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# A note on minimal normal compactifications of $\mathbb{C}^2$

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## 1 Introduction

Let $M$ be a smooth connected compact complex surface and $C$ an analytic subset $C$ of $M$. The pair $(M, C)$ is called a compactification of $\mathbb{C}^2$ if $M - C$ is biholomorphic to $\mathbb{C}^2$. Then one sees easily that $C$ is of pure codimension one. Namely, $C$ is a divisor. We set $C = \bigcup_{i=1}^{r} C_i$, where each $C_i$ is an irreducible curve. Then it is known that

(1.1) $S$ is a rational surface (cf.[3]).

(1.2) $\mathrm{H}^i(M, \mathbb{Z}) \cong \mathrm{H}^i(C, \mathbb{Z})$ for $i < 4$ , $\mathrm{H}^1(M, \mathbb{Z}) \cong \mathrm{H}^3(M, \mathbb{Z}) = 0$ and $\mathrm{H}^2(M, \mathbb{Z}) = \mathbb{Z}^r$. In particular, each $C_i$ is simply connected and its normalization is a smooth rational curve.

**Definition 1.1** *A compactification $(M, C)$ is said to be minimal normal if it satisfies the following conditions:*

(i) *each $C_i$ is smooth,*

(ii) *the singular points of $C = \bigcup_{i=1}^{r} C_i$ are ordinary double points and*

(iii) *no non-singular rational component of $C$ with self-intersection number $-1$ has at most two intersection points with other components of $C$.*

Then Morrow [4], applying the topological results of Ramanujan [7], proved the following:

**Theorem 1.1** *Let $(M, C)$ be a minimal normal compactification of $\mathbb{C}^2$ and we set $C = \bigcup_{i=1}^{r} C_i$. Then the dual graph $\Gamma(C)$ of $C$ is one of the types $(\Gamma_a) \sim (\Gamma_g)$ in* Table I.

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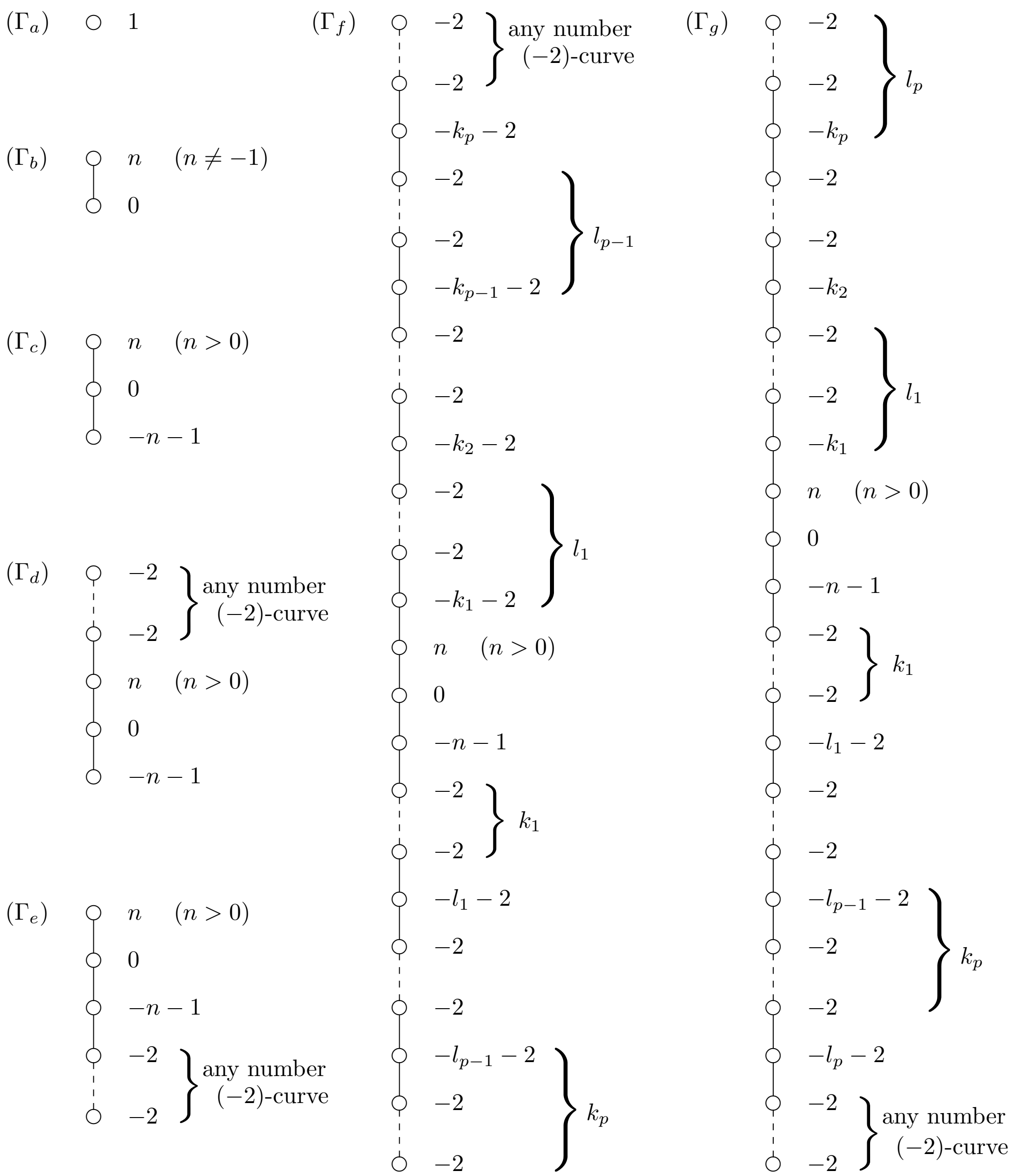
**Key words**: compactification, rational surface, cluster set.

Now, in this note, applying the theory of cluster sets of holomorphic mappings at the essential singular point according to Nishino-Suzuki [5], we shall give an elementary proof of Theorem 1.1.

Table I



# 2 The cluster set of a holomorphic mapping.

## 2.1 Analytic curves as a cluster set

Let $S$ be a nonsingular compact complex surface and $C = \bigcup_{i=1}^{s} C_i$ a compact analytic curve in $S$ satisfying

(i)' each $C_i$ $(1 \leq i \leq s)$ is a compact irreducible analytic curve,

(ii) the singular points of $C = \bigcup_{i=1}^{s} C_i$ are ordinary double points, and

(iii) no non-singular rational component of $C$ with self-intersection number $-1$ has at most two intersection points with other components of $C$.

Then Nishino-Suzuki (Théorème 5 in [6]) proved the following

**Theorem 2.1** *Assume that for each $C_i$ there exists a holomorphic mapping $\varphi_i$ : $\mathbb{C} \longrightarrow S \setminus C$ such that*

$$C_i \subset \varphi_i(\infty; S) := \bigcap_{R>0} \overline{\varphi_i(\Delta_R)} \subset C \ ,$$

*where $\Delta_R = \{z \in \mathbb{C} : |z| > R\}$ and $\overline{\varphi_i(\Delta_R)}$ is the closure of $\varphi_i(\Delta_R)$ in $S$. Then the type of $C$ is one of $(\alpha) \sim (\epsilon)$ in* Table 2*, in which for the types $(\beta_s)(s \geqq 2)$, $(\gamma)$, $(\gamma')$, $(\delta)$ and $(\epsilon)$ each irreducible component of $C$ is a non-singular rational curve and the graph $\Gamma(C)$ is depicted as* Figure 1-5 .

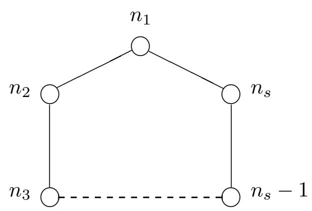



Figure 1

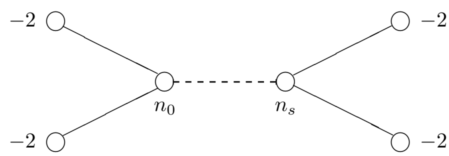



Figure 2

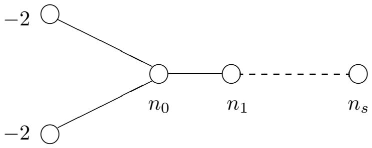



Figure 3

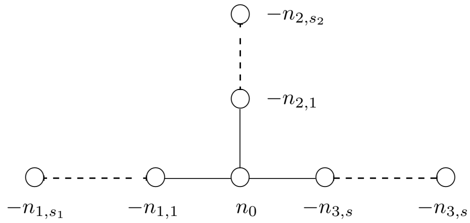



Figure 4

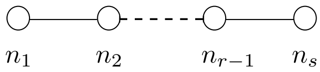



Figure 5

Table 2

| | Name of type | Explication of $C$ |
|---|---|---|
| $(\alpha)$ | $\alpha_\infty(n)$ | An irreducible non-singular elliptic curve with the self intersection number $(C^2) = n \geq 0$. |
| $(\beta)$ | $\beta_\infty(n)$ | An irreducible rational curve with only one ordinary double point and $(C^2) = n \geq 0$. |
| $(\beta_s)$ | $\beta_\infty(n_1, \cdots, n_s)(s \geq 2)$ | Figure 1, all $n_i = -2$ or $\max\{n_i\} \geq 0$. |
| $(\gamma)$ | $\gamma_\infty(n_1, \cdots, n_s)(s \geq 1)$ | Figure 2, all $n_i = -2$ or $\max\{n_0+1, n_1, \cdots, n_{s-1}, n_s+1\} \geq 0$. |
| $(\gamma')$ | $\gamma'_\infty(n_1, \cdots, n_s)(s \geq 1)$ | Figure 3, $n_i = -2$ or $\max\{n_0+1, n_1, \cdots, n_s\} \geq 0$. |
| $(\delta)$ | $\delta(n_0 \vert q_1/l_1, q_2/l_2, q_3/l_3)$ | Figure 4, (i) $n_0 \geq -2$. (ii) $(l_1, l_2, l_3) = (3,3,3), (2,4,4)$ or $(2,3,m)$ with $m = 3,4,5,6$. (iii) for each $i = 1,2,3$, $(q_i, l_i)$ is a pair of coprime integers such that $0 < q_i < l_i$ and that $\frac{l_i}{q_i} = n_{i,1} - \cfrac{1}{n_{i,2} - \cfrac{1}{n_{i,3} - \cfrac{1}{\cdots \cfrac{1}{n_{i,r_i}}}}}$ (continued fraction expansion) where $n_{i,j} \geq 2$ are integers appearing in Figure 4. |
| $(\epsilon)$ | $\epsilon_\infty(n_1, \cdots, n_s)(s \geq 1)$ | Figire 5, $\max\{n_i\} \geq 0$. |

## 2.2 Fundamental groups of the tubular neighborhoods.

Let $S$ and $C = \bigcup_{i=1}^{r} C_i$ be as in **2.1**. Let $K$ be the boundary of a tubular neighborhood of $C$ in $S$. Let $e_i$ $(1 \le i \le r)$ be a loop in $K$ that goes once around $C_i$ with positive orientation. Then one has the following (cf. Mumford [5]).

**Lemma 2.1** *The fundamental group $\pi_1(K)$ is the group generated by $e_1, e_2, \ldots, e_r$ with relations:*

$$(R_1) \qquad e_i e_j = e_j e_i \quad \text{if} \quad C_i \cap C_j \neq \emptyset$$

$$(R_2) \qquad \prod_{j=1}^{r} e_j^{s_{ij}} = 1,\ i = 1, 2, \ldots, r, \quad \text{where} \quad s_{ij} = (C_i \cdot C_j) \text{ (intersection number)}$$

*i.e.,*

$$\pi_1(K) = \big\langle e_1, e_2, \ldots, e_r \,|\, (R_1), (R_2) \big\rangle.$$

Let us introduce the integer $[n_1, n_2, \ldots, n_r] \in \mathbb{Z}$ for $n_i \in \mathbb{Z}$ $(1 \le i \le r)$ inductively as follows:

(2-a) $[\emptyset] = 1$, where $\emptyset$ denotes the empty set.

(2-b) $[n_1] = n_1$.

(2-c) $[n_1, n_2, \ldots, n_j] = n_1[n_2, \ldots, n_j] - [n_3, \ldots, n_j]$ $(2 \le j \le r)$.

Then one can easily verify the following

**Lemma 2.2** (1) $[n_1, n_2, \ldots, n_r] = [n_i, n_{i-1}, \ldots, n_1][n_{i+1}, \ldots, n_r] - [n_{i-1}, \ldots, n_1][n_{i+2}, \ldots, n_r]$.

(2) $[n_1, n_2, \ldots, n_r] = [n_r, n_{r-1}, \ldots, n_2, n_1]$.

(3) $[-n_1, -n_2, \ldots, -n_r] = (-1)^r [n_1, n_2, \ldots, n_r]$.

(4) *If $n_i \neq 0$ $(1 \le i \le r)$, then we have the continued fractional expansion*

$$\frac{[n_1, n_2, \ldots, n_r]}{[n_2, \ldots, n_r]} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \cfrac{1}{\cdots \cfrac{1}{n_r}}}}.$$

(5) $[n_1, n_2, \ldots, n_r] > [n_2, n_3, \ldots, n_r] > \cdots > [n_{r-1}, n_r] > [n_r] > 1$ if $n_i > 1$ $(1 \le i \le r)$.

(6) *Suppose that $n_i \ge 2$ $(1 \le i \le r)$. Then,*

$$[n_1, n_2, n_3 \ldots, n_r] - [n_2, n_3, \ldots, n_r] = 1 \quad \text{iff} \quad n_1 = n_2 = \cdots = n_r = 2.$$

Then we obtain the following:

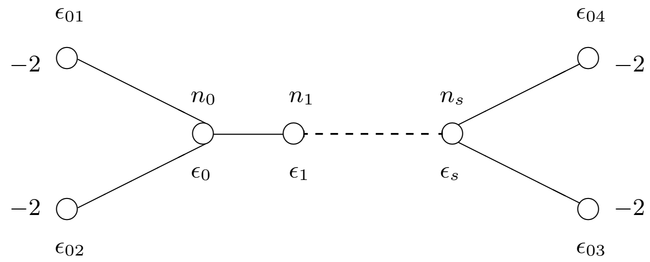
**Proposition 2.1** *Let $C$ and $K$ be as above. Assume that $\pi_1(K) = \{1\}$. Then the curve $C$ belongs to the type $(\epsilon)$.*

*Proof.* It is easy to see that $\pi_1(K) \neq 1$ for the types $(\alpha), (\beta)$ and $(\beta)_s$. Thus we need only to show that $\pi_1(K) \neq \{1\}$ for the types $(\gamma), (\gamma')$ and $(\delta)$.

*The case of type* $(\gamma)$. We set

$$C = C_{01} \cup C_{02} \cup C_{03} \cup C_{04} \cup C_0 \cup C_1 \cup \cdots \cup C_s.$$

Let us denote by $\epsilon_{0i}\,(1 \leq i \leq 4)$, $\epsilon_j\,(0 \leq i \leq s)$ the vertices of the graph $\Gamma(C)$ corresponding to the curves $C_{0i}$ and $C_j$.




$$\max\{n_0+1, n_1, \cdots, n_{s-1}, n_s+1\} \geq 0$$

Let $a_i$ $(1 \leq i \leq 4)$ and $e_j$ $(0 \leq j \leq s)$ be the loops in $K$ corresponding to the vertices $\epsilon_{0i}$ $(1 \leq i \leq 4)$, $\epsilon_j$ $(0 \leq j \leq s)$ of $\Gamma(C)$. Then we have the following relation between generators:

$(R_1)$ $a_i e_0 = e_0 a_i$ $(1 \leq i \leq 2)$, $a_i e_s = e_s a_i$ $(i = 3, 4)$, $e_i e_{i+1} e_{i+1} e_i$ $(0 \leq i \leq s-1)$.

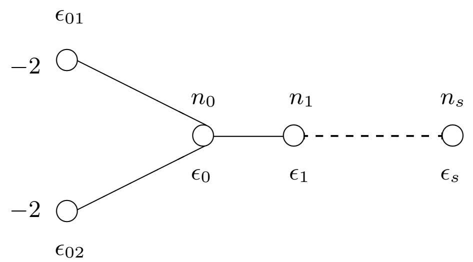
$(R_2)$ $e_1 = a_1^2 = a_2^2$, $e_s^2 = a_3^2 = a_4^2$,
$a_1 a_2 e_o^{n_0} e_1 = 1$, $e_0 e_1^{n_1} e_2 = 1, \cdots$, $e_{s-2} e_{s-1}^{n_{s-1}} e_s = 1$, $a_3 a_4 e_s^{n_s} e_{s-1} = 1$.

From this one has

$$\begin{cases} \pi_1(K)/ < e_0, e_s > \cong \mathbb{Z}_2 * \mathbb{Z}_2 \quad \text{if} \quad s > 0 \\ \pi_1(K)/ < e_0 > \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2/(a_1a_2a_3a_4 = 1) \quad \text{if} \quad s = 0\ . \end{cases}$$

This implies that $\pi_1(K) \neq \{1\}$.

*The case of type* $(\gamma')$.




$$\max\{n_0+1, n_1, \cdots, n_{s-1}, n_s\} \geq 0$$

Let $a_i$ $(1 \leq i \leq 2)$ and $e_j$ $(0 \leq j \leq s)$ be the loops in $K$ corresponding to the vertices $\epsilon_{0i}$ $(1 \leq i \leq 2)$ and $\epsilon_j$ $(0 \leq j \leq s)$ in $\Gamma(C)$.

By Lemma 2.1, we have

$$\pi_1(K) = \left\langle\ a_i(1 \le i \le 2)\ ,\ e_j\ (0 \le j \le s) \mid (R_1), (R_2)\ \right\rangle,$$

where

$(R_1)$ $a_i e_0 = e_0 a_i$ $(1 \le i \le 2), e_i e_{i+1} = e_{i+1} e_i (0 \le i \le s-1)$

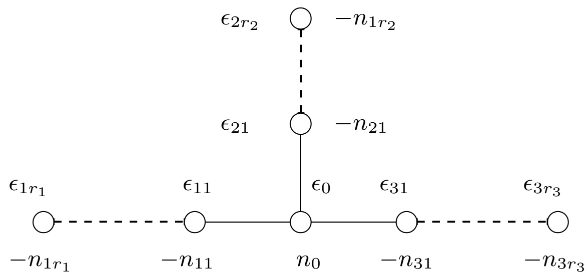
$(R_2)$ $e_0 = a_1^2 = a_2^2,\ e_0^{n_0} a_1 a_2 e_1 = 1,\ e_0 e_1^{n_1} e_2 = 1, \cdots,\ e_{s-2} e_{s-1}^{n_{s-1}} e_s,\ e_{s-1} e_s^{n_s} = 1.$

Then we have

$$\pi_1(K)/ < e_0 > \cong \left\langle a_1, a_2 | a_1^2 = a_2^2 = (a_1 a_2)^{\alpha} = 1 \right\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_{\alpha} / (a_1 a_2 b = 1) \neq \{1\},$$

where $\alpha = [n_1, \cdots n_s]$ and $b = a_1 a_2$, and hence $\pi_1(K) \neq \{1\}$.

*The case of type* $(\delta)$.




Let $a_i$ $(1 \le i \le 3)$ be a loop in $\pi_(K)$ corresponding to the vertex $\epsilon_{i1}$ $(1 \le i \le 3)$ of $\Gamma(C)$ and $e_0$ a loop corresponding to the vertex $\epsilon_0$. Then one has

$$\pi_1(K)/ < e_0 > \cong \mathbb{Z}_{\alpha_1} * \mathbb{Z}_{\alpha_2} * \mathbb{Z}_{\alpha_3} / (a_1 a_2 a_3 = 1) \neq \{1\},$$

where $\alpha_i = [n_{i1}, \cdots n_{ir_i}]$ $(1 \le i \le 3)$, hence we have $\pi_1(K) \neq \{1\}$.

Consequently the graph $\Gamma(C)$ must belongs to the type $(\epsilon)$. This completes the proof.

# 3 Rational ruled surfaces.

## 3.1 A linear trees of rational curves on a rational surface.

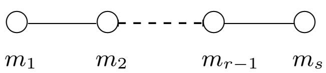
**Definition 3.1** *Let $S$ be a smooth rational surface and $A = \bigcup_{i=1}^{r} A_i$ a curve on $S$. $A$ is called a linear tree of (smooth) rational curve if*

(i) *each $A_i$ is a smooth rational curve,*

(ii) $(A_i \cdot A_{i+1}) = 1$ $(1 \le i \le r-1)$ *and*

(iii) $(A_i \cdot A_j) = 0$ if $|i - j| \ge 2$.

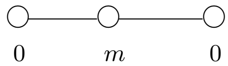
We set $m_i := (A_i^2)$ $(1 \le i \le r)$. Then the graph $\Gamma(A)$ of the curve $A$ is depicted as




Then we have

**Lemma 3.1 (**see Lemma 6 in Suzuki [8] **)** *Let $(S, A)$ be as above. Then*

(3-a) *If there exists a pair $i$, $j$ $(1 \leq \ i < j \leq r)$ such that $(A_i^2) \geq 0$, $(A_j^2) \geq 0$ and $A_i \cap A_j = \emptyset$, then $(A_i^2) = (A_j^2) = 0$ and there exists only one component $A_k$ $(k \neq i, j)$ which intersects $A_i \cup A_j$. The graph $\Gamma(A)$ is depicted as*

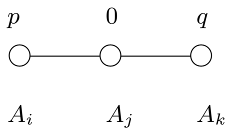



(3-b) *If there exists a pair $i$, $j$ $(1 \leq i < j \leq n)$ such that $(A_i^2) > 0$, $(A_j^2) > 0$, then $n = 2$ and $(A_i^2) = (A_j^2) = 1$. The graph $\Gamma(A)$ is depicted as*

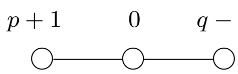

**Corollary 3.1** *Assume that $r \geq 3$. Then there exists only one component $A_{i_0}$ with $(A_{i_0}^2) > 0$. In particular, $(A_k^2) < 0$ for $A_k$ with $A_k \cap A_{i_0} = \emptyset$.*

## 3.2 An elementary transformation.

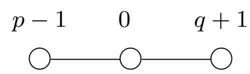
Assume that the graph $\Gamma(A)$ contains the following subgraph:




Blowing up at the intersection point of $A_j$ and $A_k$ (resp. $A_i$ and $A_j$) and contracting the proper transform of $A_j$, which is a $(-1)$-curve, to a smooth point, we obtain the following:

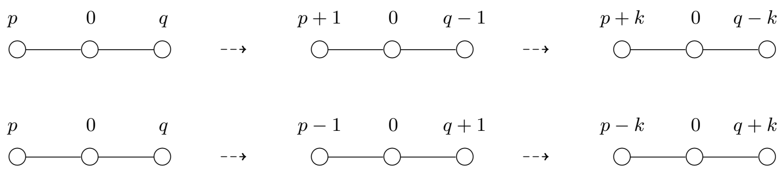



( resp.



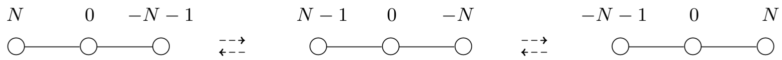

) ,

which we call the *elementary transformation.* For each $0 < k \in \mathbb{Z}$, one has the following:



In the next section **4**, we will treat the following special type of the elementary transformations:

$(\mathrm{ET})_N$ :




# 4 A classification of minimal normal compatifications of $\mathbb{C}^2$.

## 4.1 The type of the boundary divisor.

Let $(M, C)$ be a minimal normal compactification of $\mathbb{C}^2$ and set $C = \bigcup_{i=1}^{r} C_i$ , where $C_i$ is an irreducible component of $C$. According to (1.1) and (1.2) in Introduction, we know that

(4-a) $M$ is rational (see [3]).

(4-b) $\mathrm{H}^i(M, \mathbb{Z}) \cong \mathrm{H}^i(C, \mathbb{Z})$ for $i < 3$.

(4-c) $\mathrm{H}^1(M, \mathbb{Z}) = \mathrm{H}^1(C, \mathbb{Z}) = 0$ and $\mathrm{H}^2(M, \mathbb{Z}) \cong \mathrm{Pic}\, S = \bigoplus_{i=1}^{r} \mathbb{Z}\, \mathcal{O}_M(C_i)$.

Then we have

**Proposition 4.1** *For each $C_i$, there exists a holomorphic mapping $\varphi_i : \mathbb{C} \longrightarrow M \setminus C$ such that*

$$C_i \subset \varphi_i(\infty; M) \subset C,$$

*where* $\varphi_i(\infty; M) := \bigcap_{R>0} \overline{\varphi_i(\Delta_R)}$ *and* $\Delta_R = \{z \in \mathbb{C} : |z| > R\}$.

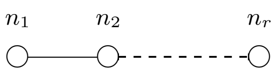
*Proof.* The proof is done by the idea similar to that of Lemma 1 in [1] (see also Lemma 2 in [8]).

By (4-c), the type of boundary $C$ is one of the types $(\alpha) - (e)$ in Table 2. Let $K$ be the boundary of a tubular neighborhood of the curve $C$ in $S$ as before. Since $\mathbb{C}^2$ is simply connected at infinity, $\pi_1(K) = \{1\}$ (trivial). By Proposition 2.1, we have

**Theorem 4.1** *The curve $C$ belongs to the type* $(\epsilon)$ *in* Table 2 (see also Figure 5).

## 4.2 The classification of $\Gamma(C)$.

First by Theorem 4.1, the graph $\Gamma(C)$ is depicted as




, $\max_{1 \le i \le r} \{n_i\} \ge 0, \; n_i \ne -1.$

By Proposition 2.1, one has $\pi_1(K) \cong \mathbb{Z}_\alpha = \{1\}$, where $\alpha = [n_1, n_2, \ldots, n_r] \in \mathbb{Z}$. Hence we have

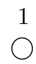
**Proposition 4.2** $[n_1, n_2, \ldots, n_r] = \pm 1$.

Applying Proposition 4.2, we shall determine the graph $\Gamma(C)$ below. Here we note that $r > 0$ is the number of vertices of the graph $\Gamma(C)$.

#### 4.2.1 The case for $r = 1$.

Since $\mathrm{b}_2(S) = \mathrm{b}_2(C) = 1$ and $S$ is rational, $S \cong \mathbb{P}^2$, and hence, $C \cong \mathbb{P}^1$ and $(C^2) \leq 2$. By Proposition 4.2, one can see that $(C^2) = 1$, and thus we obtain the graph:

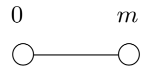
$(\Gamma_a)$




#### 4.2.2 The case for $r = 2$.

Since $\mathrm{b}_2(S) = 2$, one has $S \cong \mathbb{F}_n (n \geq 0)$ (Hirzebruch surface) with $\max\{n_1, n_2\} \geq 0$. We may assume that $n_1 \geq 0$. Since $\alpha = [n_1, n_2] = \pm 1$, one has $n_1 n_2 - 1 = \pm 1$, that is, $(n_1, n_2) = (2, 1)$ or $(0, n_2)$.By Lemma 3.2-(3-b), the case $(n_1, n_2) = (2, 1)$ cannot occur. Hence we obtain the graph:

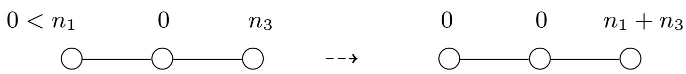
$(\Gamma_b)$




$(m \neq -1)$

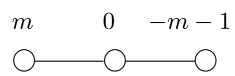
#### 4.2.3 The case for $r = 3$.

By Proposition 4.2, one has $\pm 1 = [n_1, n_2, n_3] = n_1 n_2 n_3 - n_1 - n_3$. First we claim that $n_2 = 0$. In fact, if $n_2 > 0$, then $n_1 \leq 0$ and $n_3 \leq 0$ by Lemma 3.1-(3-b). Hence $n_1 n_2 n_3 - n_1 - n_3 > 1$, which is absurd. If $n_2 < 0$, then by Lemma 3.1-(3-b), $n_1 \geq 0$ or $n_3 \geq 0$. We may assume that $n_1 \geq 0$ and $n_3 < 0$ by Lemma 3.1-(3-b). Hence $n_1(n_2 n_3 - 1) - n_3 > 1$, which is absurd. Consequently we get $n_2 = 0$. Then we have $-n_1 - n_3 = \pm 1$. Since $n_i \neq -1$ $(i = 1, 3)$, one sees easily that $n_1 n_3 < 0$. We may assume that $n_1 > 0$. Applying the elementary transformation of type $(ET)_N$, we have




By Lemma 3.1-(3-b), one has $n_1 + n_3 = -1$. Putting $n_1 = m > 0$, we have the graph:

$(\Gamma_c)$

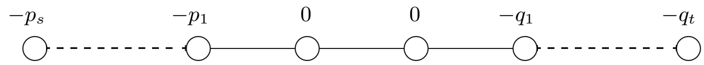



$(m > 0)$

#### 4.2.4 The case for $r \geq 4$.

**Lemma 4.1** $\max\limits_{1 \leq i \leq r} \{n_i\} > 0$.

*Proof.* Assume that $\max\limits_{1 \leq i \leq r} \{n_i\} = 0$. Let $\nu > 0$ be the number of irreducible components $C_i$ of $C$ with $n_i = (C_i^2) = 0$. Then $\nu \leq 2$ by Lemma 3.1-(3-a) since $r \geq 4$. If $\nu = 2$, then $\Gamma(C)$ can be depicted as:

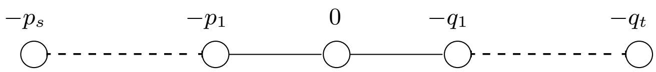



$$(s,t \geq 0,\ s+t \geq 2,\ p_i \geq 2,\ r = s+t+2,\ q_j \geq 2)$$

By Proposition 4.3, we have $-[p_s,\ldots,p_1,0,0,q_1,\ldots,q_t] = 1$. On the other hand, an easy computation yields the following:

$$\begin{aligned}
-[p_s,\ldots,p_1,0,0,q_1,\ldots,q_t] &= -[0,p_1,\ldots,p_s][0,q_1,\ldots,q_t] + [p_1,\ldots,p_s][q_1,\ldots,q_t] \\
&= [p_1,\ldots,p_s][q_1,\ldots,q_t] - [p_2,\ldots,p_s][q_2,\ldots,q_t] \\
&> [p_2,\ldots,p_s][q_1,\ldots,q_t] - [p_2,\ldots,p_s][q_2,\ldots,q_t] \\
&= [p_2,\ldots,p_s]\left([q_1,\ldots,q_t] - [q_2,\ldots,q_t]\right) \\
&> \cdots\cdots \\
&> [p_2,\ldots,p_s]\left([q_t] - 1\right) \\
&> [p_2,\ldots,p_s] > 1.
\end{aligned}$$

This contradicts the fact that $-[p_s,\ldots,p_1,0,0,q_1,\ldots,q_t] = 1$. If $\nu = 1$, then we have the graph:




$$(s,t \geq 0,\ s+t \geq 3,\ p_i \geq 2,\ r = s+t+1,\ q_j \geq 2)$$

Then we have

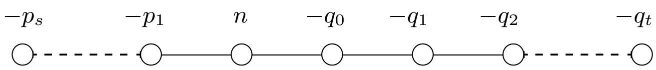
$$\begin{aligned}
\pm 1 &= -[p_s,\ldots,p_1,0,q_1,\ldots,q_t] \\
&= [p_1,\ldots,p_s][0,q_1,\ldots,q_t] - [p_2,\ldots,p_s][q_1,\ldots,q_t] \\
&= -[p_1,\ldots,p_s][q_2,\ldots,q_t] - [p_2,\ldots,p_s][q_1,\ldots,q_t].
\end{aligned}$$

This cannot occur since $[p_1,\ldots,p_s][q_2,\ldots,q_t]+[p_2,\ldots,p_s][q_1,\ldots,q_t] > 1$. Hence $\max\limits_{1\leq i\leq r}\{n_i\} > 0$. This completes the proof.

**Lemma 4.2** *There exist components $C_{i_0}$ and $C_{i_1}$ of $C$ such that*

$$n := (C_{i_0}^2) > 0,\ (C_{i_1}^2) = 0,\ (C_{i_0}\cdot C_{i_1}) = 1 \text{ and } \max_{i\neq i_1,i_2}\{n_i\} \leq -2.$$

*Proof.* Since $r \geq 4$, by Corollary 3.1 , there exists only one component $C_{i_0}$ with $n := (C_{i_0}^2) > 0$ and then the graph $\Gamma(C)$ can be depicted as:




$$(s,t \geq 0,\ s+t \geq 2,\ q_0 \geq 0,\ p_i,\ q_j \geq 2,\ r = s+t+2)$$

Let $C_{i_2}$ be the component of $C$ corresponding to the vertex $\overset{-q_0}{\bigcirc}$ . Then we only need to show that $q_0 = 0$. In fact, if we assume that $q_0 > 1$, then we have

$$\begin{aligned}
\pm 1 &= [p_s, \ldots, p_2, p_1, -n, q_1, q_2, \ldots, q_t] \\
&= [p_1, p_2, \ldots, p_s][-n, q_1, q_2, \ldots, q_t] - [p_2, \ldots, p_s][q_0, q_1, \ldots, q_t] \\
&= -[q_1, q_2, \ldots, q_t]\{n[p_1, \ldots, p_s] + q_0[p_2, \ldots, p_s]\} - [q_2, \ldots, q_t]\{[p_1, \ldots, p_s] - [p_2, \ldots, p_s]\},
\end{aligned}$$

which is absurd since

$$[q_1, q_2, \ldots, q_t]\{n[p_1, \ldots, p_s] + q_0[p_2, \ldots, p_s]\} + [q_2, \ldots, q_t]\{[p_1, \ldots, p_s] - [p_2, \ldots, p_s]\} > 2,$$

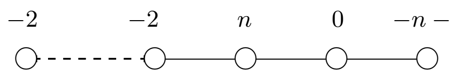
Hence $q_0 = 0$.

**Remark 4.1** *In case of $r \geq 4$, one has $t \geq 1$. In fact, if $t = 0$, then $s \geq 2$. By Proposition 4.2, we have*

$$\pm 1 = [p_s, \ldots, p_1, -n, 0] = [0, -n, p_1, \ldots, p_s] = -[p_1, \ldots, p_s]\ ,$$

*which cannot occur since $[p_1, \ldots, p_s] \geq 2$.*

**Proposition 4.3** *Assume that $t = 1$. Then $q_1 = n + 1$ and the graph $\Gamma(C)$ can be depicted as:*

$(\Gamma_d)$ 


*Proof.* We have

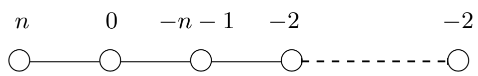
$$\begin{aligned}
\pm 1 &= [p_s, \ldots, p_1, -n, 0, q_1] \\
&= [-n, p_1, p_2, \ldots, p_s] \cdot [0, q_1] - [p_1, \ldots, p_s] \cdot q_1 \\
&= (n - q_1)[p_1, \ldots, p_s] + [p_2, \ldots, p_s] \\
&< (n + 1 - q_1)[p_1, p_2, \ldots, p_s].
\end{aligned}$$

This shows that $n - q_1 < 0$ and $n + 1 - q_1 \geq 0$, hence $q_1 = n + 1$. In particular, we have

$$[p_1, p_2, \ldots p_s] - [p_2, \ldots, p_s] = 1.$$

This implies that $p_1 = p_2 = \cdots = p_s = 2$ by Lemma 2.2-(6).

**Proposition 4.4** *Assume that $s = 0$. Then $q_1 = n + 1$ and $\Gamma(C)$ is depicted as:*

$(\Gamma_e)$ 


*Proof.* From

$$\begin{aligned}
\pm 1 &= [-n, 0, q_1, \ldots, q_t] \\
&= (n - q_1) \cdot [q_2, \ldots, q_s] + [q_3, \ldots, q_t],
\end{aligned}$$

it follows that $n - q_1 < 0$. Hence we have $(q_1 - n) \cdot [q_2, \dots, q_s] \pm 1 = [q_3, \dots, q_t] < [q_2, \dots, q_s]$, that is,

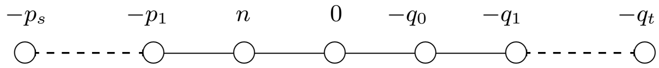
$$(q_1 - n - 1) \cdot [q_2, \dots, q_s] < \mp 1.$$

This shows that $q_1 = n + 1$ and then we have

$$[q_2, \dots, q_s] - [q_3, \dots, q_t] = 1.$$

This implies $q_2 = \cdots = q_t = 2$ by Lemma 2.2-(6).

**Proposition 4.5** *Assume that the graph $\Gamma(C)$ is depicted as:*




$$(s, t \geq 1\ ,\ p_i,\ q_j \geq 2,\ r = s + t + 3)$$

*Then we have $q_0 = n + 1$ and $p_1 = 2$ or $q_1 = 2$.*

*Proof.* From

$$\begin{aligned}
\pm 1 &= [p_s, \dots, p_1, -n, 0, q_0, q_1, \dots q_t] \\
&= [-n, p_1, \dots, p_s] \cdot [0, q_0, q_1, \dots, q_t] - [p_1, \dots, p_s] \cdot [q_0, q_1, \dots, q_t] \\
&= (n - q_0)[p_1, \dots, p_s] \cdot [q_1, \dots, q_t] + [p_2, \dots, p_s] \cdot [q_1, \dots, q_t] + [p_1, \dots, p_s] \cdot [q_2, \dots, q_t] \\
&< (n - q_0 + 2)[p_1, \dots, p_s] \cdot [q_1, \dots, q_t],
\end{aligned}$$

it follows that $n - q_0 < 0$ and $n - q_0 + 2 > 0$, hence $q_0 = n + 1$. Thus we have

$$[p_1, \dots, p_s] \cdot [q_1, \dots, q_t] \pm 1 = [p_2, \dots, p_s] \cdot [q_1, \dots, q_t] + [p_1, \dots, p_s] \cdot [q_2, \dots, q_t].$$

This implies that

$$\begin{aligned}
[p_2, \dots, p_s] \cdot [q_2, \dots, q_t] \mp 1 &= \Big([p_1, \dots, p_s] - [p_2, \dots, p_s]\Big)\Big([q_1, \dots, q_t] - [q_2, \dots, q_t]\Big) \\
&= \Big((p_1 - 1)[p_2, \dots, p_s] - [p_3, \dots, p_s]\Big)\Big((q_1 - 1)[q_2, \dots, q_t] - [q_3, \dots, q_t]\Big).
\end{aligned}$$

Since $[p_2, \dots, p_s] - [p_3, \dots, p_s] \geq 1$ and $[q_2, \dots, q_t] - [q_3, \dots, q_t] \geq 1$, one has

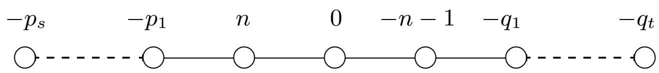
$$\Big((p_1 - 2)[p_2, \dots, p_s] + 1\Big)\Big((q_1 - 2)[q_2, \dots, q_t] + 1\Big) \leq [p_2, \dots, p_s] \cdot [q_2, \dots, q_t] \pm 1.$$

This yields

$$\{(p_1 - 2)(q_1 - 2) - 1\}[p_2, \dots, p_s][q_2, \dots, q_t] + (p_1 - 2)[p_2, \dots, p_s] + (q_1 - 2)[q_2, \dots, q_t] + 1 \leq \pm 1.$$

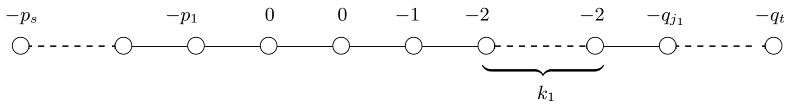
This shows that $p_1 = 2$ or $q_1 = 2$.

Finally we shall determine the graph $\Gamma(C)$ for the remaining cases where $r \geq 4$. Let us start with the following:



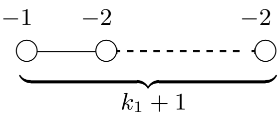

$$(s, t \geq 1\ ,\ p_i,\ q_j \geq 2,\ r = s + t + 3)$$

We have shown that $p_1 = 2$ or $q_1 = 2$. Applying the elementary transformation of type $(\mathrm{ET})_N$ in Section **3**, we may assume that $q_1 = 2$ and that $\Gamma(C)$ is represented as;




$(q_{j_1} \geq 3)$

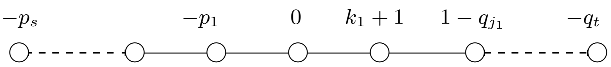
The curves corresponding to the subgraph




, called a $(-1)$-graph,

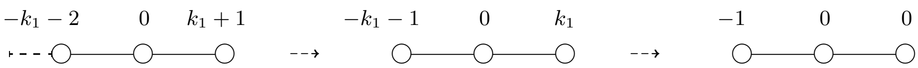
which can be contracted to a smooth point. Then one has the following:

$\Gamma^{(1)}$ :



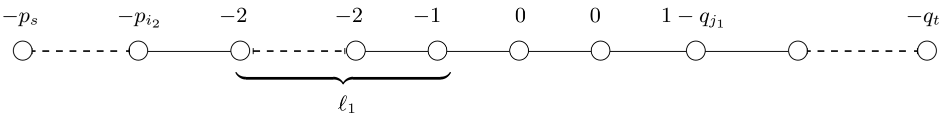

( $p_1 = k_1 + 2$ by Proposition 4.2)

Next, let us consider the elementary transformation




,

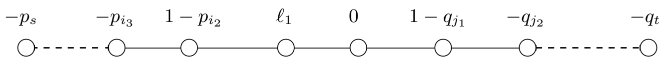
which yields




.

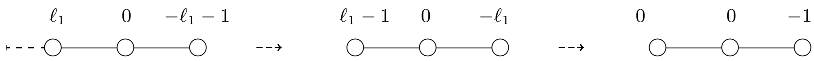
Contracting the above $(-1)$-graph to a smooth point, we obtain:

$\Gamma^{(2)}$ :

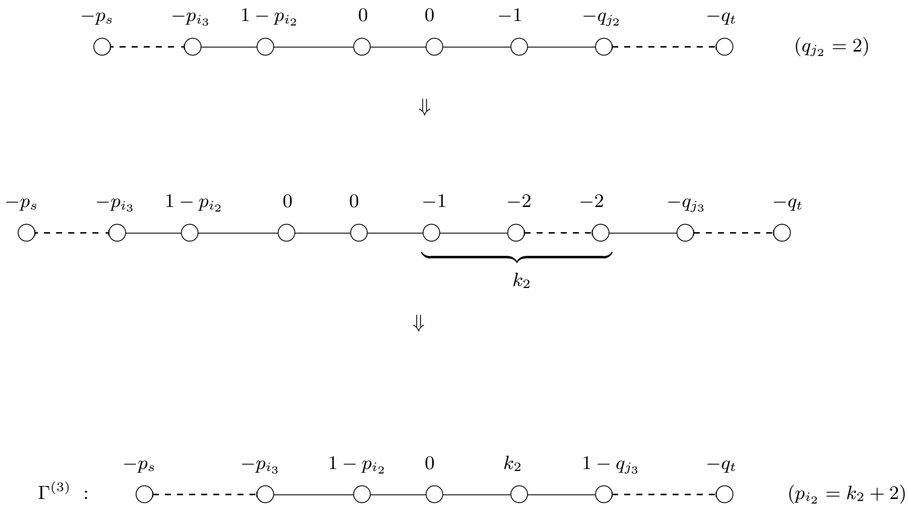



$(q_{j_1} = \ell_1 + 2)$

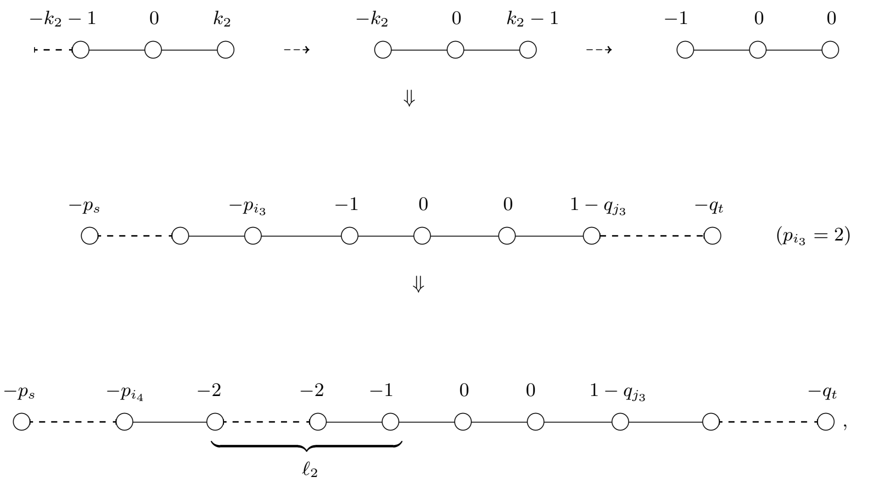
The following elementary transformation




changes $\Gamma^{(2)}$ into

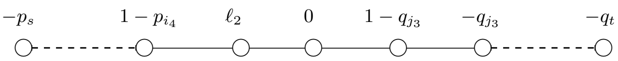



Continuing the operation as below:



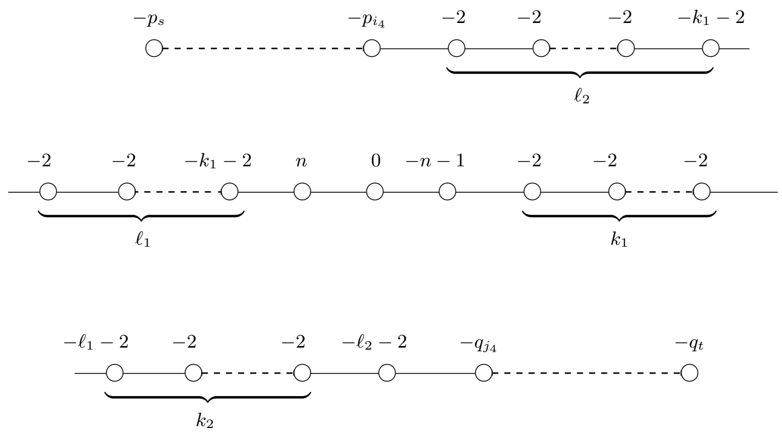

we have the graph:

$\Gamma^{(4)}$ :




$(q_{i_3} = \ell_2 + 2)$.

We note that at this stage the graph $\Gamma(C)$ is represented as

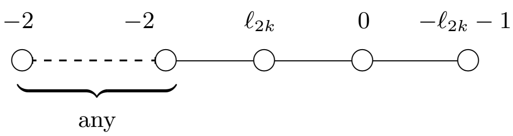



Repeating these operations, we have a sequence of birational transformations:

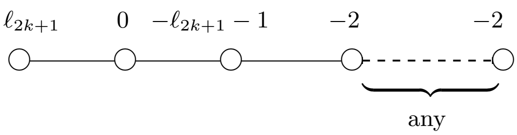
$$\Gamma^{(1)} \to \Gamma^{(2)} \to \cdots \to \Gamma^{(2k)} \quad \left(\text{or} \quad \Gamma^{(2k+1)}\right),$$

where $k > 0$ is an integer and

$\Gamma^{(2k)}$ :




$\Gamma^{(2k+1)}$ :




The inverse operation $\Gamma^{(2k)}$ ( $\Gamma^{(2k+1)}$ ) $\to \Gamma^{(1)} \to \Gamma(C)$ yields the graphs $(\Gamma_g)$ and $(\Gamma_h)$ as desired. The proof is completed. □

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