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Systems of uniformization equations related with dihedral groups[1]

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Abstract

The purpose of this paper is to study systems of uniformization equations singular along Saito free divisors which have solutions expressed in terms of hyperelliptic integrals. Saito free divisors treated here are defined by use of the discriminants of dihedral groups of order $4m$. We construct fundamental solutions of the above mentioned system by means of Gaussian hypergeometric functions in addition to a solution expressed by the hyperelliptic integral. In the last section, we discuss the cases $m = 2$ and $m = 3$ in detail.

1 Introduction

The notion of systems of uniformization equations singular along Saito free divisors was introduced by K. Saito about thirty years ago (cf. [1]). We call a divisor in $\mathbb{C}^n$ Saito free if and only if the $\mathcal{O}_{\mathbb{C}^n}$-module of its logarithmically tangent vector fields is free. In spite of its interest, such systems are not studied well and there are many problems on this theory to be done. The purpose of this paper is to study systems of uniformization equations singular along Saito free divisors which have solutions expressed in terms of hyperelliptic integrals. The same question was treated in [9] for two divisors defined by weighted homogeneous polynomials in three variables. One is defined by the discriminant of a dihedral group of order $2(2m+1)$. The other is the discriminant of the reflection group of type H_3. In the former case, we construct fundamental solutions by means of Gaussian hypergeometric functions in addition to a solution expressed by the hyperelliptic integral.

In this paper, we treat the case of a Saito free divisor in $\mathbb{C}^3$ constructed from the discriminant of a dihedral group of order $4m$ in a natural way. The argument of this paper is similar to that in [9]. The main result of this paper is to define a function of three variables which is expressed by the hyperelliptic integral and determine the system of uniformization equations which governs the function in question (cf. Theorem 1).

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We now briefly explain the contents of this paper. In section 2, we introduce a polynomial, denoted by $\sigma_n(t;u,v)$ of t whose coefficients are polynomials of u,v. We show that coefficients of $\sigma_n(t;u,v)$ are invariant by the action of a dihedral group of order $2n$ and imply a generating function of $\sigma_n(t;u,v)$. In section 3, we introduce a polynomial $\delta_m(x_1,x_2,x_3)$ of x_1,x_2,x_3 which is defined as the pull-back of the discriminant of $\sigma_{2m}(t;u,v)$ by a certain polynomial mapping. We show that the hypersurface $\delta_m(x_1,x_2,x_3)=0$ is a Saito free divisor in $\mathbb{C}^3$. In section 4, after a brief survey on a result by K. Saito on a system of uniformizations equations along the Saito free divisor in $\mathbb{C}^3$ defined as the zero set of the discriminant of type A_3, we introduce a polynomial $P_m(t;x_1,x_2,x_3)$ of t whose coefficients are polynomials of x_1,x_2,x_3 which is constructed by $\sigma_{2m}(t;u,v)$. Then we define a function $v_m(x_1,x_2,x_3)$ by

$$v_m(x_1,x_2,x_3)=\int_{\infty}^{x_1} P_m(t;x_1,x_2,x_3)^{-1/2}dt.$$

Since $2m$ is the degree of $P_m(t;x_1,x_2,x_3)$, $v_m(x_1,x_2,x_3)$ is regarded as a hyperelliptic integral. The purpose of this paper is to obtain a system of differential equations which governs $v_m(x_1,x_2,x_3)$ (cf. Theorem 1). The system in question has three linearly independent solutions outside the set $\delta_m=0$ and is regarded as a system of uniformization equations in the sense of Saito (cf. [1], §3). We also construct fundamental solutions of the system by means of Gaussian hypergeometric functions in addition to v_m (cf. Theorem 2). In section 5, we study the function v_m for the cases $m=2,3$ in detail. In the both cases, v_m is reduced to an elliptic integral.

2 Preliminaries on dihedral groups

In this section, we introduce polynomials whose coefficients are invariant polynomials of dihedral groups and study their elementary properties.

We start with introducing the polynomial of t defined by

$$\sigma_n(t;u,v)=\prod_{k=0}^{n-1}(t-(\varepsilon^k u-\varepsilon^{-k}v)) \tag{1}$$

where $\varepsilon=e^{2\pi i/n}$. It follows from the definition that

$$\begin{aligned}\sigma_n(t;-v,-u) &= \sigma_n(t;u,v)\\ \sigma_n(t;-\varepsilon v,-\varepsilon^{-1}u) &= \sigma_n(t;u,v)\end{aligned}$$

On the other hand, the group G generated by the linear transformations

$$\begin{aligned}(u,v) &\to (-v,-u)\\ (u,v) &\to (-\varepsilon v,-\varepsilon^{-1}u)\end{aligned}$$

is isomorphic to the dihedral group of order $2n$. If n is odd,

$$L=uv,\quad M=(u^n-v^n) \tag{2}$$

are basic invariants of the ring of G-invariant polynomials of u, v. On the other hand, if n is even,

$$L = uv, \quad M = (u^{n/2} - v^{n/2})^2 \tag{3}$$

are basic invariants of the ring of G-invariant polynomials of u, v. In particular,

$$\sigma_n(0; u, v) = -M$$

and the coefficients of $\sigma_n(t; u, v)$ are polynomials of L, M.

Since by virtue of the degree condition, coefficients of $\sigma_n(t; u, v) + M$ are polynomials of L and independent of M, we put

$$\chi_n(t; L) = \sigma_n(t; u, v) + M. \tag{4}$$

It is easy to see that

$$\chi_n(t; L) = \prod_{k=0}^{n-1} (t - (\varepsilon^k - \varepsilon^{-k})\sqrt{L}). \tag{5}$$

To construct a generating function of $\chi_n(t; L)$, we put

$$\Phi(z, t; L) = \frac{1}{1 - tz - Lz^2} \tag{6}$$

and define $\tau_n(t; L)$ $(n = 1, 2, \ldots)$ by

$$\Phi(z, t; L) = \sum_{n=0}^{\infty} \tau_{n+1}(t; L) z^n \tag{7}$$

Lemma 1 *The following equations hold.*

$$\left\{(t^2 + 4L)\tfrac{\partial^2}{\partial t^2} + 3t\tfrac{\partial}{\partial t}\right\}\Phi \quad = \quad \left(z^2\tfrac{\partial^2}{\partial z^2} + 3z\tfrac{\partial}{\partial z}\right)\Phi \tag{8}$$

$$\left\{(t^2 + 4L)\tfrac{d^2}{dt^2} + 3t\tfrac{d}{dt} - n(n+2)\right\}\tau_{n+1} \quad = \quad 0 \tag{9}$$

$$\tau_{n+1}(t; L) = \begin{cases} (m+1)L^m t F\left(m+2, -m, \frac{3}{2}; -\frac{t^2}{4L}\right) & (n = 2m+1) \\ L^m F\left(m+1, -m, \frac{1}{2}; -\frac{t^2}{4L}\right) & (n = 2m) \end{cases} \tag{10}$$

The proof of this lemma is straightforward.

From Lemma 1, we have the following lemma whose proof is similar to that of Lemma 1 in [9].

Lemma 2 *The polynomial $\chi_n(t; L)$ is given by*

$$\chi_n(t; L) = n \int_0^t \tau_n(t; L) dt. \tag{11}$$

As a consequence of the above lemmas, we find that

$$\chi_{n+1}(t;L) = \begin{cases} 2L^{m+1}F\left(m+1,-m-1,\frac{1}{2};-\frac{t^2}{4L}\right) - 2L^{m+1} & (n=2m+1) \\ (2m+1)L^m tF\left(m+1,-m,\frac{3}{2};-\frac{t^2}{4L}\right) & (n=2m) \end{cases} \tag{12}$$

The following lemma will be used later.

Lemma 3 *We have the following formulas.*

$$\tau_{2m}(t;L) + L\tau_{2m-2}(t;L) - \chi_{2m-1}(t;L) = 0 \quad (m=1,2,3,\ldots) \tag{13}$$

$$\tau_{2m-1}(t;L)\chi_{2m-1}(t;L) - \tau_{2m-2}(t;L)\{\chi_{2m}(t;L) + 2L^m\} = L^{2m-2}t \quad (m=1,2,3,\ldots) \tag{14}$$

Proof. The identity (13) is an easy consequence of (10) and (12).
It follows from (10) and (12) that the identity (14) is a consequence of

$$(2m+1)F(m+1,-m;\frac{3}{2};X)F(m+1,-m;\frac{1}{2};X)-$$

$$2mF(m+1,-m+1;\frac{3}{2};X)F(m+1,-m-1;\frac{1}{2};X) = 1, \tag{15}$$

which is shown by an elementary but a little tedious computation. □

The discriminant $\tilde{\Delta}_n(L,M)$ of $\sigma_n(t;L) = \chi_n(t;L) - M$ as a polynomial of t is expressed as follows: In case n is odd, then $\tilde{\Delta}_n(L,M) = n^n(M^2 + 4L^n)^{(n-1)/2}$. In case $n = 4m$, then $\tilde{\Delta}_n(L,M) = -n^n M^{n/2-1}(M + 4L^{n/2})^{n/2}$. In case $n = 4m+2$, then $\tilde{\Delta}_n(L,M) = -n^n M^{n/2}(M + 4L^{n/2})^{n/2-1}$.

3 Saito free divisors constructed by one-parameter deformation of the polynomial $y^{2m} + z^4$

From now on we assume that n is even. The odd n case is discussed in [9]. So we put $n = 2m$ with a positive integer m. The hypersurface $\tilde{\Delta}_n(L,M) = 0$ coincides with $M(M + 4L^m) = 0$ in this case. Since $M(M + 4L^m) = (M + 2L^m)^2 - 4L^{2m}$, we put $M' = M + 2L^m$ and define $\Delta_m(L,M') = M'^2 - 4L^{2m}$ and $\tilde{\sigma}_{2m}(t;L,M') = \chi_{2m}(t;L) - (M' - 2L^m)$. Introduce vector fields W_0, W_2 by

$$\begin{cases} W_0 &= 2L\partial_L + 2mM'\partial_{M'} \\ W_2 &= 2M'\partial_L + 8mL^{2m-1}\partial_{M'} \end{cases} \tag{16}$$

Both W_0, W_2 are logarithmic along $\Delta_m = 0$, that is, we have

$$W_0\Delta_m = 4m\Delta_m, \quad W_2\Delta_m = 0.$$

By direct computation we have

$$W_2\tilde{\sigma}_{2m}(t;L,M') = 4mM'\tau_{2m-1}(t;L) - 8mL^{2m-1} \tag{17}$$

Lemma 4

$$\begin{array}{rl} & \{W_2^2 + 4(m-1)^2 L^{2m-2}\}\tilde{\sigma}_{2m}^{-1/2} \\ = & \partial_t[\{8mL^{2m-1}\tau_{2m} - 4m(M'^2 - 2L^{2m})\tau_{2m-2} - 4mtL^{2m-2}M'\}\tilde{\sigma}_{2m}^{-3/2} - 4(m-1)tL^{2m-2}\tilde{\sigma}_{2m}^{-1/2}] \end{array}$$

This lemma is shown by direct computation. We omit its proof (cf. [9]).

Our concern is to introduce a Saito free divisor in $\mathbb{C}^3$ defined as the pull-back of the hypersurface $\Delta_m(L, M') = 0$ of $\mathbb{C}^2$ by a map of $\mathbb{C}^3$ to the (L, M')-space. For this purpose, we define a map of (x_1, x_2, x_3)-space to the (L, M')-space as follows. First we put

$$\left\{ \begin{array}{rcl} \lambda_m(x_1, x_2, x_3) & = & x_2 \\ \mu_m(x_1, x_2, x_3) & = & \chi_{2m}(x_1; x_2) + 2x_2^m - x_3^2. \end{array} \right. \tag{18}$$

Then

$$(x_1, x_2, x_3) \to (\lambda_m, \mu_m)$$

defines a map of (x_1, x_2, x_3)-space to the (L, M')-space. As a pull-back of $\Delta_m(L, M')$ by this map, we introduce a polynomial

$$\delta_m(x) = \mu_m(x)^2 - 4\lambda_m(x)^{2m}. \tag{19}$$

We are going to show that the hypersurface S_m of the (x_1, x_2, x_3)-space defined by $\delta_m(x) = 0$ is a Saito free divisor.

For this purpose, we construct three vector fields which are tangent to S_m. It is clear from the definition that both $\partial_{x_3}\delta_m, \partial_{x_1}\delta_m$ are divisible by $\mu_m(x)$. So we put

$$g_{21}(x) = \frac{1}{\mu_m(x)}\partial_{x_3}\delta_m, \quad g_{23}(x) = -\frac{1}{\mu_m(x)}\partial_{x_1}\delta_m, \tag{20}$$

and define vectors

$$\begin{array}{rcl} \vec{p_1} & = & (x_1, 2x_2, mx_3) \\ \vec{p_2} & = & (g_{21}(x), 0, g_{23}(x)). \end{array} \tag{21}$$

Note that $g_{21}(x) = -4x_3$.

We define $g_{31}(x),\ g_{32}(x),\ g_{33}(x)$ by

$$\vec{p_3} = (g_{31}(x), g_{32}(x), g_{33}(x)) = \frac{1}{x_3}\{(0, \partial_{x_3}\delta_m, -\partial_{x_2}\delta_m) - \chi_{2m-1}(x_1; x_2)\vec{p_2}\} \tag{22}$$

Note that $g_{31}(x), g_{32}(x), g_{33}(x)$ are polynomials of $x = (x_1, x_2, x_3)$.

Remark 1 *From the definition, $\vec{p_1}$ is the coefficient vector of the Euler vector field associated to the weighted homogeneous polynomial $\mu_m(x_1, x_2, x_3)$ and $\vec{p_2}$ (resp. $\vec{p_3}$) is the coefficient vector of a kind of Hamiltonian vector field on (x_1, x_3)-space (resp. (x_2, x_3)-space).*

Using the vectors $\vec{p_1}, \vec{p_2}, \vec{p_3}$, we define a matrix M_{δ_m} by

$$M_{\delta_m} = \begin{pmatrix} x_1 & 2x_2 & mx_3 \\ -4x_3 & 0 & g_{23}(x) \\ g_{31}(x) & g_{32}(x) & g_{33}(x) \end{pmatrix} \tag{23}$$

Introduce vector fields V_0, V_1, V_2 by

$$^t(V_0, V_1, V_2) = M_{\delta_m}{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$$

Then by direct computation we obtain

$$[V_0, V_1] = (m-1)V_1, \quad [V_0, V_2] = 2(m-1)V_2, \quad [V_1, V_2] = 4(m-1)\tau_{2m-1}(x_1; x_2)V_1, \tag{24}$$

$$V_0\delta_m = 4m\delta_m, \quad V_1\delta_m = V_2\delta_m = 0, \tag{25}$$

$$\det(M_{\delta_m}) = -16m\delta_m. \tag{26}$$

These imply that the hypersurface S_m in $\mathbb{C}^3$ is actually a Saito free divisor by virtue of (1.9) in [2].

4 A system of uniformization equations singular along S_m and its solutions

In this section, we first give a survey on the result by K. Saito concerning a system of uniformization equations singular along the divisor defined by the discriminant of the Weyl group of type A_3. (For the details, see [1], [3].) This is a model of [9] and this paper. Next we introduce a system of uniformization equations along S_m and construct solutions by hyperelliptic integrals and Gaussian hypergeometric functions.

4.1 The case of A_3. A prototype

The discriminant $\Delta(A_3)$ of the polynomial of t defined by $f(t) = t^4 + x_1t^2 + x_2t + x_3$ coincides with the determinant of the matrix

$$M_{\Delta(A_3)} = \begin{pmatrix} 2x_1 & 3x_2 & 4x_3 \\ 3x_2 & -x_1^2 + 4x_3 & -\frac{1}{2}x_1x_2 \\ 4x_3 & -\frac{1}{2}x_1x_2 & \frac{1}{4}(-3x_2^2 + 8x_1x_3) \end{pmatrix}$$

up to a constant factor. It is easy to see that $\Delta(A_3)$ defines a Saito free divisor by virtue of (1.9) in [2]. This is shown as follows. Let V_0, V_1, V_2 be vector fields defined by

$$^t(V_0, V_1, V_2) = M_{\Delta(A_3)}{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Then by direct computation, we have

$$V_0\Delta(A_3) = 12\Delta(A_3), \quad V_1\Delta(A_3) = 0, \quad V_2\Delta(A_3) = 2x_1\Delta(A_3).$$

and these imply that $\Delta(A_3) = 0$ is Saito free.

The system of differential equations

$$\begin{cases} V_0u &= -u \\ V_1V_1u &= 0 \\ V_1V_2u &= -\frac{1}{2}x_2u - \frac{1}{2}x_1V_1u \\ V_2V_2u &= -x_3u - \frac{x_2}{4}V_1u \end{cases} \tag{27}$$

is introduced in [1]. This is an example of systems of uniformization equations singular along $\Delta(A_3)=0$. We will construct solutions of (27) after K. Saito. We first introduce polynomials

$$L(x)=16x_3+\frac{4}{3}x_1^2,\quad M(x)=\frac{8}{3}x_1^3+36x_2^2-96x_1x_3.$$

It is easy to see that

$$16\Delta(A_3)=L(x)^3-\frac{1}{3}M(x)^2,$$

$$V_0L=4L,\ V_0M=6M,\ V_1L=V_1M=0,\ V_2'L=-\frac{1}{3}M,\ V_2'M=-\frac{3}{2}L^2.$$

Here we put

$$V_2'=V_2-\frac{1}{6}x_1V_0.$$

Let

$$P(t)=4t^3-L(x)t+\frac{1}{9}M(x)$$

be a cubic polynomial of t. Then the following formulas are easy to show:

$$P(-\frac{2}{3}x_1)=4x_2^2,$$

$$(V_0+2t\partial_t)P=6P,$$

$$\begin{aligned}&(9V_2'^2+\tfrac{3}{16}L(x))P(t)^{-1/2}\\ =\;&\tfrac{1}{8}\tfrac{\partial}{\partial t}\left\{\left(-12L(x)t^4-3L(x)^2t^2+\tfrac{2}{3}L(x)M(x)t-\tfrac{1}{3}M(x)^2+L(x)^3\right)P(t)^{-3/2}\right\}.\end{aligned}\tag{28}$$

We put

$$v(x)=\int_{\infty}^{-\frac{2}{3}x_1}P^{-\frac{1}{2}}dt.\tag{29}$$

Then, by an argument explained in [1], we see that

$$V_0v=-v,\quad V_1v=-1$$

and

$$V_2'^2v=\frac{1}{4}x_2V_1v-\frac{1}{48}L(x)v.$$

As a consequence, we find that the function $v(x)$ is a solution of (27).

If $u(x)$ is a solution of (27) such that $V_1u=0$, then u is a solution of the system

$$V_0u=-u,\ V_1u=0,\ V_2'^2u=-\frac{1}{48}Lu\tag{30}$$

One method to solve this differential equations is to reduce it to Gaussian hypergeometric differential equations.

In this manner, we obtain three linearly independent solutions of (27).

The main purpose of this paper is to generalize the argument of this subsection to the case of the Saito free divisor $S_m:\ \delta_m=0$.

4.2 A system of uniformization equations along S_m

We assume that m is an integer such that $m > 1$. We introduce a system of differential equations

$$\left\{\begin{array}{lcl} V_0 u & = & -(m-1)u, \\ V_1 V_1 u & = & 0, \\ V_2 V_1 u & = & 0, \\ V_2 V_2 u & = & -16(m-1)^2 x_2^{2m-2} u - 4(m-1) x_3 \tau_{2m-2}(x_1; x_2) V_1 u. \end{array}\right. \tag{31}$$

The right hand sides of the equations in the system (31) are so chosen that the system satisfies the integrability condition.

Remark 2 *The system (31) is an analogue of (27) for the Saito free divisor S_m.*

It is possible to show that (31) has three fundamental solutions outside of S_m. In particular, the system (31) is a system of uniformization equations singular along the Saito free divisor S_m in the sense of [1]. The main purpose of this subsection is to construct its fundamental solutions.

We have already introduced $\chi_{2m}(t; L)$ and $\lambda_m(x)(= x_2)$, $\mu_m(x)$ in the previous sections. Using these, we define

$$P_m(t; x_1, x_2, x_3) = \tilde{\sigma}_{2m}(t; \lambda_m(x), \mu_m(x)). \tag{32}$$

It is clear from (18) that

$$P_m(t; x_1, x_2, x_3) = \chi_{2m}(t; x_2) - \chi_{2m}(x_1; x_2) + x_3^2. \tag{33}$$

We frequently write $P_m(t)$ instead of $P_m(t; x_1, x_2, x_3)$ for simplicity.

Remark 3 *We explain here why we introduce the variables x_1, x_2, x_3. Let us consider the hyperelliptic curve C defined by*

$$y^2 = \tilde{\sigma}_{2m}(t; L, M') \tag{34}$$

on (t, y)-plane, where L, M' are parameters. If (x_1, x_3) is a point on C, then

$$x_3^2 = \tilde{\sigma}_{2m}(x_1; L, M'). \tag{35}$$

This implies that

$$x_3^2 = \chi_{2m}(x_1; L) - (M' - 2L^m). \tag{36}$$

Or equivalently,

$$M' = \chi_{2m}(x_1; L) + 2L^m - x_3^2. \tag{37}$$

If we put $L = x_2$, then (18) follows.

We note that

$$\left\{\begin{array}{ll} V_0 \lambda_m = 2\lambda_m, & V_0 \mu_m = 2m\mu_m, \\ V_1 \lambda_m = 0, & V_1 \mu_m = 0, \\ V_2 \lambda_m = -4\mu_m, & V_2 \mu_m = -16m\lambda_m^{2m-1}. \end{array}\right. \tag{38}$$

This means that the vector field V_0 (resp. V_2) acting on functions of $\lambda_m(x), \mu_m(x)$ is identified with the vector field W_0 (resp. $-2W_2$) acting functions of L, M by the correspondence $\lambda_m \to L$, $\mu_m \to M'$.

Theorem 1 *The function $v_m(x)$ defined by*

$$v_m(x) = \int_{\infty}^{x_1} P_m(t)^{-1/2} dt \tag{39}$$

is a solution of the system (31).

Proof. We regard $P_m(t)$ as a function of t. Let U be a simply connected domain in $\mathbb{C} \cup \{\infty\} - \{t \in \mathbb{C} | \, P_m(t) = 0\}$ containing the points x_1 and ∞. Since $P_m(x_1) = x_3^2$, we take a branch of $\varphi(t) = P_m(t)^{-1/2}$ on U such that

$$\varphi(x_1) = \frac{1}{x_3}. \tag{40}$$

It is clear that

$$\lim_{t\to\infty} t^k \varphi(t) = 0 \quad (k = 0, 1, 2, \ldots, m-1) \tag{41}$$

and $\lim_{t\to\infty} t^m \varphi(t)$ is bounded.

We put

$$w_m(t) = \int_C \varphi(t) dt, \tag{42}$$

where C is a path contained in U whose starting point is ∞ and terminal point is x_1. Since $v_m(x)$ is obtained by analytic continuation of $w_m(x)$, to prove the theorem it suffices to show that $w_m(x)$ is a solution of (31). We are going to prove this.

We first compute $V_0 w_m$. Since $(V_0 + t\partial_t) P_m = 2m P_m$, it follows that

$$V_0 P_m = 2m P_m - t\partial_t P_m. \tag{43}$$

On the other hand, we have

$$V_0 \varphi = -\frac{\varphi}{2P_m} V_0 P_m. \tag{44}$$

By virtue of (43) and (44), we have

$$\begin{aligned} V_0 w_m &= \int_C V_0 \varphi dt + V_0(x_1)\varphi(x_1) \\ &= -\frac{1}{2}\int_C \frac{\varphi}{P_m} V_0 P_m dt + \frac{x_1}{x_3} \\ &= -\frac{1}{2}\int_C \frac{\varphi}{P_m}(2m P_m - t\partial_t P_m) dt + \frac{x_1}{x_3} \\ &= -m w_m + \frac{1}{2}\int_C \frac{\varphi}{P_m} \cdot t\partial_t P_m dt + \frac{x_1}{x_3}. \end{aligned}$$

Integrating by t both sides of the identity equation

$$\partial_t(t\varphi) = \varphi - \frac{1}{2}\frac{\varphi}{P_m} \cdot t\partial_t P_m$$

along C, we have

$$\frac{x_1}{x_3} = w_m - \frac{1}{2}\int_C \frac{\varphi}{P_m} \cdot t\partial_t P_m dt$$

Then

$$\frac{1}{2}\int_C \frac{\varphi}{P_m} \cdot t\partial_t P_m dt = w_m - \frac{x_1}{x_3}.$$

As a consequence,

$$V_0 w_m = -m w_m + w_m = -(m-1)w_m.$$

This means that w_m satisfies the first of (31).

Next we compute $V_1 w_m$. Since

$$V_1 x_2 = 0, \quad V_1 \mu_m(x) = 0,$$

it follows that

$$V_1 P_m = 0.$$

Then

$$V_1 w_m = \int_C V_1 \varphi dt + V_1(x_1) \cdot \varphi(x_1) = -4x_3 \varphi(x_1) = -4.$$

Namely, we have

$$V_1 w_m = -4. \tag{45}$$

This implies that

$$V_1 V_1 w_m = V_2 V_1 w_m = 0, \tag{46}$$

which means that $w_m(x)$ satisfies the second and third equations of (31).

Thirdly we compute $V_2^2 w_m$. It is possible to show

$$V_2(x_1) = 4\chi_{2m-1}(x_1; x_2). \tag{47}$$

Then we find that

$$V_2 w_m = \int_C V_2 \varphi dt + V_2(x_1) \cdot \varphi(x_1) = \int_C V_2 \varphi dt + \frac{4\chi_{2m-1}(x_1; x_2)}{x_3}.$$

Consequently,

$$V_2{}^2 w_m = \int_C V_2{}^2 \varphi dt + V_2(x_1) \cdot (V_2 \varphi)|_{t=x_1} + V_2\left(\frac{4\chi_{2m-1}(x_1; x_2)}{x_3}\right).$$

Since

$$V_2 \varphi = -\frac{\varphi}{2P_m}(V_2 P_m),$$

and

$$V_2 P_m = -2(W_2 \tilde{\sigma}_{2m})|_{L=x_2, M'=\mu_m(x)} = -8m\mu_m(x)\tau_{2m-1}(t; x_2) + 16m x_2^{2m-1},$$

it follows that

$$V_2 \varphi|_{t=x_1} = \frac{4m}{x_3^3}\{\mu_m(x)\tau_{2m-1}(x_1; x_2) - 2x_2^{2m-1}\}.$$

Consequently we have

$$V_2(x_1)(V_2\varphi)|_{t=x_1} = \frac{16m}{x_3^3}\chi_{2m-1}(x_1; x_2)\{\mu_m(x)\tau_{2m-1}(x_1; x_2) - 2x_2^{2m-1}\}.$$

On the other hand, it follows from direct computation that

$$\begin{aligned} V_2(\chi_{2m-1}(x_1;x_2)) &= 4(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2)+x_1x_2^{2m-2}\}, \\ V_2x_3 &= 4mx_3\tau_{2m-1}(x_1;x_2). \end{aligned}$$

Then

$$\begin{aligned} V_2\left(\frac{4\chi_{2m-1}(x_1;x_2)}{x_3}\right) = &\frac{16(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2)+x_1x_2^{2m-2}\}}{x_3} \\ &-\frac{16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3}. \end{aligned}$$

These computations imply

$$\begin{aligned} & {V_2}^2w_m \\ = & \int_C {V_2}^2\varphi dt+\frac{16m}{x_3^3}\chi_{2m-1}(x_1;x_2)\{\mu_m(x)\tau_{2m-1}(x_1;x_2)-2x_2^{2m-1}\} \\ & +\frac{16(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2)+x_1x_2^{2m-2}\}-16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3}. \end{aligned}$$

As is shown in Lemma 4, we have

$$\begin{aligned} & \{W_2^2+4(m-1)^2L^{2m-2}\}\tilde{\sigma}_m^{-1/2} \\ = & \partial_t[\{8mL^{2m-1}\tau_{2m}-4m(M^2-2L^{2m})\tau_{2m-2}-4mtL^{2m-2}M\}\tilde{\sigma}_m^{-3/2}-4(m-1)tL^{2m-2}\tilde{\sigma}_m^{-1/2}]. \end{aligned}$$

This implies that

$$\begin{aligned} & \{\tfrac{1}{4}V_2^2+4(m-1)^2x_2^{2m-2}\}\varphi \\ = & \partial_t[\{8mx_2^{2m-1}\tau_{2m}(t;x_2)-4m(\mu_m(x)^2-2x_2^{2m})\tau_{2m-2}(t;x_2)-4mtx_2^{2m-2}\mu_m(x)\}\tfrac{\varphi}{P_m} \\ & -4(m-1)tx_2^{2m-2}\varphi]. \end{aligned}$$

Integrating both sides of this equation along C with respect to t, we have

$$\begin{aligned} & \int_C V_2^2\varphi dt+16(m-1)^2x_2^{2m-2}w_m \\ = & [\{8mx_2^{2m-1}\tau_{2m}(t;x_2)-4m(\mu_m(x)^2-2x_2^{2m})\tau_{2m-2}(t;x_2)-4mtx_2^{2m-2}\mu_m(x)\}\frac{\varphi}{P_m} \\ & -4(m-1)tx_2^{2m-2}\varphi]_{t=\infty}^{t=x_1} \\ = & \frac{16m\{2x_2^{2m-1}\tau_{2m}(x_1;x_2)-(\mu_m(x)^2-2x_2^{2m})\tau_{2m-2}(x_1;x_2)-x_1x_2^{2m-2}\mu_m(x)\}}{x_3^3} \\ & -\tfrac{16(m-1)x_1x_2^{2m-2}}{x_3}. \end{aligned}$$

Then

$$\begin{aligned}
& {V_2}^2 w_m + 16(m-1)^2 x_2^{2m-2} w_m \\
= & \left(\int_C {V_2}^2 \varphi dt + 16(m-1)^2 x_2^{2m-2} w_m\right) \\
& + \tfrac{16m}{x_3^3} \chi_{2m-1}(x_1;x_2)\{\mu_m(x)\tau_{2m-1}(x_1;x_2) - 2x_2^{2m-1}\} \\
& + \frac{16(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2) + x_1 x_2^{2m-2}\} - 16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3} \\
= & \frac{16m\{2x_2^{2m-1}\tau_{2m}(x_1;x_2) - (\mu_m(x)^2 - 2x_2^{2m})\tau_{2m-2}(x_1;x_2) - x_1 x_2^{2m-2}\mu_m(x)\}}{x_3^3} \\
& - \tfrac{16(m-1)x_1 x_2^{2m-2}}{x_3} + \frac{16m}{x_3^3}\chi_{2m-1}(x_1;x_2)\{\mu_m(x)\tau_{2m-1}(x_1;x_2) - 2x_2^{2m-1}\} \\
& + \frac{16(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2) + x_1 x_2^{2m-2}\}}{x_3} \\
& - \tfrac{16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3}.
\end{aligned}$$

At this moment, we note that the following identity holds:

$$\begin{aligned}
& \{2x_2^{2m-1}\tau_{2m}(x_1;x_2) - (\mu_m(x)^2 - 2x_2^{2m})\tau_{2m-2}(x_1;x_2) - x_1 x_2^{2m-2}\mu_m(x)\} \\
& + \chi_{2m-1}(x_1;x_2)\{\mu_m(x)\tau_{2m-1}(x_1;x_2) - 2x_2^{2m-1}\} \\
= & \ x_3^2 \tau_{2m-2}(x_1;x_2)\mu_m(x).
\end{aligned}$$

This is a consequence of Lemma 3 and (18). Then

$$\begin{aligned}
& {V_2}^2 w_m + 16(m-1)^2 x_2^{2m-2} w_m \\
= & -\frac{16(m-1)x_1 x_2^{2m-2}}{x_3} + \frac{16m\tau_{2m-2}(x_1;x_2)\mu_m(x)}{x_3} \\
& + \frac{16(2m-1)\{x_3^2\tau_{2m-2}(x_1;x_2) + x_1 x_2^{2m-2}\} - 16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3} \\
= & -\frac{16(m-1)x_1 x_2^{2m-2}}{x_3} + \frac{16m\tau_{2m-2}(x_1;x_2)\mu_m(x)}{x_3} \\
& + \frac{16(2m-1)x_1 x_2^{2m-2} - 16m\chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)}{x_3} + 16(2m-1)x_3\tau_{2m-2}(x_1;x_2) \\
= & \frac{16m\{\tau_{2m-2}(x_1;x_2)\mu_m(x) + x_1 x_2^{2m-2} - \chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)\}}{x_3} \\
& + 16(2m-1)x_3\tau_{2m-2}(x_1;x_2).
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
& {V_2}^2 w_m \\
= & -16(m-1)^2 x_2^{2m-2} w_m \\
& + \frac{16m\{\tau_{2m-2}(x_1;x_2)\mu_m(x) + x_1 x_2^{2m-2} - \chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2)\}}{x_3} \\
& + 16(2m-1)x_3\tau_{2m-2}(x_1;x_2).
\end{aligned}$$

At this moment, we note an identity equation

$$\tau_{2m-2}(x_1;x_2)\mu_m(x) + x_1 x_2^{2m-2} - \chi_{2m-1}(x_1;x_2)\tau_{2m-1}(x_1;x_2) = -x_3^2\tau_{2m-2}(x_1;x_2),$$

which is also a consequence of Lemma 3 and (18). Then

$$\begin{aligned} & {V_2}^2 w_m \\ = \; & -16(m-1)^2 x_2^{2m-2} w_m - 16m x_3 \tau_{2m-2}(x_1; x_2) + 16(2m-1) x_3 \tau_{2m-2}(x_1; x_2) \\ = \; & -16(m-1)^2 x_2^{2m-2} w_m + 16(m-1) x_3 \tau_{2m-2}(x_1; x_2). \end{aligned}$$

Since $V_1 w_m = -4$, it follows that

$${V_2}^2 w_m = -16(m-1)^2 x_2^{2m-2} w_m - 4(m-1) x_3 \tau_{2m-2}(x_1; x_2) V_1 w_m.$$

This means that $w_m(x)$ satisfies the last of the system (31).

Therefore we find that $w_m(x)$ is a solution of (31) and the theorem is proved. □

Solutions of (31) with the condition $V_1 u = 0$ are expressed in terms of Gaussian hypergeometric functions.

Theorem 2 *Put*

$$u(x) = \delta_m(x)^{\frac{1-m}{4m}} F\left(\frac{m-1}{4m}, \frac{m-1}{4m}; \frac{2m-1}{2m}; -\frac{4x_2^{2m}}{\delta_m(x)}\right).$$

Then $u(x)$ is a solution of (31) such that $V_1 u = 0$.

Proof. The proof of this theorem is similar to that in [9]. For the sake of completeness, we give here its outline. First we note that

$$V_0 \delta_m(x) = 4m \delta_m(x), \quad V_1 \delta_m = V_2 \delta_m(x) = 0.$$

Put

$$y = -\frac{4x_2^{2m}}{\delta_m(x)}$$

and let $\varphi(t)$ be a function of t. Let $u(x)$ be a solution of (31) such that $V_1 u = 0$. Then

$$V_0 u = -(m-1)u, \quad V_1 u = 0, \quad {V_2}^2 u = -16(m-1)^2 x_2^{2m-2} u. \tag{48}$$

Assume that $u(x) = \delta_m(x)^p \varphi(y)$. Then

$$V_0 u = \delta_m(x)^p (4mp + V_0)\varphi(y).$$

Since $V_0 u = -(m-1)u$, it follows that

$$(V_0 + 4pm)\varphi(y) = -(m-1)\varphi(y).$$

Now assume that $p = \frac{1-m}{4m}$. Then

$$V_0 \varphi(y) = 0.$$

Since $V_1 \delta_m(x) = V_2 \delta_m(x) = 0$, we find that

$$V_0 \varphi(y) = V_1 \varphi(y) = 0, \quad {V_2}^2 \varphi(y) = -16(m-1)^2 x_2^{2m-2} \varphi(y). \tag{49}$$

It is straightforward to show that

$$V_2^2\varphi(y) = (V_2 y)^2\varphi''(y) + (V_2{}^2 y)\varphi'(y).$$

It is also straightforward that

$$V_2 y = \frac{8\mu_m(x)}{x_2} y$$

and

$$V_2{}^2 y = \frac{32m(4m x_2^{2m} + (2m-1)\mu_m(x)^2)}{x_2^2} y.$$

Noting these, we find from

$$V_2{}^2\varphi(y) = -16(m-1)^2 x_2^{2m-2}\varphi(y)$$

that

$$64m^2\mu_m(x)^2 y^2\varphi''(y) + 32m\{4m x_2^{2m} + (2m-1)\mu_m(x)^2\} y\varphi'(y) = -16(m-1)^2 x_2^{2m}\varphi(y). \quad (50)$$

We note here that

$$x_2^{2m} = -\frac{1}{4}\delta_m(x) y, \quad \mu_m(x) = -\delta_m(x)(y-1).$$

Then (50) turns out to be

$$64m^2(1-y)y^2\varphi''(y) + 32m\{-my - (2m-1)(y-1)\} y\varphi'(y) - 4(m-1)^2 y\varphi(y) = 0. \quad (51)$$

This is equivalent to

$$\left\{\vartheta_y\left(\vartheta_y - \frac{1}{2m}\right) - y\left(\vartheta_y + \frac{m-1}{4m}\right)^2\right\}\varphi(y) = 0. \quad (52)$$

It is clear that

$$\varphi(y) = F\left(\frac{m-1}{4m}, \frac{m-1}{4m}, \frac{2m-1}{2m}; y\right)$$

is a solution of (52) and the theorem is proved. □

5 The Cases $m = 2, 3$

The hyperelliptic integral $v_m(x)$ introduced in Theorem 1 is reduced to an elliptic integral, when $m = 2$ and $m = 3$. We treat these cases in detail. In particular, we explain the relationship between these cases and the classical theory of elliptic functions.

5.1 The case $m = 2$

In this subsection, we treat the case $m = 2$. In particular, we determine the relationship between the variables x_1, x_2, x_3 used in the previous sections and the coefficients g_2, g_3 of the elliptic curve defined by $y^2 = 4x^3 - g_2 x - g_3$.

In this case

$$P_2(t; x_1, x_2, x_3) = t^4 + 4x_2 t^2 - (x_1^4 + 4x_1^2 x_2) + x_3^2.$$

We put

$$L = x_2, \quad M = x_1^4 + 4x_1^2 x_2 - x_3^2 \tag{53}$$

as before. Then the hypersurface $M(4L^2 + M) = 0$ is a Saito free divisor in the (x_1, x_2, x_3)-space.

We define the integral

$$v(x) = \int_\infty^{x_1} \frac{dt}{\sqrt{t^4 + 4Lt^2 - M}} \tag{54}$$

By taking $r = t^2$, we have

$$v(x) = \frac{1}{2} \int_\infty^{x_1^2} \frac{dr}{\sqrt{r^3 + 4Lr^2 - Mr}}$$

We note here that

$$4(r^3 + 4Lr^2 - Mr) = 4p^3 - g_2 p - g_3,$$

if there are relations among p, L, M and r, g_2, g_3 by

$$\begin{aligned} p &= r + \frac{4}{3}L, \\ g_2 &= 4(M + \frac{16}{3}L^2), \\ g_3 &= -\frac{16}{3}L(M + \frac{32}{9}L^2). \end{aligned}$$

Substitution by (53) implies that

$$\begin{cases} g_2 = \frac{4}{3}(3x_1^4 + 12x_1^2 x_2 + 16x_2^2 - 3x_3^2) \\ g_3 = -\frac{16}{27}x_2(9x_1^4 + 36x_1^2 x_2 + 32x_2^2 - 9x_3^2) \end{cases} \tag{55}$$

and that

$$4p^3 - g_2 p - g_3|_{p = x_1^2 + \frac{4}{3}x_2} = 4x_1^2 x_3^2. \tag{56}$$

Moreover, if g_2, g_3 are defined by (55), we have

$$v(x) = \int_\infty^{x_1^2 + \frac{4}{3}x_2} \frac{dp}{\sqrt{4p^3 - g_2 p - g_3}}. \tag{57}$$

We now introduce Weierstrass elliptic function

$$\wp(z;u_1,u_2)=\frac{1}{z^2}+\sum_{(m,n)\in\mathbf{Z}^2-\{(0,0)\}}\left(\frac{1}{(z+mu_1+nu_2)^2}-\frac{1}{(mu_1+nu_2)^2}\right). \tag{58}$$

As is well-known, the relations among g_2, g_3 and u_1, u_2 are

$$g_2=60\sum{}'\frac{1}{(mu_1+nu_2)^4}\quad g_3=140\sum{}'\frac{1}{(mu_1+nu_2)^6}. \tag{59}$$

In the rest of this subsection, we show how x_1, x_2, x_3 are defined as functions of $\wp(z;u_1,u_2)$ and u_1, u_2. For this purpose, we introduce e_1, e_2, e_3 by

$$4p^3-g_2p-g_3=4(p-e_1)(p-e_2)(p-e_3).$$

Theorem 3 *We assume $g_2, g_3, v(x)$ are defined by (55) and (57) and put $z=v(x)$. Let $\wp(z)$ be Weierstrass elliptic function satisfying $\wp'(z)^2=4\wp(z)^3-g_2\wp(z)-g_3$. Then*

$$\begin{cases} x_1^2 &= \wp(z)-\alpha \\ x_2 &= \dfrac{3}{4}\alpha \\ x_3^2 &= \dfrac{\wp'(z)^2}{4(\wp(z)-\alpha)} \\ 2x_1x_3 &= \wp'(z) \end{cases} \tag{60}$$

where α is one of e_1, e_2, e_3.

Moreover, if L, M are polynomials of x_1, x_2, x_3 defined by (53) and $\alpha=e_j$, then

$$L=\frac{3}{4}e_j,\quad M=-(e_j-e_{k_1})(e_j-e_{k_2}), \tag{61}$$

where $\{j,k_1,k_2\}=\{1,2,3\}$.

Proof. To prove this relation, we first note that the relation (55) implies the identity equation

$$(\frac{4}{3}x_2)g_2+g_3=4(\frac{4}{3}x_2)^3.$$

This means that $p=\frac{4}{3}x_2$ is a solution of the equation $4p^3-g_2p-g_3=0$ for p. So we put

$$x_2=\frac{3}{4}e_j$$

for $j=1,2,3$. On the other hand, the definition of $v(x)$ implies that

$$\wp(z)=x_1^2+\frac{4}{3}x_2=x_1^2+e_j.$$

Then we may put

$$x_1=\sqrt{\wp(z)-e_j}.$$

Noting (56), we may put

$$\wp'(z)=2x_1x_3.$$

Then

$$x_3=\frac{\wp'(z)}{2x_1}=\frac{\wp'(z)}{2\sqrt{\wp(z)-e_j}}.$$

The rest of the theorem is easy to show. □

5.2 The case $m = 3$

In this subsection, we treat the case $m = 3$. In this case

$$P_3(t; x_1, x_2, x_3) = t^6 + 6x_2t^4 + 9x_2^2t^2 - x_1^2(x_1^2 + 3x_2)^2 + x_3^2.$$

We put

$$L = x_2, \quad M = x_1^2(x_1^2 + 3x_2)^2 - x_3^2. \tag{62}$$

The hypersurface $M(4L^3 + M) = 0$ is a Saito free divisor on the (x_1, x_2, x_3)-space.

In the following argument, we assume that L, M are algebraically independent, but don't assume the relationship (62) for L, M and x_1, x_2, x_3, and state explicitly when we assume it.

We define the integral

$$v_a(x_1; L, M) = \int_0^{x_1} \frac{dt}{\sqrt{t^6 + 6Lt^4 + 9L^2t^2 - M}} \tag{63}$$

By an argument parallel to the proof of Theorem 1, it is possible to show that $v_a(x_1, L, M)$ is a solution of the system (31) for the case $m = 3$. (We note that the integration path for $v_a(x_1; L, M)$ starts from 0 and terminates at x_1. This is different from the definition of the function $v_m(x)$ in Theorem 1.)

By changing the integration variable $s = t^2$, we have

$$v_a(x_1; L, M) = \frac{1}{2}\int_0^{x_1^2} \frac{ds}{\sqrt{s(s^3 + 6Ls^2 + 9L^2s - M)}} \tag{64}$$

This means that $v_a(x_1; L, M)$ is reduced to an elliptic integral. By changing integration variable $s = 1/r$, we have

$$v_a(x_1; L, M) = \int_{1/x_1^2}^{\infty} \frac{dr}{\sqrt{-4Mr^3 + 36L^2r^2 + 24Lr + 4}} \tag{65}$$

Moreover by changing integration variable $r = \frac{-p+3L^2}{M}$, we have

$$v_a(x_1; L, M) = \int_{\infty}^{3L^2 - \frac{M}{x_1^2}} \frac{dr}{\sqrt{4p^3 - g_2p - g_3}}, \tag{66}$$

if g_2, g_3 satisfy the relation

$$\begin{cases} g_2 &= 12L(9L^3 + 2M), \\ g_3 &= -4(54L^6 + 18L^3M + M^2). \end{cases} \tag{67}$$

We note here that there are two relations among x_1, x_2, x_3 and g_2, g_3

$$\begin{cases} 4M^2 &= 4(3L^2)^3 - g_2(3L^2) - g_3, \\ \left(\frac{2x_3M}{x_1^3}\right)^2 &= 4\left(3L^2 - \frac{M}{x_1^2}\right)^3 - g_2\left(3L^2 - \frac{M}{x_1^2}\right) - g_3, \end{cases} \tag{68}$$

if we assume the conditions (62) and (67) for L, M and g_2, g_3.

We are now going to express x_1, x_2, x_3 as functions of g_2, g_3 and Weierstrass elliptic function. We regard g_2, g_3, z as variables and let $\wp(z)$ be Weierstrass elliptic function satisfying

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{69}$$

We introduce L, M so that L, M satisfy the condition (67). It follows from the first of (67) that

$$M = \frac{g_2}{24L} - \frac{9}{2}L^3 \tag{70}$$

Substituting this to the second of (67), we find that $u = L$ is a solution of the equation

$$u^8 - 6 \cdot \frac{g_2}{2^2 \cdot 3^3} \cdot u^4 - \frac{g_3}{3^3} \cdot u^2 - 3 \cdot \frac{g_2^2}{2^4 \cdot 3^6} = 0. \tag{71}$$

For this reason, we first take a solution L of (71) and define M by (70). Next we define x_1, x_2, x_3 by the relations

$$\wp(z) = 3L^2 - \frac{M}{x_1^2}, \quad x_2 = L, \quad \wp'(z) = \frac{2x_3M}{x_1^3} \tag{72}$$

In other words,

$$\begin{cases} x_1 &= \dfrac{\sqrt{M}}{\sqrt{3L^2 - \wp(z)}}, \\ x_2 &= L, \\ x_3 &= \dfrac{x_1^3\wp'(z)}{2M}. \end{cases} \tag{73}$$

The equation (72) is compatible with the second of (68). In this manner, x_1, x_2, x_3 are expressed by $g_2, g_3, \wp(z)$.

We are going to specify the meaning of L in the argument above. As we will see at the end of this subsection, the value of the 3-division point of a fundamental period of $\wp(z)$ is written by L. Recall the duplication formula for the elliptic function

$$\wp(2z) = -2\wp(z) + \frac{(3\wp(z)^2 - g_2/4)^2}{(4\wp(z)^3 - g_2\wp(z) - g_3)}.$$

The substitution of both $Z = \wp(z)$ and $Z = \wp(2z)$ in this identity implies an equation

$$Z = -2Z + \frac{(3Z^2 - g_2/4)^2}{4Z^3 - g_2Z - g_3}. \tag{74}$$

This turns out to be an algebraic equation

$$48Z^4 - 24g_2Z^2 - 48g_3Z - g_2^2 = 0 \tag{75}$$

By the substitution $Z = 3u^2$, (75) becomes (71), which means that if L is a solution of (71) and $\wp(a_0) = 3L^2$ for a constant a_0, then $\wp(2a_0) = \wp(a_0)$. We now define g_2, g_3 by (67) and substitute them on (75). Then we obtain

$$Z^4 - 6L(9L^3 + 2M)Z^2 + 4(54L^6 + 18L^3M + M^2)Z - 3L^2(9L^3 + 2M)^2 = 0, \tag{76}$$

which is equivalent to

$$(Z-3L^2)(Z^3+3L^2Z^2-3L(15L^3+4M)Z+(9L^3+2M)^2)=0. \tag{77}$$

If z_0 is a solution of

$$Z^3+3L^2Z^2-3L(15L^3+4M)Z+(9L^3+2M)^2=0,$$

then

$$Z^3+3L^2Z^2-3L(15L^3+4M)Z+(9L^3+2M)^2=(Z-z_0)(Z-z_1)(Z-z_2),$$

where

$$\begin{cases} z_1 &= \frac{1}{2}(-3L^2-z_0-\frac{\sqrt{-3}}{M}(54L^5+15L^2M-12L^3z_0-Mz_0-2Lz_0^2)),\\ z_2 &= \frac{1}{2}(-3L^2-z_0+\frac{\sqrt{-3}}{M}(54L^5+15L^2M-12L^3z_0-Mz_0-2Lz_0^2)).\end{cases} \tag{78}$$

We take a_0 such that $\wp(a_0)=3L^2$, where L is a solution of (71). Then $\wp(2a_0)=\wp(a_0)$, which implies that $2a_0=\pm a_0+\omega$ for a period ω. If $2a_0=a_0+\omega$, then $3L^2=\wp(a_0)=\wp(\omega)=\wp(0)$. This contradicts $\wp(0)=\infty$. If $2a_0=-a_0+\omega$, then $a_0=\omega/3$. Let ω_1,ω_2 be fundamental periods of $\wp(z)$. Then $a_0=(m+j_1/3)\omega_1+(n+j_2/3)\omega_2$ for some integers m,n and $j_1,j_2\in\{0,1,2\}$. This implies that $\wp(a_0)=\wp((j_1\omega_1+j_2\omega_2)/3)$.

The possibilities of $(j_1\omega_1+j_2\omega_2)/3$ are

$$0,\ \frac{\omega_1}{3},\ \frac{\omega_2}{3},\ \frac{2\omega_1}{3},\ \frac{2\omega_2}{3},\ \frac{\omega_1+\omega_2}{3},\ \frac{2\omega_1+\omega_2}{3},\ \frac{\omega_1+2\omega_2}{3},\ \frac{2\omega_1+2\omega_2}{3}.$$

Moreover,

$$\wp\left(\frac{\omega_1}{3}\right)=\wp\left(\frac{2\omega_1}{3}\right),\quad \wp\left(\frac{\omega_2}{3}\right)=\wp\left(\frac{2\omega_2}{3}\right),\quad \wp\left(\frac{\omega_1+\omega_2}{3}\right)=\wp\left(\frac{2(\omega_1+\omega_2)}{3}\right),$$

$$\wp\left(\frac{\omega_1+2\omega_2}{3}\right)=\wp\left(\frac{2(\omega_1+2\omega_2)}{3}\right)=\wp\left(\frac{2\omega_1+\omega_2}{3}\right).$$

Then

$$\begin{aligned} &\left(Z-\wp\left(\tfrac{\omega_1}{3}\right)\right)\left(Z-\wp\left(\tfrac{\omega_2}{3}\right)\right)\left(Z-\wp\left(\tfrac{\omega_1+\omega_2}{3}\right)\right)\left(Z-\wp\left(\tfrac{2\omega_1+\omega_2}{3}\right)\right)\\ =\ &(Z-z_0)(Z-z_1)(Z-z_2)(Z-3L^2)\end{aligned} \tag{79}$$

We put

$$w_j=2\sqrt{-3}(3L^3+M-Lz_j)\quad (j=0,1,2). \tag{80}$$

Then it is easy to show that

$$w_j^2=z_j^3-g_2z_j-g_3$$

under the relation (67).

Now we recall the addition formula for $\wp(z)$;

$$\begin{cases} \wp(z_1+z_2) &= -\wp(z_1)-\wp(z_2)+\dfrac{1}{4}\dfrac{(\wp'(z_2)-\wp'(z_1))^2}{(\wp(z_2)-\wp(z_1))^2}\\ \wp'(z_1+z_2) &= \dfrac{\wp'(z_1)(\wp'(z_1+z_2)-\wp(z_2))+\wp'(z_2)(\wp(z_1)-\wp(z_1+z_2))}{\wp(z_2)-\wp(z_1)}\end{cases} \tag{81}$$

Then by the argument above, we may take fundamental periods ω_1, ω_2 of $\wp(z)$ so that

$$\begin{cases} (\wp(\omega_1/3), \wp'(\omega_1/3)) & = & (3L^2, 2M), \\ (\wp(\omega_2/3), \wp'(\omega_2/3)) & = & (z_0, w_0). \end{cases} \tag{82}$$

Then it follows from the addition formula (81) that

$$\begin{cases} (\wp((\omega_1+\omega_2)/3), \wp'((\omega_1+\omega_2)/3)) & = & (z_1, w_1), \\ (\wp((2\omega_1+\omega_2)/3), \wp'((2\omega_1+\omega_2)/3)) & = & (z_2, w_2). \end{cases} \tag{83}$$

As a conclusion, we find that there is a fundamental period ω_1 such that

$$\wp(\omega_1/3) = 3L^2, \quad \wp'(\omega_1/3) = 2M. \tag{84}$$

This is a relation between L and a fundamental period of $\wp(z)$.

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