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Nonexistence for extremal Type II $\mathbb{Z}_{2k}$-Codes [1]

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Abstract

In this paper, we show that an extremal Type II $\mathbb{Z}_{2k}$-code of sufficently large length n does not exist if $k = 2, 3, 4$.

1 Introduction

Let $\mathbb{Z}_{2k}$ $(= \{0, 1, 2, \ldots, 2k-1\})$ be the ring of integers modulo $2k$, where k is a positive integer. We sometimes regard the elements of $\mathbb{Z}_{2k}$ as those of $\mathbb{Z}$. A $\mathbb{Z}_{2k}$-code C of length n (or a code C of length n over $\mathbb{Z}_{2k}$) is a $\mathbb{Z}_{2k}$-submodule of $\mathbb{Z}_{2k}^n$. A code C is *self-dual* if $C = C^{\perp}$ where the dual code $C^{\perp}$ of C is defined as $C^{\perp} = \{x \in \mathbb{Z}_{2k}^n \mid x \cdot y = 0 \quad \text{for all} \quad y \in C\}$ under the standard inner product $x \cdot y$. The Euclidean weight of a codeword $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^{n} \min\{x_i^2, (2k - x_i)^2\}$. The minimum Euclidean weight $d_E(C)$ of C is the smallest Euclidean weight among all nonzero codewords of C.

A binary doubly even self-dual code is often called Type II. For $\mathbb{Z}_4$-codes, Type II codes were first defined in [4] as self-dual codes containing a (± 1)-vector and with the property that all Euclidean weights are divisible by eight. Then it was shown in [10] that, more generally, the condition of containing a (± 1)-vector is redundant. Type II $\mathbb{Z}_{2k}$-codes was defined in [3] as a self-dual code with the property that all Euclidean weights are divisible by $4k$. It is known that a Type II $\mathbb{Z}_{2k}$-code of length n exists if and only if n is divisible by eight.

In [9], we show the following theorem:

Theorem 1.1 (cf. [9]). *Let C be a Type II $\mathbb{Z}_{2k}$-code of length n. If $k \le 6$ then the minimum Euclidean weight $d_E(C)$ of C is bounded by*

$$d_E(C) \le 4k \left\lfloor \frac{n}{24} \right\rfloor + 4k. \tag{1}$$

Remark 1.1. *The upper bound (1) has been known for the cases $k = 1$ [13] and $k = 2$ [4]. For $k \ge 3$, the bound (1) was known under the assumption that $\lfloor n/24 \rfloor \le k - 2$ [3].*

In [9], we define that a Type II $\mathbb{Z}_{2k}$-code meeting the bound (1) with equality is *extremal* for $k \le 6$.

The aim of this paper is to show the following theorem.

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Theorem 1.2. *For $k \leq 4$, an extremal Type II $\mathbb{Z}_{2k}$-code of length n does not exist for all sufficiently large n.*

Remark 1.2. *For the case $k = 1$, the above result in Theorem 1.2 was shown in [13].*

2 Preliminaries

An n-dimensional (Euclidean) lattice Λ is a subset of $\mathbb{R}^n$ with the property that there exists a basis $\{e_1, e_2, \dots, e_n\}$ of $\mathbb{R}^n$ such that $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$, i.e., Λ consists of all integral linear combinations of the vectors $e_1, e_2, \dots, e_n$. The dual lattice Λ^* of Λ is the lattice $\{x \in \mathbb{R}^n \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all y } \in \Lambda\}$, where $\langle x, y\rangle$ is the standard inner product. A lattice with $\Lambda = \Lambda^*$ is called *unimodular*. The norm of x is $\langle x, x\rangle$. A unimodular lattice with even norms is said to be *even*, otherwise *odd*. An n-dimensional even unimodular lattice exists if and only if $n \equiv 0 \pmod 8$, while an odd unimodular lattice exists for every dimension. The minimum norm $\min(\Lambda)$ of Λ is the smallest norm among all nonzero vectors of Λ. For Λ and a positive integer m, the shell Λ_m of norm m is defined as $\{x \in \Lambda \mid \langle x, x\rangle = m\}$.

The theta series of Λ is

$$\Theta_\Lambda(z) = \Theta_\Lambda(q) = \sum_{x\in\Lambda} q^{\langle x,x\rangle} = \sum_{m=0}^{\infty} |\Lambda_m| q^m, \quad q = e^{\pi i z},\ \mathrm{Im}(z) > 0.$$

For example, let Λ be the E_8-lattice. Then,

$$\begin{aligned}\Theta_\Lambda(q) = E_4(q) &= 1 + 240\sum_{m=1}^{\infty} \sigma_3(m) q^{2m}\\ &= 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots,\end{aligned}$$

where $\sigma_3(m)$ is a divisor function $\sigma_3(m) = \sum_{0<d|m} d^3$.

It is well-known that if Λ is an n-dimensional even unimodular lattice, then Θ_Λ is a modular form of weight $n/2$ for the full modular group $SL_2(\mathbb{Z})$ (see [8]). For example, $E_4(q)$ is a modular form of weight 4 for $SL_2(\mathbb{Z})$. Moreover the following theorem is known (see [8, Chap. 7]).

Theorem 2.1. *If Λ is an even unimodular lattice, then*

$$\Theta_\Lambda(q) \in \mathbb{C}[E_4(q), \Delta_{24}(q)],$$

where $\Delta_{24}(q) = q^2 \prod_{m=1}^{\infty}(1 - q^{2m})^{24}$ which is a modular form of weight 12 for $SL_2(\mathbb{Z})$.

We now give a method to construct even unimodular lattices from Type II codes, which is called Construction A [3]. Let ρ be a map from $\mathbb{Z}_{2k}$ to $\mathbb{Z}$ sending $0, 1, \dots, k$ to $0, 1, \dots, k$ and $k+1, \dots, 2k-1$ to $1-k, \dots, -1$, respectively. If C is a self-dual $\mathbb{Z}_{2k}$-code of length n, then the lattice

$$A_{2k}(C) = \frac{1}{\sqrt{2k}}\{\rho(C) + 2k\mathbb{Z}^n\}$$

is an n-dimensional unimodular lattice, where

$$\rho(C) = \{(\rho(c_1), \ldots, \rho(c_n)) \mid (c_1, \ldots, c_n) \in C\}.$$

The minimum norm of $A_{2k}(C)$ is $\min\{2k, d_E(C)/2k\}$. Moreover, if C is Type II, then the lattice $A_{2k}(C)$ is an even unimodular lattice.

The symmetrized weight enumerator of a $\mathbb{Z}_{2k}$-code C is

$${}_C(x_0, x_1, \ldots, x_k) = \sum_{c \in C} x_0^{n_0(c)} x_1^{n_1(c)} \cdots x_{k-1}^{n_{k-1}(c)} x_k^{n_k(c)},$$

where $n_0(c)$, $n_1(c)$, $\ldots$, $n_{k-1}(c)$, $n_k(c)$ are the number of $0, \pm 1, \ldots, \pm k-1, k$ components of c, respectively. Then the theta series of $A_{2k}(C)$ can be found by replacing x_1, x_2, $\ldots$, x_k by

$$f_0 = \sum_{x \in 2k\mathbb{Z}} q^{x^2/2k},\ f_1 = \sum_{x \in 2k\mathbb{Z}+1} q^{x^2/2k},\ \ldots,\ f_k = \sum_{x \in 2k\mathbb{Z}+k} q^{x^2/2k}.$$

respectively. Let C be a Type II $\mathbb{Z}_{2k}$-code of length n. Then, the even unimodular lattice $A_{2k}(C)$ contains the sublattice $\Lambda_0 = \sqrt{2k}\mathbb{Z}^n$ which has minimum norm $2k$. We set $\Theta_{\Lambda_0}(q) = \theta_0$, $n = 8j$ and $j = 3\mu + \nu$ $(\nu = 0, 1, 2)$, that is, $\mu = \lfloor n/24 \rfloor$. We denote $E_4(q)$ and $\Delta_{24}(q)$ by E_4 and Δ, respectively. By Theorem 2.1, the theta series of $A_{2k}(C)$ can be written as

$$\Theta_{A_{2k}(C)}(q) = \sum_{s=0}^{\mu} a_s E_4^{j-3s} \Delta^s = \sum_{r \geq 0} |A_{2k}(C)_r| q^r = \theta_0 + \sum_{r \geq 1} \beta_r q^r.$$

Let C be an extremal Type II $\mathbb{Z}_{2k}$-code for $1 \leq k \leq 6$, namely, $d_E(C) = 4k(\mu+1)$. We remark that a codeword of Euclidean weight $4km$ gives a vector of norm $2m$ in $A_{2k}(C)$. Then we choose the $a_0, a_1, \ldots, a_\mu$ so that

$$\Theta_{A_{2k}(C)}(q) = \theta_0 + \sum_{r \geq 2(\mu+1)} \beta_r^* q^r.$$

Here, we set b_{2s} as $E_4^{-j}\theta_0 = \sum_{s=0}^{\infty} b_{2s}(\Delta/E_4^3)^s$. That is, $\theta_0 = \sum_{s=0}^{\infty} b_{2s} E_4^{j-3s} \Delta^s$. Then

$$\sum_{s=0}^{\mu} a_s E_4^{j-3s} \Delta^s = \Theta_{A_{2k}(C)}(q) = \sum_{s=0}^{\infty} b_{2s} E_4^{j-3s} \Delta^s + \sum_{r \geq 2(\mu+1)} \beta_r^* q^r.$$

Comparing the coefficients of q^i $(0 \leq i \leq 2\mu)$, we get $a_s = b_{2s}$ $(0 \leq s \leq \mu)$. Hence we have

$$-\sum_{r \geq (\mu+1)} b_{2r} E_4^{j-3r} \Delta^r = \sum_{r \geq 2(\mu+1)} \beta_r^* q^r.$$

In (2), comparing the coefficients of $q^{2(\mu+1)}$ and $q^{2(\mu+2)}$, we have

$$\begin{cases} \beta_{2(\mu+1)}^* &= -b_{2(\mu+1)}, \\ \beta_{2(\mu+2)}^* &= -b_{2(\mu+2)} + b_{2(\mu+1)}(24\mu - 240\nu + 744). \end{cases} \tag{2}$$

All the series are in $q^2 = t$, and Bürman's formula [15, page 128] shows that

$$b_{2s} = \frac{1}{s!}\frac{d^{s-1}}{dt^{s-1}}\left(\left(\frac{d}{dt}(E_4^{-j}\theta_0)\right)(tE_4^3/\Delta)^s\right)_{\{t=0\}}.$$

In [9], we show that

$$\beta^*_{2(\mu+1)} > 0 \tag{3}$$

and we remark that the inequality (3) is a crucial part of the proof of Theorem 1.1.

Finally, we quote the two theorems needed later:

Theorem 2.2 (cf. [14, page 18, Theorem 1.64])**.** *Let* $\eta(z) = t^{1/24}\prod_{m=1}^{\infty}(1-t^m)$ *be the Dedekind eta function, where* $t = e^{2\pi iz}$*, the same for several places and* $Im(z) > 0$*. If* $f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta}$ *with* $k = (1/2)\sum_{\delta|N} r_\delta \in \mathbb{Z}$*, with the additional properties that*

$$\sum_{\delta|N}\delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N}\frac{N}{\delta}r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*. Here the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$*, where* $\left(\frac{\cdot}{\cdot}\right)$ *is the usual Jacobi symbol and* $s := \prod_{\delta|N}\delta^{r_\delta}$*.*

Theorem 2.3 (cf. [14, page 18, Theorem 1.65])**.** *Let* c*,* d *and* N *be positive integers with* $d|N$ *and* $\gcd(c,d) = 1$*. If* $f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta}$ *satisfying the conditions of Theorem 2.2 for* N*, then the order of vanishing of* $f(z)$ *at the cusp* c/d *is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}.$$

3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Our proof is an analogue of that of [12]. Before we give the proof of Theorem 1.2, we give two lemmas. First, we quote the following lemma from [12]. In [11], Ibukiyama remarked that in [12, Lemma 1] 2π (p. 70, l. -1) should be $(2\pi)^{1/2}$.

Lemma 3.1 ([12, Lemma 1], [11, Theorem 12]). *Suppose that $G(q)$, $H(q)$ are analytic inside the circle $|q| = 1$ and satisfy:*

(i) $H(q) = \sum_{s=0}^{\infty} H_s q^s$ *with* $H_0 > 0$, $H_1 > 0$, *and* $H_s \geq 0$ *for* $s \geq 2$,

(ii) *if* $F(y) = e^{2\pi y} H(e^{-2\pi y})$, *then* $F'(y) = 0$ *has a solution* $y = y_0$ *in the range* $y > 0$, *with* $F(y_0) = c_1 > 0, F''(y_0)/F(y_0) = c_2 > 0, G(e^{-2\pi y_0}) \neq 0$.

Then β_r, the coefficient of q^r in $G(q)H(q)^r$, satisfies

$$\beta_r \sim \frac{(2\pi)^{1/2}}{(rc_2)^{1/2}} G(e^{-2\pi y_0}) c_1^r, \ \textit{as}\ r \to \infty.$$

Second, we show the following lemma:

Lemma 3.2. *We set $t = q^2 = e^{2\pi i z}$ and $f_0(k,t) = \sum_{x \in \mathbb{Z}} t^{kx^2}$. Let $Z(k,t) := [f_0(k,t)^8, E_4(t)]/4 = f_0(k,t)^8 E_4(t)' - (f_0(k,t)^8)' E_4(t)$, where $[\ ,\ ]$ is the Rankin-Cohen bracket and $f(t)' = t(df/dt)$. Then, for $1 \leq k \leq 4$ and a positive real number y, $Z(k, e^{-2\pi y}) \neq 0$.*

Proof. Let f (resp. g) be a modular form of weight k (resp. ℓ) for a group Γ. Then, $[f,g] := kfg' - \ell f'g$ is a modular form of weight $k + \ell + 2$ for Γ [6, page 53].

We remark that $f_0(1,t)$ is a modular form of weight 1/2 for $\Gamma_0(4)$ [14, page 12]. Therefore, $f_0(1,t)^4$ is a modular form of weight 2 for $\Gamma_0(4)$. Moreover, $f_0(k,t)^4$ is a modular form of weight 2 for $\Gamma_0(4k)$ [14, page 28, Proposition 2.22].

- The case of $k = 1$:

  We remark that $Z(1,t) \in \Gamma_0(4)$ and define the functions:

$$\begin{cases} \Delta_4^{\infty}(t) = \eta^8(4z)/\eta^4(2z), \\ \Delta_4^{0}(t) = \eta^8(z)/\eta^4(2z), \\ J_4(t) = \Delta_4^{0}(t)/\Delta_4^{\infty}(t), \end{cases}$$

  Note that $J_4(t)$ is an isomorphism from a fundamental domain of $\Gamma_0(4)$ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and a generator of the function field of $\mathbb{H}^*/\Gamma_0(4)$, where $\mathbb{H}$ be the upper half plane and $\mathbb{H}^*/\Gamma_0(4)$ is a compactification of $\mathbb{H}/\Gamma_0(4)$ [5, page 407], [2, page 16]. Then, we have the following equality:

$$\frac{Z(1,t)}{\Delta_4^{\infty}(t)^5} = 224X^4 + 11264X^3 + 188416X^2 + 1048576X,$$

  where $X := J_4(t)$. It is easy to check that there are no positive real roots of the right-hand side (3). Here, we remark that $J_4(e^{2\pi i z})$ takes real values on the imaginary axis. Using Theorem 2.2 and 2.3, we have $\Delta_4^{\infty}(e^{2\pi i 0}) \neq 0$ and $\Delta_4^{0}(e^{2\pi i 0}) = 0$, namely $J_4(e^{2\pi i 0}) = 0$. Therefore, the values of the $J_4(t)$ on the imaginary axis are positive real numbers and we have $Z(1,t) \neq 0$ on the imaginary axis.

The other cases can be proved similarly. We only mention the functions which could be used for the proofs of the cases $k = 2$, 3 and 4.

- The case of $k = 2$:

$$\begin{cases} \Delta_8^\infty(t) = \eta^4(8z)/\eta^2(4z), \\ \Delta_8^0(t) = \eta^4(z)/\eta^2(2z), \\ J_8(t) = \Delta_8^0(t)/\Delta_8^\infty(t), \end{cases}$$

where $J_8(t)$ is Hauptmodul for type "8−" [7, page 331].

$$\begin{aligned} &Z(2,t)/\Delta_8^\infty(t)^{10} = \\ &240X^9 + 12928X^8 + 283136X^7 + 3358720X^6 \\ &+23883776X^5 + 105086976X^4 + 281018368X^3 \\ &+419430400X^2 + 268435456X \end{aligned}$$

where $X := J_8(t)$.

- The case of $k = 3$:

$$\begin{cases} \Delta_{12}^\infty(t) = \eta(2z)\eta^{-2}(4z)\eta^{-3}(6z)\eta^6(12z), \\ \Delta_{12}^0(t) = \eta^6(z)\eta^{-3}(2z)\eta^{-2}(3z)\eta(6z), \\ J_{12}(t) = (\Delta_{12}^0(t)/\Delta_{12}^\infty(t))^{1/2}, \end{cases}$$

where$J_{12}(t)$ is Hauptmodul for type "12−" [7, page 331].

$$\begin{aligned} &Z(3,t) = \\ &240X^{19} + 18000X^{18} + 616032X^{17} + 12860832X^{16} \\ &+184227840X^{15} + 1927623168X^{14} + 15293558784X^{13} \\ &+94189206528X^{12} + 456914313216X^{11} + 1760257683456X^{10} \\ &+5401844490240X^9 + 13181394788352X^8 + 25400510447616X^7 \\ &+38149727846400X^6 + 43699899727872X^5 + 36857648775168X^4 \\ &+21565588635648X^3 + 7815347306496X^2 + 1320903770112X \end{aligned}$$

where $X := J_{12}(t)$.

- The case of $k = 4$:

$$\begin{cases} \Delta_{16}^\infty(t) = \eta(16z)^2/\eta(8z), \\ \Delta_{16}^0(t) = \eta^2(z)/\eta(2z), \\ J_{16}(t) = \Delta_{16}^0(t)/\Delta_{16}^\infty(t), \end{cases}$$

where $J_{16}(t)$ is Hauptmodul for type "16−" [7, page 331].

$$
\begin{aligned}
&Z(3,t) = \\
&240X^{19} + 13440X^{18} + 339840X^{17} + 5259776X^{16} \\
&+56422912X^{15} + 448143360X^{14} + 2741043200X^{13} \\
&+13230211072X^{12} + 51153629184X^{11} + 159735971840X^{10} \\
&+403939164160X^{9} + 825259589632X^{8} + 1351740293120X^{7} \\
&+1750333390848X^{6} + 1751407132672X^{5} + 1305938493440X^{4} \\
&+682899800064X^{3} + 223338299392X^{2} + 34359738368X
\end{aligned}
$$

where $X := J_{16}(t)$.

$\square$

Proof of Theorem 1.2. Using the equation (2) and the fact that $\theta_0 = \theta_1^j$ where θ_1 is the theta series of the lattice $(2k\mathbb{Z})^8/\sqrt{2k}$, we have $b_{2s} = \frac{-j}{s!}\frac{d^{s-1}}{dt^{s-1}}\left(E_4^{3s-j-1}\theta_1^{j-1}\ (\theta_1 E_4' - \theta_1' E_4)(t/\Delta)^s_{\{t=0\}}\right.$, where f' is the derivation of f with respect to $t = q^2$.

We show that $\beta^*_{2(\mu+2)} < 0$ for sufficiently large n. We recall here that $\mu = \lfloor n/24 \rfloor$. When we set $h(t) = \prod_{r=1}^{\infty}(1-t^r)^{-24}$, we have

$$
\begin{aligned}
&b_{2(\mu+1)} \\
&= \tfrac{-j}{(\mu+1)!}\tfrac{d^{\mu}}{dt^{\mu}}\left(E_4^{2-\nu}\theta_1^{j-1}(\theta_1 E_4' - \theta_1' E_4)(h(q))^{\mu+1}\right)_{\{t=0\}}, \\
&b_{2(\mu+2)} \\
&= \tfrac{-j}{(\mu+2)!}\tfrac{d^{\mu+1}}{dt^{\mu+1}}\left(E_4^{5-\nu}\theta_1^{j-1}(\theta_1 E_4' - \theta_1' E_4)(h(q))^{\mu+2}\right)_{\{t=0\}}.
\end{aligned}
$$

We show that $|b_{2(\mu+2)}/b_{2(\mu+1)}|$ is bounded, which implies that $\beta^*_{2(\mu+2)} < 0$ as $n \to \infty$ since the equations (2) and the inequality (3) hold.

We now apply Lemma 3.1 with $G(t) = G_1(t) = E_4^{2-\nu}\theta_1^{j-1}(\theta_1 E_4' - \theta_1' E_4)h(t)$ and $H(t) = h(t)$. Then, as is shown in [12], and using Lemma 3.2, the hypotheses (i) and (ii) in Lemma 3.1 are satisfied. So,

$$b_{2(\mu+1)} \sim -(2\pi)^{1/2} j c_2^{-1/2} \mu^{-3/2} G_1(e^{-2\pi y_0}) c_1^{\mu},\ \textit{as}\ r \to \infty.$$

where c_1 and c_2 are constants. Similarly with $G(q) = G_2(q) = E_4^{5-\nu}\theta_1^{j-1}(\theta_1 E_4' - \theta_1' E_4)h(q)$ and $H(q) = h(q)$.

$$b_{2(\mu+2)} \sim -(2\pi)^{1/2} j c_2^{-1/2} \mu^{-3/2} G_2(e^{-2\pi y_0}) c_1^{\mu+1},\ \textit{as}\ r \to \infty.$$

Hence $|b_{2(\mu+2)}/b_{2(\mu+1)}|$ is bounded (In fact, it approaches a limit of about 1.64×10^5 as $\mu \to \infty$). $\square$

Remark 3.1. *Using the equations (2), the coefficient* $\beta^*_{2(\mu+2)}$ *first attains a negative value as* n *is about* 1.64×10^5*.*

Remark 3.2. *For* $k=5$ *and* 6*, we could not show* $G(e^{-2\pi y_0})\neq 0$ *in the hypothesis (ii) in Lemma 3.1. The method of Lemma 3.2 does not work because there are no Hauptmoduls for the groups* $\Gamma_0(20)$ *and* $\Gamma_0(24)$ *since the groups are not genus zero.*

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References

[1] E. Bannai, S. T. Dougherty, M. Harada and M. Oura, Type II codes, even unimodular lattices, and invariant rings, *IEEE Trans. Inform. Theory* **45** (1999), 1194–1205.

[2] E. Bannai, K. Kojima, T. Miezaki, On the zeros of Hecke-type Faber polynomials, *Kyushu J. Math.* **62-1** (2008), 15–61.

[3] E. Bannai, S. T. Dougherty, M. Harada and M. Oura, Type II codes, even unimodular lattices, and invariant rings, *IEEE Trans. Inform. Theory* **45** (1999), 1194–1205.

[4] A. Bonnecaze, P. Solé, C. Bachoc and B. Mourrain, Type II codes over $\mathbb{Z}_4$, *IEEE Trans. Inform. Theory* **43** (1997), 969–976.

[5] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109-2** (1992), 405–444.

[6] J. H. Bruinier, G. van der Geer, G. Harder, D. Zagier, *The 1-2-3 of modular forms*, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004. Edited by Kristian Ranestad. Universitext. Springer-Verlag, Berlin, 2008.

[7] J. H. Conway, S. P. Norton, Monstrous Moonshine, *Bull. Lond. Math. Soc.* **11** (1979), 308–339.

[8] J. H. Conway and N. J. A. Sloane, *Sphere Packing, Lattices and Groups (3rd ed.)*, Springer-Verlag, New York, 1999.

[9] M. Harada and T. Miezaki, An Upper Bound on the Minimum Weight of Type II $\mathbb{Z}_{2k}$-Codes, *J. Combin. Theory Ser. A*, (to appear).

[10] M. Harada, P. Solé and P. Gaborit, Self-dual codes over $\mathbb{Z}_4$ and unimodular lattices: a survey, Algebras and combinatorics (Hong Kong, 1997), 255–275, Springer, Singapore, 1999.

[11] T. Ibukiyama, Application of Modular Forms to Lattices, Proceedings of the Second Spring Conference on Modular Forms and Related Topics, Careac Hamamatsu, February (2003) (in Japanese).

[12] C. L. Mallows, A. M. Odlyzko and N. J. A. Sloane, Upper bounds for modular forms, lattices, and codes, *J. Algebra* **36** (1975), 68–76.

[13] C. L. Mallows and N. J. A. Sloane, An upper bound for self-dual codes, *Inform. Control* **22** (1973), 188–200.

[14] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[15] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th edition, Cambridge University Press, Cambridge, 1963.

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