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Minimal Surface Area Related to Kelvin's Conjecture [1]

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Abstract

How can a space be divided into cells of equal volume so as to minimize the surface area of the boundary? L. Kelvin conjectured that the partition made by a tiling (packing) of congruent copies of the truncated octahedron with slightly curved faces is the solution ([4]). In 1994, D. Phelan and R. Weaire [6] showed a counterexample to Kelvin's conjecture, whose tiling consists of two kinds of cells with curved faces.

In this paper, we study the orthic version of Kelvin's conjecture: the truncated octahedron has the minimum surface area among all polyhedral space-fillers, where a polyhedron is called a *space-filler* if its congruent copies fill space with no gaps and no overlaps. We study the new conjecture by restricting a family of space-fillers to its subfamily consisting of *unfoldings* of *doubly covered* cuboids (rectangular parallelepipeds), we show that the truncated octahedron has the minimum surface area among all polyhedral unfoldings of a doubly covered cuboid with relation $\sqrt{2}$: $\sqrt{2}$: 1 for its edge lengths. We also give the minimum surface area of polyhedral unfoldings of a doubly covered cube.

1 Introduction.

The problem of foams, first raised by L. Kelvin, is easy to state and hard to solve ([4]). How can space be divided into cells of equal volume so as to minimize the surface area of the boundary? Kelvin conjectured that the partition made by a tiling (packing) of congruent copies of the truncated octahedron with slightly curved faces is the solution. This tiling satisfies the conditions discovered by Plateau more than a century ago for minimal soap bubbles.

In 1994, the physicists D. Phelan and R. Weaire [6] showed a counter-example to Kelvin's conjecture. They produced foam with cells of equal volume with a smaller surface area than the Kelvin foam. The Phelan-Weaire foam contains two different types of cells with 14 faces and 12 faces. Recently, R. Gabbrielli [1] proposed a new counterexample to Kelvin's conjecture. The Gabbrielli foam contains four different types of cells and its surface area is smaller than the Kelvin foam but larger than the Phelan-Weaire foam.

For the two dimensional case (the plane), the classical honeycomb conjecture that

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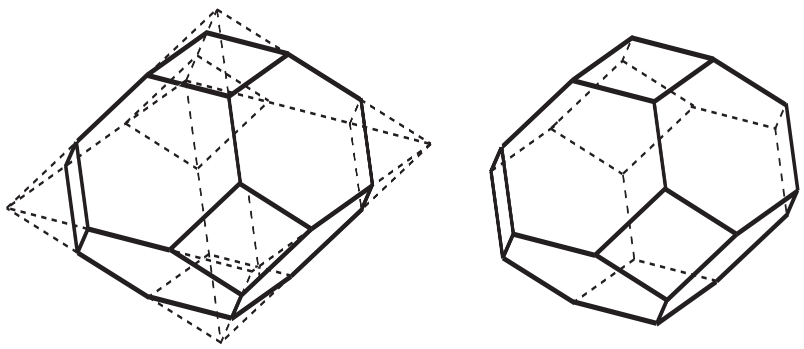


Figure 1: The truncated octahedron

any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling, was proved by F. Tóth [5], under the hypothesis that cells are convex, and it was proved completely by T. C. Hales [2] without the convexity hypothesis.

For the three dimensional case, we consider the minimal surface area problem under the hypothesis that all cells are congruent and convex. A body (a compact set homeomorphic to the closed ball in $\mathbb{R}^3$) is called a *space-filler* if its congruent copies fill the space with no gaps and no interior overlaps. It is easy to see that if a space-filler is convex, it should be a polyhedron. Our question is if any partition of $\mathbb{R}^3$, by any space-filler, has boundary area at least that of the tiling by the truncated octahedron with equal volume. In other words, does the truncated octahedron have the minimum surface area among convex space-fillers?.

The orthic version of Kelvin's conjecture. *Any convex space-filler has the surface area at least that of the truncated octahedron with equal volume.*

In this paper, we study the conjecture mentioned above by restricting the set of convex space-fillers to a smaller subset which consists of *unfoldings* of *doubly covered cuboids* (see Definition 1 and Definition 2 for detail). We show that the truncated octahedron has the minimum surface area among all convex unfoldings of doubly covered cuboid, with relation $\sqrt{2} : \sqrt{2} : 1$ for its edge lengths.

2 Definitions and Theorems.

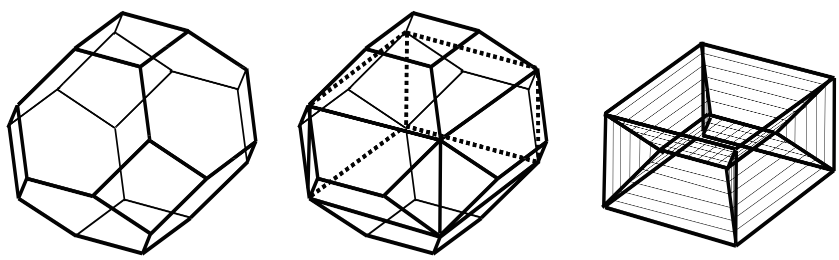
We call a compact set, $W \subset \mathbb{R}^3$, a *body* if W is homeomorphic to a closed unit ball in $\mathbb{R}^3$. For a body W in $\mathbb{R}^3$, we denote the surface area of W by $area(W)$ and denote the volume of W by $vol(W)$.

Definition 1. Let P be a polyhedron. The *doubly covered P* (denoted by $D(P)$) is the degenerated polytope in 4-space consisting of P and its congruent copy (denoted by P^*)

whose corresponding faces are identified.

Definition 2. Let P be a polyhedron. A body W is called an *unfolding* of $D(P)$ if W is mapped onto $D(P)$ by a locally isometric map (denoted by $\phi_{W,D(P)}$) with no 3-dimensional overlaps. We call $\phi_{W,D(P)}$ a *folding map* of W onto $D(P)$, and the image of the boundary of W a *cut 2-complex* of $D(P)$ for W. Then the map $\phi_{W,D(P)}$ is one-to-one in the interior of W, but not one-to-one on the boundary of W. For the sake of simplicity, we assume that the cut 2-complex is included in P^*.

Fig. 2 shows the unfolding map of the truncated octahedron W onto the doubly covered cuboid with relation $\sqrt{2} : \sqrt{2} : 1$ for its edge lengths. W is divided into a cuboid and six pieces (Fig. 2 (2)), each of the six pieces is reflected in the common face with the cuboid, and we obtain the doubly covered cuboid (Fig. 2 (3), see [3] for more details).



(1) (2) (3)

Figure 2: The Folding map of the truncated octahedron onto the doubly covered cuboid

Theorem 1. *Let α be the surface area of the truncated octahedron with unit volume, that is, $\alpha = 5.31472\ldots$. Let P be a cuboid with relation $\sqrt{2} : \sqrt{2} : 1$ for its edge lengths. Then for any convex unfolding W of the doubly covered cuboid $D(P)$, we have*

$$area(W) \geq \alpha \cdot (vol(W))^{2/3}.$$

Equality is attained if and only if W is the truncated octahedron.

For cubes, we obtain the following theorem.

Theorem 2. For any convex unfolding W of a doubly covered cube, we have

$$area(W) \geq \beta \cdot (vol(W))^{2/3},$$

where $\beta = 5.34525\ldots$.

3 Preliminaries.

In the rest of this paper, we denote by P a cuboid with the vertices v_i $(1 \leq i \leq 4)$ and w_i $(1 \leq i \leq 4)$ in Fig. 3 (1). The geometric properties of convex unfoldings of $D(P)$ are characterized in [3], and shown in Lemma 1 below.

Definition 3. Draw a rectangle $\Box p_1p_2p_3p_4$ in P^*, which is parallel to a face $(\Box v_1v_2v_3v_4)$ of P^* and whose edges are parallel to v_1v_2 or v_2v_3 (Fig. 3 (1)). Then we have the cut 2-complex of $D(P)$ consisting of the set $\{\Box v_iv_{i+1}p_{i+1}p_i, \Box w_iw_{i+1}p_{i+1}p_i, \Delta v_ip_iw_i : i = 1, 2, 3, 4\}$, where $v_5 = v_1$ and $w_5 = w_1$ and some of them may be empty ((Fig. 3 (2)). Divide P^* by the cut 2-complex and reflect each piece in the face common with P^*. Then we get an unfolding of $D(P)$ (Fig. 3 (3)). We call such unfolding a *generalized truncated octahehedron*. It is possible that $\Box xyzw$ is included in the surface of P^*. The rectangle in Fig. 3 (1) may be a line segment (for example, when $p_1 = p_4$, $p_2 = p_3$, $p_1 \neq p_2$) or a single point when $p_1 = p_2 = p_3 = p_4$.

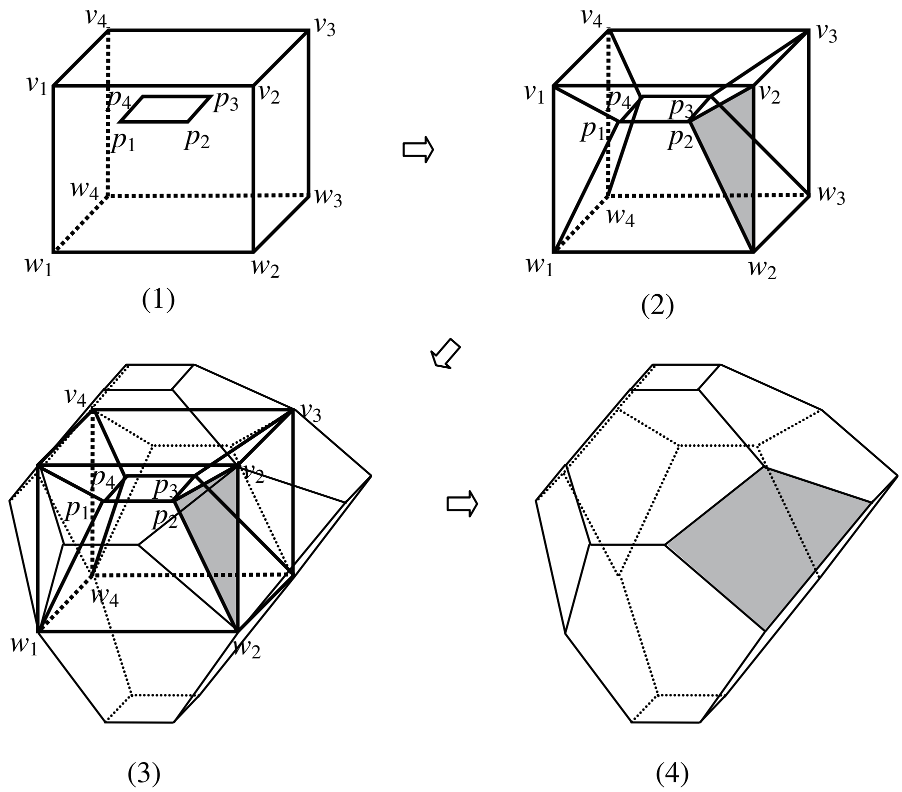


Figure 3: A generalized truncated octahedron

Lemma 1 [3]. *Let W be a convex unfolding of $D(P)$. Then the cut 2-complex has a rectangle $p_1p_2p_3p_4$ (which may be a line segment or a single point) in Fig. 2 (2) or a parallelogram in Fig. 3. The unfolding W is a generalized truncated octahedron or a* *parallelepiped.*

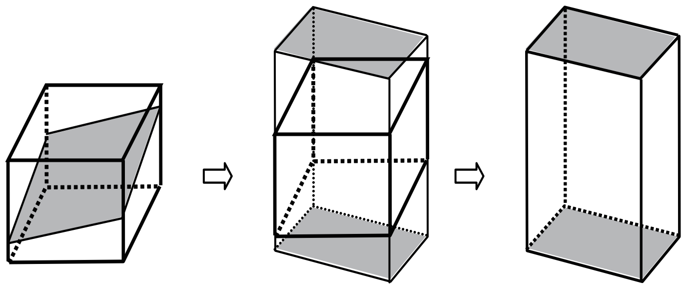


Figure 4: A parallelepiped

Let $\Box p_1p_2p_3p_4$ be a rectangle in Fig. 2 (1). We let

$$|v_1v_2| = a, |v_1v_4| = b, |v_1w_1| = c, |p_1p_2| = a - x, |p_1p_4| = b - y,$$

where $|uv|$ means the edge length of an edge uv. We denote by a function $f_{a,b,c}(x, y)$ the area of a cut 2-complex of $D(P)$ in Fig. 3 (2).

Notice that the area of the cut 2-complex for an unfolding W of $D(P)$ is half of the surface area of W. So, to find the minimal surface area of W is equivalent to minimizing $f_{a,b,c}(x, y)$.

Lemma 2. *If $f_{a,b,c}(x, y)$ attains the minimum among all cut 2-complexes of $D(P)$, then the center of $\Box p_1p_2p_3p_4$ is identical to the center of P.*

Proof. For two trapezoids $v_1p_1p_2v_2$ and $w_3p_3p_4w_4$, the sum of their heights is at least $\sqrt{y^2 + c^2}$ and equality holds when the two trapezoids are parallel. Hence the sum of their area is at least $\{a + (a - y)\} \sqrt{y^2 + c^2}\,/2$ and equality holds if and only if the center of $\Box p_1p_2p_3p_4$ is identical to the center of P. $\Box$

So, we assume the center of $\Box p_1p_2p_3p_4$ is the center of P in the rest of paper.

Lemma 3. *Let a, b, c be given positive real numbers. The notation $f(x, y)$ stands for $f_{a,b,c}(x, y)$, that is, the area of a cut 2-complex of $D(P)$ in Fig. 3 (2). We have*

(1) $f(x, y) = \sqrt{c^2 + x^2}\,(2b - y) + \sqrt{c^2 + y^2}\,(2a - x) + \sqrt{x^2 + y^2}\,c + (a - x)(b - y).$

(2) $\dfrac{\partial f}{\partial x}(x, y) = \dfrac{x}{\sqrt{c^2 + x^2}}\,(2b - y) - \sqrt{c^2 + y^2} + \dfrac{cx}{\sqrt{x^2 + y^2}} - (b - y)$

(3) $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{y}{\sqrt{c^2+x^2}}\ (2a-x) - \sqrt{c^2+x^2} + \dfrac{cy}{\sqrt{x^2+y^2}} - (a-x)$

(4) $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = 1 - \dfrac{x}{\sqrt{c^2+x^2}} - \dfrac{y}{\sqrt{c^2+y^2}} - \dfrac{xy}{(x^2+y^2)^{3/2}}.$

(5) $\dfrac{\partial^2 f}{(\partial x)^2}(x,y) = \dfrac{(2b-y)c^2}{(c^2+x^2)^{3/2}} + \dfrac{cy^2}{(x^2+y^2)^{3/2}} > 0$

(6) $\dfrac{\partial^2 f}{(\partial y)^2}(x,y) = \dfrac{(2a-x)c^2}{(c^2+y^2)^{3/2}} + \dfrac{cx^2}{(x^2+y^2)^{3/2}} > 0$

Proof. We denote by the area of a polygon A by $|A|$. Then
$f(x,y) = |\Box v_1v_2p_2p_1| + |\Box w_1w_2p_2p_1| + |\Box v_2v_3p_3p_2| + |\Box w_2w_3p_3p_2| +$
$|\Box v_3v_4p_4p_3| + |\Box w_3w_4p_4p_3| + |\Box v_4v_1p_1p_4| + |\Box w_4w_1p_1p_4| +$
$|\Delta v_1w_1p_1| + |\Delta v_2w_2p_2| + |\Delta v_3w_3p_3| + |\Delta v_4w_4p_4| + |\Box p_1p_2p_3p_4|.$
By Lemma 2, we have
$f(x,y) = 4|\Box v_1v_4p_4p_1| + 4|\Box v_1v_2p_2p_1| + 4|\Delta v_1w_1p_1| + |\Box p_1p_2p_3p_4|$
$= \sqrt{c^2+x^2}\ (2b-y) + \sqrt{c^2+y^2}\ (2a-x) +$
$\sqrt{x^2+y^2}\ c + (a-x)(b-y).$
Hence (1) holds. By standard calculations we have (2) through (6).

□

4 The minimum of the function $f_{a,b,c}(x,y)$ in the case $a = b \geq c = 1$.

In this section we suppose $a = b \geq c = 1$ and let $L = a = b$. Then $L \geq 1$. The notation $f(x,y)$ stands for the function $f_{L,L,1}(x,y)$. We will prove the following proposition.

Proposition 1. *Let $L \geq 1$. The function $f_{L,L,1}(x,y)$ for $0 \leq x,y \leq L$ attains its minimum when $x = y$. For the case $L = \sqrt{2}$, $f_{\sqrt{2},\sqrt{2},1}(x,y)$ attains its minimum when $x = y = 1/\sqrt{2}$, for which the corresponding unfolding of $D(P)$ is the truncated octahedron.*

By equations (1) through (6), we have the following:

(7) $f(x,y) = \sqrt{1+x^2}\ (2L-y) + \sqrt{1+y^2}\ (2L-x) + \sqrt{x^2+y^2} + (L-x)(L-y).$

(8) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{x}{\sqrt{1+x^2}}\ (2L-y) - \sqrt{1+y^2} + \dfrac{x}{\sqrt{x^2+y^2}} - (L-y)$

(9) $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{y}{\sqrt{1+x^2}}\,(2L-x) - \sqrt{1+y^2} + \dfrac{y}{\sqrt{x^2+y^2}} - (L-x)$

(10) $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = 1 - \dfrac{x}{\sqrt{1+x^2}} - \dfrac{y}{\sqrt{1+y^2}} - \dfrac{xy}{(x^2+y^2)^{3/2}}.$

(11) $\dfrac{\partial^2 f}{(\partial x)^2}(x,y) = \dfrac{2L-y}{(1+x^2)^{3/2}} + \dfrac{y^2}{(x^2+y^2)^{3/2}} > 0$

(12) $\dfrac{\partial^2 f}{(\partial y)^2}(x,y) = \dfrac{2L-x}{(1+y^2)^{3/2}} + \dfrac{x^2}{(x^2+y^2)^{3/2}} > 0.$

We prove Proposition 1 by dividing the domain of the function $f(x,y) = f_{L,L,1}(x,y)$ into four subsets $\{(x,y):\ 1/\sqrt{3} \le x, y \le L\}$, $\{(x,y):\ 0 \le x \le 1/\sqrt{3} \le y \le L\}$, $\{(x,y):\ 0 \le y \le 1/\sqrt{3} \le x \le L\}$, and $\{(x,y):\ 0 \le x, y \le 1/\sqrt{3}\}$.

Lemma 4. *Let x_o and y_o satisfy $1/\sqrt{3} \le x_o, y_o \le L$. If*

$$\frac{\partial f}{\partial x}(x_o, y_o) = \frac{\partial f}{\partial y}(x_o, y_o) = 0,$$

then

$$x_o = y_o.$$

Moreover, there exists at least one t ($1/\sqrt{3} \le t \le L$) which satisfies $\dfrac{\partial f}{\partial x}(t,t) = \dfrac{\partial f}{\partial y}(t,t) = 0.$

Proof. Suppose $1/\sqrt{3} \le x, y \le L$. Since by the equation (10)

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(L,y) &= 1 - \frac{L}{\sqrt{1+L^2}} - \frac{y}{\sqrt{1+y^2}} - \frac{Ly}{(L^2+y^2)^{3/2}} \\ &< 1 - \frac{1}{\sqrt{2}} - \frac{1}{2} - \frac{Ly}{(L^2+y^2)^{3/2}} < 0,\end{aligned}$$

we have $\dfrac{\partial}{\partial x} f(L,y)$ is decreasing with respect to y for $1/\sqrt{3} \le y \le L$. Since $\dfrac{\partial f}{\partial x}(L,L) = \dfrac{L^2}{\sqrt{1+L^2}} - \sqrt{1+L^2} + \dfrac{1}{\sqrt{2}} = -\dfrac{1}{\sqrt{1+L^2}} + \dfrac{1}{\sqrt{2}} \ge -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = 0$ by $L \ge 1$, we have

(13) $\dfrac{\partial f}{\partial x}(L,y) > 0 \quad$ for $1/\sqrt{3} \le y \le L.$

Since

$$\frac{\partial f}{\partial x}(\frac{1}{\sqrt{3}}, y) = \frac{1}{2}(2L - y) - \sqrt{1+y^2} + \frac{1}{\sqrt{1+3y^2}} - (L - y)$$
$$= \frac{y}{2} - \sqrt{1+y^2} + \frac{1}{\sqrt{1+3y^2}}$$

and

$$(\sqrt{1+y^2})^2 - (\frac{1}{2}y + \frac{1}{\sqrt{1+3y^2}})^2 = 1 + \frac{3}{4}y^2 - \frac{y}{\sqrt{1+3y^2}} - \frac{1}{1+3y^2}$$
$$= (\frac{1}{2} - \frac{1}{1+3y^2}) + (\frac{1}{2} + \frac{3}{4}y^2 - \frac{y}{\sqrt{1+3y^2}}) \geq 0 + (\frac{1}{2} + \frac{1}{4} - \frac{1}{\sqrt{3}}) > 0,$$

we have

(14) $\frac{\partial f}{\partial x}(\frac{1}{\sqrt{3}}, y) < 0 \quad$ for $1/\sqrt{3} \leq y \leq L$.

By (13), (14), and the Implicit Function Theorem, there is a function $g(y)$ defined on $1/\sqrt{3} \leq y \leq L$, which satisfies

(15) $\frac{\partial f}{\partial x}(g(y), y) = 0,$

(16) $1/\sqrt{3} \leq g(y) \leq L$, and

(17) $g'(y) = -\frac{(\partial^2 f/\partial x \partial y)(g(y), y)}{(\partial^2 f/(\partial x)^2)(g(y), y)}.$

Since for $1/\sqrt{3} \leq x, y \leq L$,

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = 1 - \frac{x}{\sqrt{1+x^2}} - \frac{y}{\sqrt{1+y^2}} - \frac{xy}{(x^2+y^2)^{3/2}}$$
$$< 1 - \frac{1}{\sqrt{1+1/x^2}} - \frac{1}{\sqrt{1+1/y^2}} \leq 0,$$

we have by (17) $g'(y) > 0$ for $1/\sqrt{3} \leq y \leq L$, and hence

(18) $g(y)$ is strictly increasing on $1/\sqrt{3} \leq y \leq L$.

Since $f(x, y) = f(y, x)$, for $1/\sqrt{3} \leq x, y \leq L$

(19) $\frac{\partial f}{\partial x}(g(y), y) = \frac{\partial f}{\partial y}(y, g(y)) = 0.$

Hence if $\dfrac{\partial f}{\partial x}(x_o, y_o) = \dfrac{\partial f}{\partial y}(x_o, y_o) = 0$ for some (x_o, y_o) with $1/\sqrt{3} \le x_o, y_o \le L$, then $x_o = g(y_o)$, for, if $x_o \le y_0$ (resp. $x_o \ge y_0$) , then $x_o = g(y_o) \ge g(x_o) = y_o$ (resp. $x_o = g(y_o) \le g(x_o) = y_o$) by (18) and so $x_o = y_o$. Moreover, there exists at least one $t\,(1/\sqrt{3} \le t \le L)$ which satisfies $\dfrac{\partial f}{\partial x}(t,t) = \dfrac{\partial f}{\partial y}(t,t) = 0$.

□

Lemma 5. *For x and y satisfying $0 \le x \le 1/\sqrt{3} \le y \le L$, we have*

$$f(x, y) \ge f(1/\sqrt{3}, y),$$

and for x and y satisfying $0 \le y \le 1/\sqrt{3} \le x \le L$, we have

$$f(x, y) \ge f(x, 1/\sqrt{3}).$$

Proof. Let x and y satisfy $0 \le x \le 1/\sqrt{3} \le y \le L$. By (14), $\dfrac{\partial f}{\partial x}(1/\sqrt{3}, y) < 0$. Since $\dfrac{\partial f}{\partial x}(x, y)$ is increasing with respect to x for $0 \le x \le 1/\sqrt{3}$ by (8), we have $\dfrac{\partial f}{\partial x}(x, y) < 0$ for all $0 \le x \le 1/\sqrt{3}$. Hence $f(x, y)$ is decreasing with respect to x for $0 \le x \le 1/\sqrt{3}$ and so $f(x, y) \ge f(1/\sqrt{3}, y)$.

By symmetry $f(x, y) = f(y, x)$, the latter part of Lemma 5 holds. □

Lemma 6. *For any x and y satisfying $0 \le x, y \le 1/\sqrt{3}$*

$$f(x, y) \ge \min\{f(t,t) :\ 0 \le t \le 1/\sqrt{3}\}.$$

Proof. Let x and y satisfy $0 \le x, y \le 1/\sqrt{3}$ and suppose $x \le y$. Since $\dfrac{\partial^2 f}{(\partial x)^2}(x, y) > 0$ by (11), $\dfrac{\partial f}{\partial x}(x, y)$ is increasing with respect to x for $0 \le x \le L$. By $x \le y$

$$\frac{\partial f}{\partial x}(x, y) \le \frac{\partial f}{\partial x}(y, y) = \frac{y}{\sqrt{1+y^2}}(2L - y) - \sqrt{1+y^2} + \frac{1}{\sqrt{2}} - (L - y)$$

$$= \left(\frac{2y}{\sqrt{1+y^2}} - 1\right) L - \frac{y^2}{\sqrt{1+y^2}} - \sqrt{1+y^2} + \frac{1}{\sqrt{2}} + y.$$

Since $\frac{2y}{\sqrt{1+y^2}} - 1 \le 0$ by $0 \le y \le \frac{1}{\sqrt{3}}$ and $L \le 1$,

$$\frac{\partial f}{\partial x}(x, y) \le \frac{2y}{\sqrt{1+y^2}} - 1 - \frac{y^2}{\sqrt{1+y^2}} - \sqrt{1+y^2} + \frac{1}{\sqrt{2}} + y$$

$$= -\frac{(y-1)^2}{\sqrt{1+y^2}} - 1 - \frac{y^2}{\sqrt{1+y^2}} + \frac{1}{\sqrt{2}} + y < -1 + \frac{1}{\sqrt{2}} + y\left(1 - \frac{y}{\sqrt{1+y^2}}\right)$$

$$\leq -1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\left(1 - \frac{1/\sqrt{3}}{(1/\sqrt{3})^2+1}\right) < 0.$$

Hence for any fixed y ($0 \leq y \leq 1/\sqrt{3}$), $f(x,y)$ is decreasing with respect to x for $0 \leq x \leq 1/\sqrt{3}$ with $x \leq y$, and so $f(x,y) \geq f(y,y)$. By the symmetry $f(x,y) = f(y,x)$, we have for any fixed x ($0 \leq x \leq 1/\sqrt{3}$), $f(x,y) \geq f(x,x)$ for any x and y satisfying $0 \leq y \leq 1/\sqrt{3}$ with $x \geq y$. Therefore $f(x,y) \geq \min\{f(t,t) : \ 0 \leq t \leq 1/\sqrt{3}\}$. $\square$

Lemma 7. *Suppose $L = \sqrt{2}$. The function $f(t,t)$ for $0 \leq t \leq \sqrt{2}$ attains its minimum at $t = 1/\sqrt{2}$.*

Proof. Let $h(t) = f(t,t)$ for $0 \leq t \leq \sqrt{2}$. Then

$$h'(t) = \frac{2t}{\sqrt{1+t^2}}(2\sqrt{2} - t) - 2\sqrt{1+t^2} - \sqrt{2} + 2t$$

and $h'(t) = 0$ if and only if $t = 1/\sqrt{2}$. Therefore, the function $f(t,t)$ attains the minimum at $t = 1/\sqrt{2}$. $\square$

Proof of Proposition 1. Let $L \geq 1$. By Lemma 4 through Lemma 7 $f_{L,L,1}(x,y)$ for $0 \leq x, y \leq L$ attains its minimum when $x = y$. In particular, when $L = \sqrt{2}$, we get

$$\frac{\partial f}{\partial x}(1/\sqrt{2}, 1/\sqrt{2}) = \frac{\partial f}{\partial y}(1/\sqrt{2}, 1/\sqrt{2}) = 0$$

and

$$\frac{\partial^2 f}{(\partial x)^2}(1/\sqrt{2}, 1/\sqrt{2}) \times \frac{\partial^2 f}{(\partial y)^2}(1/\sqrt{2}, 1/\sqrt{2}) - \left(\frac{\partial^2 f}{\partial x \partial y}(1/\sqrt{2}, 1/\sqrt{2})\right)^2$$
$$= 2.3\cdots > 0.$$

Hence the function $f(x,y)$ attains a minimal value at $x = y = 1/\sqrt{2}$. The value $f(1/\sqrt{2}, 1/\sqrt{2})$ is not only minimal but also the minimum value by the fact mentioned above.

If $a = b = \sqrt{2}$ and $x = y = 1/\sqrt{2}$, then trapezoids $v_i v_{i+1} p_{i+1} p_i$ ($1 \leq i \leq 4$), where suffixes are considered modulo 4, are congruent to a half of a regular hexagon, and $\Delta v_i p_i w_i$ ($1 \leq i \leq 4$) are congruent to a half square, which implies that the corresponding unfolding of $D(P)$ is the truncated octahedron. Therefore, by Lemma 4 through Lemma 7, we have proved Proposition 1. $\square$

5 The minimum of the function $f_{a,b,c}(x,y)$ in the case $a = c = \sqrt{2}$ and $b = 1$.

In this section we suppose $a = c = \sqrt{2}$ and $b = 1$, and the notation $f(x,y)$ stands for the function $f_{\sqrt{2},1,\sqrt{2}}(x,y)$. We will prove the following proposition.

Proposition 2. *For $0 \le x \le \sqrt{2}$ and $0 \le y \le 1$,*

$$f_{\sqrt{2},1,\sqrt{2}}(x,y) \;\ge\; f_{\sqrt{2},\sqrt{2},1}(1/\sqrt{2},1/\sqrt{2}).$$

Proof. We will prove Proposition 2 by dividing the domain of the function $f(x,y) = f_{\sqrt{2},1,\sqrt{2}}(x,y)$ into four subsets $\{(x,y) :\; 2\sqrt{2}/3 \le x \le \sqrt{2},\; 0 \le y \le 1\}$, $\{(x,y) :\; 0 \le x \le 2\sqrt{2}/3,\; 2/3 \le y \le 1\}$, $\{(x,y) :\; 0 \le x \le 2\sqrt{2}/3,\; 0 \le y \le 2/3,\; y \le x/\sqrt{2}\}$, and $\{(x,y) :\; 0 \le x \le 2\sqrt{2}/3,\; 0 \le y \le 2/3,\; y \ge x/\sqrt{2}\}$.

Since the process of the proof is a somewhat tedious, we state the outline briefly.

Case 1 : Let $(x,y) \in \{(x,y) :\; 2\sqrt{2}/3 \le x \le \sqrt{2},\; 2/3 \le y \le 1\}$.
Since

$$\frac{\partial f^2}{(\partial x)(\partial y)}(x,1) = -\frac{x}{\sqrt{2+x^2}} - \frac{1}{\sqrt{3}} - \frac{\sqrt{2}x}{(x^2+1)^{3/2}} + 1 < 0,$$

$\dfrac{\partial f}{\partial y}(x,1)$ is decreasing with respect to x. Since

$$\frac{\partial f}{\partial y}(x,1) \le \frac{\partial f}{\partial y}(2\sqrt{2}/3,1) = -0.05\cdots < 0,$$

by (6), $\dfrac{\partial f}{\partial y}(x,y) < \dfrac{\partial f}{\partial y}(x,1) < 0$, which implies $f(x,y) \ge f(x,1)$. When $y=1$, we have $|p_1p_4| = 1 - y = 0$ and the rectangle $p_1p_2p_3p_4$ is a line segment. This case is included in the case $a = b = \sqrt{2}$ and $c = 1$ studied in the section 4.

Case 2 : Let $(x,y) \in \{(x,y) :\; 0 \le x \le 2\sqrt{2}/3,\; 2/3 \le y \le 1\}$.
Since by (5) $\dfrac{\partial f}{\partial x}(x,y)$ is increasing, we have

$$\frac{\partial f}{\partial x}(x,y) < \frac{\partial f}{\partial x}(2\sqrt{2}/3,y)$$

$$= (1 - 2/\sqrt{13})y - \sqrt{2+y^2} + 4/\sqrt{13} + 4/\sqrt{8+9y^2} - 1 < -0.002\cdots.$$

Hence $f(x,y) \ge f(2\sqrt{2}/3,y)$ and the point $(2\sqrt{2}/3,y)$ is included in Case 1.

Case 3 : Let $(x,y) \in \{(x,y) :\; 0 \le x \le 2\sqrt{2}/3,\; 0 \le y \le 2/3,\; y \le x/\sqrt{2}\}$.
By (6), we have

$$\frac{\partial f}{\partial y}(x,y) \le \frac{\partial f}{\partial y}(x,x/\sqrt{2}) = -\sqrt{2+x^2} + \frac{x(2\sqrt{2}-x)}{\sqrt{4+x^2}} + \frac{\sqrt{2}}{\sqrt{3}} - \sqrt{2} + x$$

$$< -\sqrt{2} + \frac{4}{\sqrt{11}} + \frac{\sqrt{2}}{\sqrt{3}} - \sqrt{2} + (1 - \frac{\sqrt{2}}{\sqrt{11}} \times \frac{2\sqrt{2}}{3} = -0.2\ldots < 0,$$

and $f(x,y) \ge f(x,x/\sqrt{2})$. Since $f(x,x/\sqrt{2})$ is a function of one variable, we get $f(x,x/\sqrt{2}) \ge f_{\sqrt{2},\sqrt{2},1}(1/\sqrt{2},1/\sqrt{2})$ (we used Mathematica for calculation).

Case 4 : Let $(x, y) \in \{(x, y) : \ 0 \le x \le 2\sqrt{2}/3, \ 0 \le y \le 2/3, \ y \ge x/\sqrt{2}\}$.
By (5), we have

$$\frac{\partial f}{\partial x}(x,y) \le \frac{\partial f}{\partial x}(\sqrt{y}, y) = \frac{\sqrt{2}y(2-y)}{\sqrt{2+2y^2}} - \sqrt{2+y^2} + \frac{2y}{\sqrt{3y^2}} - (1-y)$$

$$= (\frac{y}{\sqrt{1+y^2}} - 1)(1-y) + \frac{y}{\sqrt{1+y^2}} - \sqrt{2+y^2} + \frac{2}{\sqrt{3}}$$

$$< \frac{2}{\sqrt{13}} - 1 + \frac{2}{\sqrt{13}} - \sqrt{2} + \frac{2}{\sqrt{3}} = -0.15\cdots < 0,$$

and $f(x, y) \ge f(\sqrt{y}, y)$. Since $f(\sqrt{y}, y)$ is a function of one variable, we get $f(\sqrt{y}, y) \ge f_{\sqrt{2},\sqrt{2},1}(1/\sqrt{2}, 1/\sqrt{2})$ (we used Mathematica for calculation).

This complete the proof of Proposition 2. $\square$

6 Proofs of theorems.

Proof of Theorem 1. Let P be a cuboid with relation $\sqrt{2} : \sqrt{2} : 1$ for its edge lengths. Since the surface area of the truncated octahedron obtained by the cut 2-complex whose area is $f_{\sqrt{2},\sqrt{2},1}(1/\sqrt{2}, 1/\sqrt{2})$, is twice of $f_{\sqrt{2},\sqrt{2},1}(1/\sqrt{2}, 1/\sqrt{2})$, its ratio $r = 5.31472\ldots$. By Proposition 1 and Proposition 2, Theorem 1 has been proved.

Proof of Theorem 2. Let P be a cube. Then by Proposition 1 the function $f_{1,1,1,}(x, y)$ for $0 \le x, y \le \sqrt{2}$ attains its minimum on the subset $\{(t, t) : \ 0 \le t \le 1\}$. Hence by calculating the one variable function $f_{1,1,1,}(t, t) = 2\sqrt{1+t^2}(2-t) + \sqrt{2}t + (1-t)^2$, we get the minimum value $4.24\cdots$ and the ratio $r = 2 \times 4.24\cdots/2^{2/3} = 5.345\cdots$. $\square$

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