be a prime number. A group G can be the union of $p+1$ proper normal subgroups but can not be the union of i proper normal subgroups with $i \leq p$, if and only if G is homomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ but not homomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$ for some prime number k strictly smaller than p.

(2) Suppose that a group G can be the union of n proper normal subgroups but not the union of i proper normal subgroups ($i < n$). Then $n-1$ is a prime and G is homomorphic to $\mathbb{Z}_{n-1} \times \mathbb{Z}_{n-1}$.

(3) Suppose that a group G can be the union of $p+1$ proper normal subgroups $G_1, G_2, \cdots, G_{p+1}$ but can not be the union of i proper normal subgroups for $i \leq p$. Then $G/G_1 \cap G_2 \cap \cdots \cap G_{p+1} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

1 Introduction

As we know, a group can not be the union of two proper subgroups, however, according to [1] or [2] we see that a group can be the union of three proper subgroups. Finite groups that can be the union of 4,5,6 proper subgroups are characterized in [3], however, Tomkinson [4] proved that there is no group that can be the union of 7 proper subgroups. Cohn [3] showed that for any prime power p^a there exists solvable group G that can be the union of $p^a + 1$ proper subgroups. But, how about the non-solvable groups? The results for non-solvable groups seems to be totally different from solvable ones. In fact, it is difficult for us to determine the group that can be the union of given number proper subgroups. But according to [1] or [2] a group that can be the union of three proper subgroups is also the union of three proper normal subgroups. It is natural to ask what kind of group can be expressed as the union of given number of normal subgroups. In fact, [5] has given the result in a constructive way, but it is somewhat regrettable that we ignore the character of the homomorphic kernel in [5]. In this paper we give a description of the group that can be the union of proper normal subgroups and give the homomorphic kernel in rather a clear way.

All notations used are standard.

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2 Auxiliary lemmas

In this section we collect some lemmas that will be used to prove the main results in the next section.

Lemma 2.1 Suppose that a group G can be the union of n proper subgroups $G_1, G_2, \cdots$, G_n but G can not be the union of i proper normal subgroups ($i \leq n$). Then

$$G_1 \cap G_2 \cap \cdots \cap G_n = G_2 \cap G_3 \cap \cdots \cap G_n = G_1 \cap G_3 \cap \cdots \cap G_n = \cdots = G_1 \cap G_2 \cap \cdots \cap G_{n-1}$$

Proof. It is enough to prove the first equality. For an element $g_1 \in G_1 \setminus (G_2 \cup G_3 \cup \cdots \cup G_n)$, and an arbitrary element $x \in G_2 \cap G_3 \cap \cdots \cap G_n$, we see that $g_1 x \in G_1$, hence $x \in G_1$, which implies that,

$$G_1 \cap G_2 \cap \cdots \cap G_n = G_2 \cap G_3 \cap \cdots \cap G_n.$$

This concludes the lemma. $\square$

Lemma 2.2 $\mathbb{Z}_p \times \mathbb{Z}_p$ can be the union of $p+1$ proper normal subgroups,where p is a prime.

Proof. As we know, $\mathbb{Z}_p \times \mathbb{Z}_p$ have only $p+1$ proper normal subgroups, and each one is not contained in the union of other p proper normal groups. $\square$

3 Main results

Now we discuss the group that can be the union of n proper normal subgroups, first we give the following Theorem 3.1 which essentially belongs to [5] although our proof seems to be slightly simpler than the one given there.

Theorem 3.1 Suppose that a group G can be the union of $p+1$ proper normal subgroups but can not be the union of i proper normal subgroups for $i \leq p$. Then p is a prime and G is homomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ but not homomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$, where k is a prime number strictly smaller than p.

Proof. By [5], we may assume that $p > 2$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_{p+1}$, and $H = G_1 \cap G_2 \cap \cdots \cap G_{p+1}$.

We can assume that $H = G_1 \cap G_2 \cap \cdots \cap G_{p+1} = 1$, otherwise, if $H \neq 1$,since G_i are normal in G, we can consider G/H.

Further, we can also assume that each G_i is a maximal normal subgroup, otherwise, if some G_i is not maximal normal, there will be a maximal normal subgroup $M_i > G_i$, we may substitute M_i for G_i, in this way, G can always be the union of $p+1$ maximal normal subgroups.

According to Lemma 2.1 we can get the following equations:

$$G_1 \cap G_2 \cap \cdots \cap G_{p+1} = G_2 \cap G_3 \cap \cdots \cap G_{p+1} = G_1 \cap G_3 \cap \cdots \cap G_{p+1} = \cdots = G_1 \cap G_2 \cap \cdots \cap G_p = 1$$

Let Ω be the set of intersections of some G_i $(i = 1, 2, \cdots, p+1)$, we choose $N \in \Omega$ the intersection of $G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_r}$ with r maximal and $N \neq 1$, and without loss of generality we can assume these r maximal normal subgroups are $G_1, G_2, \cdots, G_r$ and

$$N = G_1 \cap G_2 \cap \cdots \cap G_r.$$

It is clear that $r \leq p-1$, so we can take two distinct subgroups $G_m, G_n(m, n \geq r+1)$. By the maximality of G_m, one has $G = G_m N$. By the maximality of r, one has that $G_m \cap N = 1$, hence $G = G_m \times N$, by the same reason,it follows that $G = G_n \times N$,therefore $G/G_m \cong N \cong G/G_n$. Again, by the maximal normality of G_m, N is simple.

It is clear that the element of N computes the element of G_m and G_n, so the element of N computes the element of $G = G_m G_n$, that is to say $N \leq Z(G)$, thus N is an abelian group.

Therefore N is a cyclic group of prime order, let $|N| = q$, then $|G/G_m| = |G/G_n| = q$, and hence we can get

$$G/G_m \cap G_n \cong G/G_m \times G/G_n \cong \mathbb{Z}_q \times \mathbb{Z}_q.$$

By Lemma 2.2, G can be the union of $q+1$ proper normal subgroups, according to our condition in the Theorem 3.1, we have $q \geq p$, hence it is enough to show that $q \leq p$.

Take $x \in G \setminus (G_1 \cup G_2 \cup \cdots \cup G_r)$, for any $n \in N$, we have $nx^{-1} \in G_i$ $(i \geq r+1)$, so

$$N \subseteq (G_{r+1} \cup G_{r+2} \cup \cdots \cup G_{p+1})x = G_{r+1}x \cup G_{r+2}x \cup \cdots \cup G_{p+1}x.$$

For any $n \in N$, we claim that n belongs to exact one $G_i x$ $(i \geq r+1)$. Otherwise, suppose that $n_1, n_2 \in G_j x$ $(j \geq r+1)$, then there exist $g_{j1}, g_{j2} \in G_j$ such that $n_1 = g_{j1}x, n_2 = g_{j2}x$, thus

$$n_1 n_2^{-1} = g_{j1} g_{j2}^{-1} \in N \cap G_j = 1.$$

now we obtained that $q \leq p+1-r \leq p$ and $q = p$, therefore G is homomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

Now we will show that G is not homomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$ for k a prime strictly smaller than p. Otherwise, by Lemma 2.2, G can be the union of $k+1$ proper normal subgroups, that leads to a contradiction. □

Now we proved the Theorem 3.1.

Theorem 3.2 Suppose that group G can be the union of n proper normal subgroups but not be the union of i proper normal subgroups($i < n$).Then $n-1$ is a prime and G is homomorphism to $\mathbb{Z}_{n-1} \times \mathbb{Z}_{n-1}$.

Proof. Use the similar method of Theorem 3.1, substitute the number n for the number $p+1$. □

The following theorem obviously follows from Theorem 3.2.

Theorem 3.3 Suppose that G is homomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ but not homomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$. Then group G can be the union of $p+1$ proper normal subgroups but can not be the union of i proper normal subgroups, where, $i \leq p, k \leq p-1$, k and p are primes.

Proof. Let f be a homomorphic from G to $\mathbb{Z}_p\times\mathbb{Z}_p$, by Lemma 2.2, $\mathbb{Z}_p\times\mathbb{Z}_p$ can be the union of $T_1, T_2, \cdots$, and T_{p+1}, where, T_i are proper normal subgroups, let $G_i = f^{-1}(T_i)$, it is clear that

$$G = G_1 \cup G_2 \cup \cdots \cup G_{p+1}.$$

We assert that G can not be the union of i proper normal subgroups($i \leq p$),otherwise,suppose that G can be the union of i normal subgroups($i \leq p$), obviously, there may be many desired i, but we can always pick the minimal one, say s, by Theorem 3.2, $s-1$ is a prime and G is homomorphic to $\mathbb{Z}_{s-1}\times\mathbb{Z}_{s-1}$, but $s-1 \leq p-1$, which contracts to the condition of Theorem 3.3. This is the end of the proof. $\square$

Theorem 3.4 A group G can be the union of $p+1$ proper normal subgroups but can not be the union of i proper normal subgroups if and only if G is homomorphic to $\mathbb{Z}_p\times\mathbb{Z}_p$ but not homomorphic to $\mathbb{Z}_k \times \mathbb{Z}_k$, where, $i \leq p, k \leq p-1$, k and p are primes.

Proof. It follows from Theorem 3.1 and Theorem 3.3 $\square$

Up to now, we only know a group can be the union of $p+1$ proper normal subgroups is homomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, but what on earth the homomorphic kernel could be? Or in another word, what the relationship between the kernel and the G_i's and can me give the kernel in a elegant(or uniform) way? Now we will give the relationship in a uniform way, at first, we give the following theorem.

Theorem 3.5 Suppose that a finite group that can be the union of $p+1$ proper normal subgroups $G_1, G_2, \cdots, G_{p+1}$. Then

$$G_1 \cap G_2 \cap \cdots \cap G_{p+1} = G_i \cap G_j$$

where $i \neq j \in \{1, 2, \cdots, p+1\}$, and

$$G/G_1 \cap G_2 \cap \cdots \cap G_{p+1} \cong G/G_1 \cap G_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

Proof. Obviously, from Theorem 3.4, we can infer that $[G : G_i] = p, i \in \{1, 2, \cdots, p+1\}$. So

$$\begin{aligned}
(1-\frac{1}{p})|G| &= |(G_2 \cup G_3 \cup \cdots \cup G_{p+1}) \setminus G_1| \\
&\leq |G_2 \setminus G_1| + |G_3 \setminus G_1| + \cdots + |G_{p+1} \setminus G_1| \\
&= |G_2| - |G_2 \cap G_1| + |G_3| - |G_3 \cap G_1| + \cdots + |G_{p+1}| - |G_{p+1} \cap G_1| \\
&= \frac{|G|}{p} - \frac{|G|}{p^2} + \frac{|G|}{p} - \frac{|G|}{p^2} + \cdots + \frac{|G|}{p} - \frac{|G|}{p^2} \\
&= (1-\frac{1}{p})|G|
\end{aligned}$$

the equality occurs,thus $(G_i \setminus G_1) \cap (G_j \setminus G_1) = \phi$ and $G_i \cap G_j \leq G_1$, for $i \neq j \in \{2, 3, \cdots, p+1\}$, it follows that $G_i \cap G_j \leq G_1 \cap G_i$,for $i \in \{2, 3, \cdots, p+1\}$.

But $|G_i \cap G_j| = |G_1 \cap G_i|$, so $G_i \cap G_j = G_1 \cap G_i$, in same way, we can obtain $G_i \cap G_j = G_m \cap G_n$, for distinct $i, j, m, n \in \{1, 2, \cdots, p+1\}$. This concludes that $G_i \cap G_j \leq G_1 \cap G_2 \cap \cdots \cap G_{p+1}$ and completes the proof. □

Remark If we denote H_i={The intersection of i distinct proper normal subgroups}.Then from the proof of above theorem, we have

$$H_{p+1} = H_p = \cdots = H_2.$$

Theorem 3.6 Suppose that a group that can be the union of $p+1$ proper normal subgroups $G_1, G_2, \cdots$,andG_{p+1}. Then

$$G/G_1 \cap G_2 \cap \cdots \cap G_{p+1} \cong G/G_1 \cap G_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

Proof. Let $H = G_1 \cap G_2 \cap \cdots \cap G_{p+1}$. Obviously, $G/H = G_1/H \cup G_2/H \cup \cdots G_{p+1}/H$, so G/H can be the union of $p+1$ proper normal subgroups, furthermore, from

$$G/H \lesssim G/G_1 \times G/G_2 \times \cdots \times G/G_{p+1}$$

G/H is a finite group. According to Theorem 3.5 ,

$$(G/H)/(G_1/H) \cap (G_2/H) \cap \cdots \cap (G_{p+1}/H) \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

But$(G_1/H) \cap (G_2/H) \cap \cdots \cap (G_{p+1}/H) = H/H = 1$, so $G/H \cong \mathbb{Z}_p \times \mathbb{Z}_p$. This completes the proof. □

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