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On the proportion of quadratic twists for non-vanishing and vanishing central values of L-functions attached to newforms

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Abstract. In this article, we prove several results on non-vanishing or vanishing of central values of quadratic twist L-functions attached to newforms, which were originally conjectured in [Gol79]. Let p be a prime congruent to 1 modulo 4. For such p, we show that there exists a newform f of level p such that a positive proportion of quadratic twists of its central L-value are non-zero. This result is a generalization of [Koh99] to the case of prime levels under the condition that the weight of f is 12, 16 or 20. Let N be a positive odd integer such that the exponent of each prime divisor of N is odd. We prove that any newform of level N of weight $k \geq 2$ has the property that a positive proportion of quadratic twists of its central L-value are zero.

Contents

1 Introduction

For $X > 0$, let $F(X)$ denote the set of fundamental discriminants D satisfying $|D| < X$. We denote by $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ the space spanned by newforms of weight $2k$ on $\Gamma_0(N)$. Let k and N be positive integers and $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ a newform. Let r be either 0 or 1. Then a conjecture of Goldfeld [Gol79] states that

$$\sharp\{D \in F(X) \mid \operatorname{ord}_{s=k} L(s, f \otimes \chi_D) = r\} \gg X \tag{1.1}$$

(see also Ono [Ono04, Conjecture 9.10]), that is, there exists a positive constant c such that, for sufficiently large $X > 0$, we have

$$\sharp\{D \in F(X) \mid \operatorname{ord}_{s=k} L(s, f \otimes \chi_D) = r\} \geq cX, \tag{1.2}$$

where $\chi_D := \left(\frac{D}{\cdot}\right)$ is the Kronecker character attached to the quadratic field with discriminant D and $\operatorname{ord}_{s=k} L(s, f \otimes \chi_D)$ is the order of the L-function $L(s, f \otimes \chi_D)$ attached to the χ_D-twist $f \otimes \chi_D$ of f at $s = k$. The automorphic L-function $L(s, f \otimes \chi_D)$ has a functional equation relating $L(s, f \otimes \chi_D)$ with $L(2k - s, f \otimes \chi_D)$ and $L(k, f \otimes \chi_D)$ is called the *central L-values.*

Remark 1.1 *(see [Vat98, Theorem* II*]) The conjecture is true for any newform $f \in S_2^{\mathrm{new}}(\Gamma_0(19))$ corresponding to an elliptic curve over the rational number field $\mathbb{Q}$ with conductor* 19*.*

1.1 Results on non-vanishing of the central L-values

We recall some known results on the non-vanishing of the central L-values $L(k, f \otimes \chi_D)$. We put

$$\mathcal{N}_{k,f}(X) := \sharp\{D \in F(X) \mid L(k, f \otimes \chi_D) \neq 0\}. \tag{1.3}$$

Currently, it seems that the best estimate is due to Ono and Skinner [OS98], who showed that

$$\mathcal{N}_{k,f}(X) \gg \frac{X}{\log X} \tag{1.4}$$

(see [OS98, Corollary 3]). Galois representations attached to modular forms and a theorem of Waldspurger play important roles in the proof of this result. The estimate above arises from the Chebotarev Density Theorem. By Waldspurger [Wal81], under certain conditions, the central values $L(k, f \otimes \chi_D)$ are proportional to the squares of Fourier coefficients for a modular form of weight $k + 1/2$ corresponding to f under the Shimura correspondence.

James [Jam98] gave the first example of (k, N, f) satisfying

$$\mathcal{N}_{k,f}(X) \gg X. \tag{1.5}$$

In his paper, the estimate (1.5) is reduced to that of the proportion of imaginary quadratic fields whose class numbers are not divisible by 3. This proportion can be estimated by the work of Davenport and Heilbronn [DH71], and was refined by Nakagawa and Horie [NH88]. James's method was employed by Vatsal [Vat99] and Kohnen [Koh99] in the proofs of their results on the non-vanishing.

Vatsal proved that $\mathcal{N}_{1,f}(X) \gg X$ for each newform f corresponding to an elliptic curve over $\mathbb{Q}$ with a rational point of order 3 and good ordinary reduction at 3 (see [Vat99, Theorem 0.3]). Roughly speaking, he showed a congruence between the algebraic part of the central L-value $L(k, f \otimes \chi_D)$ and the class number of the imaginary quadratic field with discriminant D modulo 3, up to 3-adic units (see [Vat99, Theorem 3.3]). Therefore, he obtained the estimate in a fashion similar to [Jam98].

Suppose that k is even. Let $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(1)) = S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Since the sign of the functional equation of $L(k, f \otimes \chi_D)$ is $\chi_D(-1)$ (see [Ono04, Lemma 9.2 and Remark 9.3]), it is natural to suppose that $\chi_D(-1) = +1$, that is, $D > 0$. For $X > 0$, let $\mathcal{N}_{k,1}^{+}(X)$ be the number of fundamental discriminants D with $0 < D < X$ such that there exists a Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ satisfying $L(k, f \otimes \chi_D) \neq 0$. For $\varepsilon > 0$, let $X \gg_\varepsilon 0$ mean that there exists a constant $c > 0$ depending on ε and $X > c$. Kohnen [Koh99] proved that for any even integer $k \geq 6$, if $\varepsilon > 0$ and $X \gg_\varepsilon 0$, then

$$\mathcal{N}_{k,1}^{+}(X) \geq \left(\frac{9}{16\pi^2} - \varepsilon\right) X \tag{1.6}$$

(see [Koh99, Theorem]). Kohnen's result above immediately implies that there exists a Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ such that

$$\mathcal{N}_{k,f}^{+}(X) \geq \left(\frac{1}{d_{k,1}} \cdot \frac{9}{16\pi^2} - \varepsilon\right) X \tag{1.7}$$

(see [Koh99, Corollary 1]), where $d_{k,1} := \dim_{\mathbb{C}}(S_{2k}(\mathrm{SL}_2(\mathbb{Z})))$. Moreover, he pointed out that (1.7) holds for any Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (see [Koh99, Corollary 2]) assuming a conjecture of Maeda (see [HM97, Conjecture 1.2]) with respect to each even integer $k \geq 6$. Currently, this conjecture has been verified for all weights $k \leq 14000$ (see [GM12]).

We will improve Kohnen's result in that the assertion similar to (1.7) holds for a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(p))$ with a prime p congruent to 1 modulo 4 (see Theorem 2.1, Corollary 2.2 and Corollary 2.3), while Kohnen's result asserts the non-vanishing only for the level 1. However, we need to assume that the weights $2k$ must be either 12, 16 or 20 in our statements.

1.2 Results on vanishing of the central L-values

We recall some known results on the vanishing of the central L-values. Currently, it seems that the best estimate is due to Perelli and Pomykala [PP97], who showed that for any N and any f corresponding to an elliptic curve over $\mathbb{Q}$, if $\varepsilon > 0$, then

$$\sharp\{D \in F(X) \mid \operatorname{ord}_{s=1} L(s, f \otimes \chi_D) = 1\} \gg_\varepsilon X^{1-\varepsilon}. \tag{1.8}$$

We will prove

$$\sharp\{D \in F(X) \mid \operatorname{ord}_{s=1} L(s, f \otimes \chi_D) \geq 1\} \gg X \tag{1.9}$$

for a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ with $k \geq 1$ under the assumption that the exponent of each prime divisor of N is odd (see Theorem 2.4).

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2 Statements of results

For a positive integer N, we put

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}. \tag{2.1}$$

We denote by $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ the space spanned by newforms of weight $2k$ on $\Gamma_0(N)$. For $X > 0$, let $F^+(X)$ be the set of fundamental discriminants D satisfying $0 < D < X$ and $\mathcal{N}_{k,N}^+(X)$ the number of elements $D \in F^+(X)$ such that there exists a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ satisfying $L(k, f \otimes \chi_D) \neq 0$. For a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$, we put

$$\mathcal{N}_{k,f}^+(X) := \sharp\{D \in F^+(X) \mid L(k, f \otimes \chi_D) \neq 0\}. \tag{2.2}$$

One of the main results on the non-vanishing is stated as follows:

Theorem 2.1 *Let p be a prime with $p \equiv 1 \pmod 4$. Assume that $k = 6, 8$ or 10. Then we have $\mathcal{N}_{k,p}^+(X) \gg X$. More precisely, if $\varepsilon > 0$ and $X \gg_\varepsilon 0$, then*

$$\mathcal{N}_{k,p}^+(X) \geq \left(\frac{9p}{32\pi^2(p+1)} - \varepsilon \right) X. \tag{2.3}$$

We put

$$d_{k,p} := \dim_{\mathbb{C}}(S_{2k}^{\mathrm{new}}(\Gamma_0(p))). \tag{2.4}$$

By an argument similar to [Koh99], we obtain the following:

Corollary 2.2 *Let the assumptions be the same as in Theorem 2.1. Then there exists a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(p))$ such that $\mathcal{N}_{k,f}^+(X) \gg X$ holds. More precisely, if $\varepsilon > 0$ and $X \gg_\varepsilon 0$, then*

$$\mathcal{N}_{k,f}^+(X) \geq \left(\frac{1}{d_{k,p}} \cdot \frac{9p}{32\pi^2(p+1)} - \varepsilon \right) X. \tag{2.5}$$

Let $f = \sum_{n\geq 1} a(n)q^n \in S_{2k}^{\mathrm{new}}(\Gamma_0(p))$ be a newform. We denote by $\bar{\mathbb{Q}}$ an algebraic closure of the rational number field $\mathbb{Q}$. It is well-known that $a(n) \in \bar{\mathbb{Q}}$ for all $n \geq 1$ (see [Shi72, Proposition 1.3]). For $\sigma \in G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we define

$$f^\sigma := \sum_{n\geq 1} a(n)^\sigma q^n. \tag{2.6}$$

Then $f^\sigma \in S_{2k}^{\mathrm{new}}(\Gamma_0(p))$ is a newform (see [Shi72, Proposition 1.2]). Since it is known that the central L-value of f is not zero if and only if the central L-value of f^σ is not zero by [Shi77, Theorem 1], Corollary 2.2 can be extended to a result for each newform f^σ as follows:

Corollary 2.3 *Let the assumptions be the same as in Theorem 2.1 and $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(p))$ a newform satisfying $\mathcal{N}_{k,f}^+(X) \gg X$. Then, the estimate $\mathcal{N}_{k,f^\sigma}^+(X) \gg X$ holds for all $\sigma \in G_{\mathbb{Q}}$.*

Remark 2.1 *In general, for positive integers N and k, the space $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is not necessarily spanned by a single Galois orbit. A conjecture about the cardinality of Galois orbits in $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is formulated in [Tsa14]. As another related topic, we refer to [KSW08] on the density of the set of primes ℓ such that the ℓ-th Fourier coefficient $a(\ell)$ of f generates the Hecke field $\mathbb{Q}(\{a(n)\}_{n\geq 1})$.*

For $X > 0$, let $F^-(X)$ be the set of fundamental discriminants D satisfying $-X < D < 0$ and $F_N(X)$ the set of fundamental discriminants D satisfying $|D| < X$ and $(D, N) = 1$ for a positive integer N. For $\epsilon \in \{\pm\}$, we put

$$F_N^\epsilon(X) := F^\epsilon(X) \cap F_N(X). \tag{2.7}$$

We denote by $\mathbb{R}$ the real number field. We define the function $\mathrm{sgn} : \mathbb{R}^\times \to \{\pm\}$ by

$$\mathrm{sgn}(x) := \begin{cases} + & \text{if } x > 0, \\ - & \text{if } x < 0. \end{cases} \tag{2.8}$$

For a newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$, we put

$$\mathcal{V}_{k,f}^\epsilon(X) := \sharp\{D \in F_N^\epsilon(X) \mid L(k, f \otimes \chi_D) = 0\}, \tag{2.9}$$

where $\epsilon := \mathrm{sgn}((-1)^k)$. For a prime p, we denote by ord_p the p-adic additive valuation. Let $\nu(N)$ denote the number of distinct prime divisors of N.

Theorem 2.4 *Let $k \geq 2$ be an integer and N a positive odd integer such that $\mathrm{ord}_p(N)$ is odd for any prime divisor p of N . For any newform $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$, we have $\mathcal{V}_{k,f}^\pm(X) \gg X$. More precisely, if $\varepsilon > 0$ and $X \gg_\varepsilon 0$, then*

$$\mathcal{V}_{k,f}^\pm(X) \geq \left(\frac{3\nu(N)}{2^{\nu(N)}\pi^2} \prod_{p|N:\mathrm{prime}} \frac{p}{p+1} - \varepsilon \right) X. \tag{2.10}$$

3 The proportion of indivisibility for class numbers of quadratic fields

For $\mathbb{R}$-valued functions $F(X)$ and $G(X)$ on $\mathbb{R}$, we use the notation

$$F(X) \sim G(X) \ (X \to \infty), \tag{3.1}$$

which means

$$\lim_{X\to\infty} \frac{F(X)}{G(X)} = 1. \tag{3.2}$$

We use the following lemma to prove Proposition 3.3 and Theorem 2.4.

Lemma 3.1 *Let $F(X)$ be an $\mathbb{R}$-valued function on $\mathbb{R}$.*

(1) Assume that $F(X) \sim cX$ $(X \to \infty)$ for a positive number c. Then for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$,

$$F(X) > (c - \varepsilon)X. \tag{3.3}$$

(2) Let $G_1(X)$ and $G_2(X)$ be $\mathbb{R}$-valued functions on $\mathbb{R}$. We put $G(X) := G_1(X)+G_2(X)$. Assume that $G(X) > 0$, $F(X) \sim 2G(X)$ $(X \to \infty)$ and $F(X) \geq 3G_1(X) + G_2(X)$. Then for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$,

$$G_2(X) > \left(\frac{1}{2} - \varepsilon\right) G(X). \tag{3.4}$$

Proof.

(1) By the assumption, for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$, we have $-\varepsilon < \dfrac{F(X)}{cX} - 1$, and hence have $(c - \varepsilon)\, X < F(X)$.

(2) By the assumption, for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$, we have $F(X) < 2(1 + \varepsilon)G(X)$. By combining the assumption with this, we have

$$G(X) \leq F(X) - 2G_1(X) < 2(1+\varepsilon)G(X) - 2G_1(X) = 2\varepsilon G(X) + 2G_2(X).$$

Therefore, we have $2G_2(X) > (1 - 2\varepsilon)G(X)$. This implies the assertion.

For two positive integers m and N, a positive number X and a signature $\epsilon \in \{\pm\}$, we put

$$\begin{aligned} F^\epsilon(X, m, N) :=& \{D \in F^\epsilon(X) \mid D \equiv m \pmod N\}, & (3.5)\\ N_2^\epsilon(X, m, N) :=& \sharp F^\epsilon(X, m, N), & (3.6)\\ S_2^-(X, m, N) :=& \sum_{D\in F^-(X,m,N)} h_3^*(D), & (3.7) \end{aligned}$$

where $h_3^*(D)$ is the order of the 3-torsion subgroup of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ for the quadratic field $\mathbb{Q}(\sqrt{D})$ with discriminant D.

Lemma 3.2 *Let m and N be positive integers and X a positive number. Assume that N is odd and that $p^2 \mid N$ and $p^2 \nmid m$ for any odd prime divisor p of (m, N). Then we have the following:*

$$S_2^-(X, m, N) \sim 2N_2^-(X, m, N) \ (X \to \infty), \tag{3.8}$$

$$N_2^+(X, m, N) \sim N_2^-(X, m, N) \sim \frac{3}{\varphi(N)\pi^2} \prod_{p \mid N:\text{prime}} \frac{p}{p+1} X \ (X \to \infty), \tag{3.9}$$

where φ is Euler's totient function.

For this see [NH88, Theorem 1 and Proposition 2]. We denote by $h(D) := |\operatorname{Cl}(\mathbb{Q}(\sqrt{D}))|$ the class number of $\mathbb{Q}(\sqrt{D})$. In later discussion, we need the following proposition, whose proof is similar to [Koh99].

Proposition 3.3 *Let N be a positive odd square-free integer not divisible by 3 and m an integer with $(m, N) = 1$. For $\varepsilon > 0$ and $X \gg_\varepsilon 0$, we have*

$$\begin{aligned} &\sharp \left\{ D \in F^+(X, m, N) \cap F^+(X, -1, 3) \mid 3 \nmid h(-3D) \right\} \\ &\qquad \geq \left(\frac{9}{16\pi^2\varphi(N)} \prod_{p \mid N:\text{prime}} \frac{p}{p+1} - \varepsilon \right) X. \end{aligned} \tag{3.10}$$

In particular, if N is a prime $p \neq 3$, then

$$\sharp \left\{ D \in F^+(X, m, p) \cap F^+(X, -1, 3) \mid 3 \nmid h(-3D) \right\} \geq \left(\frac{9p}{16(p^2-1)} - \varepsilon \right) X. \tag{3.11}$$

Proof. Fix $D_0 \in \mathbb{Z}$ satisfying $D_0 \equiv m \pmod{N}$ and $D_0 \equiv -1 \pmod{3}$, and put $\Delta_0 := -3D_0$. Set

$$\begin{aligned} F(X) :=& S_2^-(3X, \Delta_0, 9N), \\ G_1(X) :=& \sharp \left\{ \Delta \in F^-(3X, \Delta_0, 9N) \mid 3 \mid h(\Delta) \right\}, \\ G_2(X) :=& \sharp \left\{ \Delta \in F^-(3X, \Delta_0, 9N) \mid 3 \nmid h(\Delta) \right\}, \\ G(X) :=& G_1(X) + G_2(X) = N_2^-(3X, \Delta_0, 9N). \end{aligned}$$

Since $h_3^*(\Delta) \geq 3$ if $3 \mid h(\Delta)$, we have $F(X) \geq 3G_1(X) + G_2(X)$. Since we have $F(X) \sim 2G(X)$ $(X \to \infty)$ by Lemma 3.2 (3.8), we can apply Lemma 3.1 (2) to these functions. Consequently, we have

$$G_2(X) \geq \left(\frac{1}{2} - \varepsilon \right) G(X)$$

for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$. Since the mapping $D \mapsto -3D$ from $F^+(X, D_0, 3N)$ into $F^-(3X, \Delta_0, 9N)$ is surjective, we have

$$\sharp \left\{ D \in F^+(X, D_0, 3N) \mid 3 \nmid h(-3D) \right\} \geq G_2(X).$$

Thus,

$$\sharp\left\{D \in F^+(X, D_0, 3N) \mid 3 \nmid h(-3D)\right\} \geq \left(\frac{1}{2} - \varepsilon\right) G(X)$$

for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$. By Lemma 3.2 (3.9), we have

$$\begin{aligned} G(X) &= N_2^-(3X, \Delta_0, 9N) \\ &\sim \frac{3}{\varphi(9N)\pi^2} \prod_{p|9N:\text{prime}} \frac{p}{p+1}(3X) = \frac{9}{8\pi^2\varphi(N)} \prod_{p|N:\text{prime}} \frac{p}{p+1} X \ (X \to \infty). \end{aligned}$$

Therefore, we have

$$\sharp\left\{D \in F^+(X, D_0, 3N) \mid 3 \nmid h(-3D)\right\} \geq \left(\frac{9}{16\pi^2\varphi(N)} \prod_{p|N:\text{prime}} \frac{p}{p+1} - \varepsilon\right) X$$

for any $\varepsilon > 0$ and $X \gg_\varepsilon 0$. This inequality is nothing but the formula in the assertion.

4 The central L-values and modular forms of half-integral weight

In this section, we review the formula by Kohnen [Koh85] which generalizes the well-known Kohnen-Zagier formula [KZ81] to the case of modular forms with levels. Throughout the paper, we let $\sqrt{z}$ be the branch of the square root having argument in $(-\pi/2, \pi/2]$ and put $q := e^{2\pi\sqrt{-1}z}$. We denote by $S_{k+1/2}(\Gamma_0(4N), \chi)$ the space of cusp forms of half-integral weight $k + 1/2$ on $\Gamma_0(4N)$ with Nebentypus χ (see [Shi73]). Let $g(z) = \sum_{n\geq 1} b(n)q^n \in S_{k+1/2}(\Gamma_0(4N), \chi)$. For a prime p, the action of the Hecke operator $T(p^2)$ on $g(z)$ in the sense of Shimura [Shi73] is defined by

$$g(z) \mid T(p^2) := \sum_{n\geq 1} \left(b(p^2 n) + \chi(p)\chi_{(-1)^k n}(p)p^{k-1}b(n) + \chi(p^2)p^{2k-1}b(n/p^2)\right) q^n, \tag{4.1}$$

where $b(n/p^2) := 0$ for $n/p^2 \notin \mathbb{Z}$.

We now recall the Kohnen plus space defined in [Koh82]. Let d be a positive integer. For a formal q-series $\sum_{n\geq 0} a(n)q^n \in \mathbb{C}[[q]]$, the V-operator $V(d)$ and U-operator $U(d)$ is defined by

$$(\sum_{n\geq 0} a(n)q^n) \mid V(d) := \sum_{n\geq 0} a(n)q^{dn} \tag{4.2}$$

$$(\sum_{n\geq 0} a(n)q^n) \mid U(d) := \sum_{n\geq 0} a(dn)q^n. \tag{4.3}$$

Suppose that N is a positive odd square-free integer and that χ is a Dirichlet character modulo N satisfying $\chi(-1) = 1$ and $\chi^2 = \mathbf{1}_N$, where $\mathbf{1}_N$ is the trivial character modulo N. Then we have $\chi = \chi_{N_0} (= \left(\frac{N_0}{\cdot}\right))$ for some positive divisor N_0 of N. The *Kohnen plus space* $S^+_{k+1/2}(\Gamma_0(4N), \chi)$ is the subspace of cusp forms $g(z) \in S_{k+1/2}(\Gamma_0(4N), \chi)$ with the Fourier expansion

$$g(z) = \sum_{\substack{n \geq 1 \\ \chi_{(2)}(-1)(-1)^k n \equiv 0,1 \pmod 4}} b(n) q^n, \tag{4.4}$$

where $\chi_{(2)}$ is the 2-primary component of χ. We denote the conductor of χ by c_χ. For a positive divisor d of N, we put

$$S^+_{k+1/2}(\Gamma_0(4d), \chi) := S^+_{k+1/2}(\Gamma_0(4d)) \mid U(c_\chi) := \{f \mid U(c_\chi) \mid f \in S^+_{k+1/2}(\Gamma_0(4d))\} \tag{4.5}$$

(see [Koh82, Section 2, Proposition 3]). Let $g(z) = \sum_{n \geq 1} b(n) q^n \in S^+_{k+1/2}(\Gamma_0(4N), \chi)$. For a prime $p \nmid N$, the action of the Hecke operator $T(p^2)^+$ on $g(z)$ is defined by

$$g(z) \mid T(p^2)^+ := \sum_{n \geq 1} \left(b(p^2 n) + \chi(p)\chi_{(-1)^k n}(p) p^{k-1} b(n) + p^{2k-1} b(n/p^2) \right) q^n \tag{4.6}$$

(see [Koh82, Section 3, Proposition]). We note that, by the assumption that χ is quadratic, the Hecke operator $T(p^2)^+$ coincides with $T(p^2)$ as (4.1) except $p = 2$. We define the space of *oldforms* in $S^+_{k+1/2}(\Gamma_0(4N), \chi)$ to be

$$\sum_{d \mid N, d < N} \left(S^+_{k+1/2}(\Gamma_0(4d), \chi) + S^+_{k+1/2}(\Gamma_0(4d), \chi) \mid U(N^2/d^2) \right). \tag{4.7}$$

If $f(z)$ and $g(z)$ are cusp forms in $S_{k+1/2}(\Gamma_0(4N))$, then their *Petersson inner product* is defined by

$$\langle f, g \rangle := \frac{1}{[\Gamma_0(4) : \Gamma_0(4N)]} \int_{\Gamma_0(4N) \backslash \mathfrak{H}} f(z) \overline{g(z)} y^{k-3/2} dx dy, \tag{4.8}$$

where $z = x + \sqrt{-1} y$ denotes a variable for the complex upper half plane $\mathfrak{H}$. Define the space $S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4N), \chi)$ of *newforms* in $S^+_{k+1/2}(\Gamma_0(4N), \chi)$ to be the orthogonal complement of the space of oldforms with respect to the Petersson inner product. We simply write

$$S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4N)) := S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4N), \mathbf{1}_{4N}). \tag{4.9}$$

We refer to $g \in S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4N), \chi)$ as a *Kohnen newform* if g is a common eigenvector for all operators $T(p^2)^+$ (respectively $U(p^2)$) for primes $p \nmid N$ (respectively $p \mid N$). The space $S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4N), \chi)$ has an orthogonal basis consisting of Kohnen newforms (see [Koh82, Theorem 2.ii)]). We denote by $w_\ell(f) \in \{\pm 1\}$ the eigenvalue of a newform $f \in S^{\mathrm{new}}_{2k}(\Gamma_0(N))$ under the Atkin-Lehner involution for a prime divisor ℓ of N. Kohnen [Koh85] related $L(k, f \otimes \chi_D)$ to $b(|D|)^2$ explicitly as follows:

Theorem 4.1 *([Koh85, Corollary 1 and Remark]) Let N be a positive odd square-free integer. Let $g(z) = \sum_{n\geq 1} b(n)q^n \in S_{k+1/2}^{+\mathrm{new}}(\Gamma_0(4N))$ be a Kohnen newform and $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ the newform corresponding to g under the Shimura correspondence. Let D be a fundamental discriminant with $(-1)^k D > 0$ and $(D,N) = 1$.*

(i) If $\chi_D(\ell) = w_\ell(f)$ for all prime divisors ℓ of N, then

$$\frac{b(|D|)^2}{\langle g, g\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(k, f\otimes\chi_D)}{\langle f, f\rangle}, \tag{4.10}$$

where $\nu(N)$ denotes the number of distinct prime divisors of N.

(ii) If $\chi_D(\ell) = -w_\ell(f)$ for some divisor ℓ of N,

$$b(|D|) = L(k, f\otimes\chi_D) = 0. \tag{4.11}$$

We note that this result implies that the central L-value $L(k, f\otimes\chi_D)$ is not zero if and only if the Fourier coefficient $b(|D|)$ is not zero.

5 Proof of the non-vanishing

5.1 Proof of Theorem 2.1

For an even integer $k \geq 6$, we define

$$\delta_k(z) := \frac{1}{4\pi\sqrt{-1}}\left(\left(\frac{k}{2}-1\right) G_{k-2}(4z)\frac{d}{dz}\theta(z) - \frac{d}{dz}G_{k-2}(4z)\theta(z)\right) =: \sum_{n\geq 1}\alpha_k(n)q^n \tag{5.1}$$

(see [KZ81, Proof of Corollary 2]), where

$$G_k(z) := \frac{\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}(n)q^n \text{ with } \sigma_{k-1}(n) := \sum_{0<d\mid n} d^{k-1}, \tag{5.2}$$

$$\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2\sum_{n\geq 1} q^{n^2}. \tag{5.3}$$

Lemma 5.1 *For an even integer $k \geq 6$, we have $\delta_k(z) \in S_{k+1/2}^{+}(\Gamma_0(4))$.*

Proof. We recall that $G_{k-2}(4z) \in M_{k-2}(\Gamma_0(4))$ and $\theta(z) \in S_{1/2}(\Gamma_0(4))$. Let $F_1(\ ,\ )$ be the first Rankin-Cohen bracket as in [Coh75, Theorem 7.1]. We have

$$\begin{aligned} F_1(G_{k-2}(4z), \theta(z)) &= \frac{1}{2}\frac{d}{dz}G_{k-2}(4z)\theta(z) - (k-2)G_{k-2}(4z)\frac{d}{dz}\theta(z) \\ &= -8\pi\sqrt{-1}\delta_k(z). \end{aligned}$$

Since $F_1(G_{k-2}(4z), \theta(z)) \in S_{k+1/2}(\Gamma_0(4))$ (see [Coh75, Corollary 7.2]), so is $\delta_k(z)$. The Fourier coefficients $\alpha_k(n)$ of $\delta_k(z)$ are given by

$$\alpha_k(n) = -\frac{n}{8}\sigma_{k-3}\left(\frac{n}{4}\right) + \sum_{\substack{x\in\mathbb{Z} \\ 0<x\leq\sqrt{n}}} \left(\left(\frac{k}{2}-1\right)x^2 + \frac{x^2-n}{4}\right)\sigma_{k-3}\left(\frac{n-x^2}{4}\right), \tag{5.4}$$

where we define $\sigma_{k-3}(n)$ by 0 for $n \notin \mathbb{Z}$ and $\sigma_{k-3}(0)$ by $\dfrac{\zeta(3-k)}{2}$ (see [KZ81, Proof of Corollary 2]). This completes the proof.

Lemma 5.2 *([Koh99, (6)]) Let $k \geq 6$ be an even integer with $k \not\equiv 1 \pmod 3$. For any fundamental discriminants $D > 1$, the Fourier coefficients $\alpha_k(D)$ of $\delta_k(z)$ are in $\mathbb{Z}$. Moreover, if $D \equiv -1 \pmod 3$, then*

$$\alpha_k(D) \equiv -u_k h(-3D) \pmod 3 \tag{5.5}$$

with

$$u_k := \begin{cases} 1 & \text{if } k \equiv 0 \pmod 3, \\ -1 & \text{if } k \equiv 2 \pmod 3. \end{cases} \tag{5.6}$$

For a prime p with $p \equiv 1 \pmod 4$, we have

$$\delta_k(z)|V(p) = \sum_{n\geq 1}\alpha_k(n/p)q^n \in S^+_{k+1/2}(\Gamma_0(4p), \chi_{4p}) \tag{5.7}$$

where $\alpha_k(x) := 0$ if $x \notin \mathbb{Z}$. For $X > 0$ and $\epsilon \in \{\pm 1\}$, we put

$$F^+_{k,p}(X) := \{D \in F^+_p(X) \mid \alpha_k(D) \not\equiv 0 \pmod 3\}, \tag{5.8}$$
$$F^{+,\epsilon}_{k,p}(X) := \{D \in F^+_{k,p}(X) \mid \chi_D(p) = \epsilon\}. \tag{5.9}$$

To prove Theorem 2.1, we use the following lemma:

Lemma 5.3 *Let p be a prime with $p \equiv 1 \pmod 4$. Assume that $k = 6, 8$ or 10, which implies that $\dim_{\mathbb{C}}(S^+_{k+1/2}(\Gamma_0(4))) = 1$. Then there exists $\epsilon \in \{\pm 1\}$ such that for any $X > 0$, we have*

$$\mathcal{N}^+_{k,p}(X) \geq \sharp F^{+,\epsilon}_{k,p}(X) \tag{5.10}$$

Proof. It suffices to prove that there exists $\epsilon \in \{\pm 1\}$ such that for any $X > 0$ and any $D \in F^+_{k,p}(X)$ with $\chi_D(p) = \epsilon$, there exists a newform $f \in S^{\text{new}}_{2k}(\Gamma_0(p))$ satisfying $L(k, f \otimes \chi_D) \neq 0$. By [Koh82, Theorem 2.i) and Lemma], we have a decomposition

$$\begin{aligned} &S^+_{k+1/2}(\Gamma_0(4p), \chi_{4p})) \\ &\quad = S^{+\text{new}}_{k+1/2}(\Gamma_0(4)) \mid U(p) \oplus S^{+\text{new}}_{k+1/2}(\Gamma_0(4)) \mid U(p^3) \oplus S^{+\text{new}}_{k+1/2}(\Gamma_0(4p)) \mid U(p). \end{aligned}$$

Since δ_k is non-zero by Lemma 5.2 and $\dim_{\mathbb{C}}(S^+_{k+1/2}(\Gamma_0(4))) = 1$ by the assumption that $k = 6, 8$ or 10, we have $S^+_{k+1/2}(\Gamma_0(4)) = \mathbb{C}\delta_k$. We thus have

$$\delta_k \mid V(p) = a\delta_k \mid U(p) + b\delta_k \mid U(p^3) + g \mid U(p) \tag{5.11}$$

for some $a, b \in \mathbb{C}$ and $g \in S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$. Applying the both sides by $V(p)$, we have

$$\delta_k \mid V(p^2) = a\delta_k + b\delta_k \mid U(p^2) + g. \tag{5.12}$$

We denote by $b_g(n)$ the n-th Fourier coefficient of g. It follows that for all positive integers n,

$$\alpha_k(n/p^2) = a\alpha_k(n) + b\alpha_k(p^2 n) + b_g(n). \tag{5.13}$$

Let

$$\Delta_k(z) = \sum_{n\geq 1} \tau_k(n) q^n$$

be the cusp form in $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ corresponding to δ_k under the Shimura correspondence (see [Shi73]). We recall that the Shimura correspondence gives an isomorphism between $S^+_{k+1/2}(\Gamma_0(4))$ and $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (see [Koh82, Theorem 2]). Since $\dim_{\mathbb{C}}(S_{2k}(\mathrm{SL}_2(\mathbb{Z}))) = 1$ by the assumption that $k = 6, 8$ or 10, we deduce that Δ_k is a Hecke eigenform. Hence, we have $\tau_k(1) = 1$ and $\tau_k(p^2) = \tau_k(p)^2 - p^{2k-1}$. Let $D > 0$ be a fundamental discriminant with $(D, p) = 1$. By [Koh85, (11)], we have

$$\alpha_k(n^2 D) = \alpha_k(D) \sum_{d\mid n} \mu(d)\chi_D(d) d^{k-1} \tau_k(n/d) \tag{5.14}$$

for all positive integers n. In particular, we have

$$\alpha_k(p^2 D) = \alpha_k(D)\left(\tau_k(p) - \chi_D(p)p^{k-1}\right). \tag{5.15}$$

$$\alpha_k(p^4 D) = \alpha_k(D)\left(\tau_k(p)^2 - p^{2k-1} - \chi_D(p)p^{k-1}\tau_k(p)\right), \tag{5.16}$$

where we use $\tau_k(1) = 1$ and $\tau_k(p^2) = \tau_k(p)^2 - p^{2k-1}$. Putting $n = D$ in (5.13), we have

$$\begin{aligned} b_g(D) &= \left\{-a - b\left(\tau_k(p) - \chi_D(p)p^{k-1}\right)\right\}\alpha_k(D) \\ &= \left\{b\chi_D(p)p^{k-1} - (a + b\tau_k(p))\right\}\alpha_k(D). \end{aligned} \tag{5.17}$$

Putting $n = p^2 D$ in (5.13), we have

$$\begin{aligned} b_g(p^2 D) &= \left\{1 - a\left(\tau_k(p) - \chi_D(p)p^{k-1}\right) - b\left(\tau_k(p)^2 - p^{2k-1} - \chi_D(p)p^{k-1}\tau_k(p)\right)\right\}\alpha_k(D) \\ &= \left\{1 + bp^{2k-1} - (\tau_k(p) - \chi_D(p)p^{2k-1})(a + b\tau_k(p))\right\}\alpha_k(D). \end{aligned} \tag{5.18}$$

Recall that $S^{+\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ has a basis of Kohnen newforms $\{g_i\}_{i=1}^s$ (see [Koh82, Theorem 2.ii)]). We write $g = \sum_{i=1}^s c_i g_i$ with $c_i \in \mathbb{C}$. We denote by $b_i(n)$ the

n-th Fourier coefficient of g_i. Let f_i be the newform corresponding to g_i under the Shimura correspondence. Note that the eigenvalues of the Atkin-Lehner involution at f_i and g_i coincide (see [Koh82, Theorem 2.iii)]). Recall that we denote by $w_p(f_i) \in \{\pm 1\}$ the eigenvalue of f_i under the Atkin-Lehner involution for p. Since the eigenvalue of g_i under $T(p^2) = U(p^2)$ is $-w_p(f_i)p^{k-1}$ (see [Koh82, Theorem 1)]), we have

$$b_i(p^2D) = -w_p(f_i)p^{k-1}b_i(D).$$

We thus have

$$b_g(p^2D) = -p^{k-1}\sum_{i=1}^{s} w_p(f_i)c_ib_i(D). \tag{5.19}$$

Assume that there exists $D_0 \in F^+_{k,p}(X)$ such that $b_i(D_0) = 0$ for all i. We remark that if any $D \in F^+_{k,p}(X)$ satisfies $b_i(D) \neq 0$ for some i, then we have $L(k, f_i \otimes \chi_D) \neq 0$ by Theorem 4.1, that is, the assertion follows. Since $b_g(p^2D_0) = 0$ by (5.19), we have

$$1 + bp^{2k-1} = (\tau_k(p) - \chi_{D_0}(p)p^{k-1})(a + b\tau_k(p)) \tag{5.20}$$

by (5.18). We put $\epsilon := -\chi_{D_0}(p)$. Let $D \in F^+_{k,p}(X)$ with $\chi_D(p) = \epsilon$. By (5.18) and (5.20), we see that

$$\begin{aligned} b_g(p^2D) &= \left\{(\tau_k(p) - \chi_{D_0}(p)p^{k-1})(a + b\tau_k(p)) - (\tau_k(p) - \chi_D(p)p^{k-1})(a + b\tau_k(p))\right\}\alpha_k(D) \\ &= \left\{(\tau_k(p) - \chi_{D_0}(p)p^{k-1}) - (\tau_k(p) - \chi_D(p)p^{k-1})\right\}(a + b\tau_k(p))\alpha_k(D) \\ &= 2\chi_D(p)p^{k-1}\left(a + b\tau_k(p)\right)\alpha_k(D) \end{aligned}$$

If $b_i(D) = 0$ for all i, then we have $b_g(D) = 0$ and $b_g(p^2D) = 0$, so $a + b\tau_k(p) = 0$. By (5.17), we have $0 = b\chi_D(p)p^{k-1}\alpha_k(D)$. Since $b \neq 0$ by (5.20), this contradicts $\alpha_k(D) \neq 0$. Thus, there exists i for which $b_i(D) \neq 0$. This yields $L(k, f_i \otimes \chi_D) \neq 0$ by Theorem 4.1. We have completed the proof.

We put $\varpi_p := (p-1)/2$. Let $\{m_i\}_{i=1}^{\varpi_p}$ be a set of representatives of all quadratic residue classes in $(\mathbb{Z}/p\mathbb{Z})^\times$. Since

$$F^{+,\epsilon}_{k,p}(X) = \bigsqcup_{i=1}^{\varpi_p} F^{+,\epsilon}_{k,p}(X) \cap F^+(X, m_i, p),$$

we have $\sharp F^{+,\epsilon}_{k,p}(X) = \sum_{i=1}^{\varpi_p} \sharp\left(F^{+,\epsilon}_{k,p}(X) \cap F^+(X, m, p)\right)$. Suppose that $k \not\equiv 1 \pmod 3$, that is, $k = 6, 8$. Then $F^{+,\epsilon}_{k,p}(X) \cap F^+(X, m_i, p)$ contains

$$\{D \in F^+(X, m_i, p) \mid D \equiv -1 \pmod 3,\ 3 \nmid h(-3D)\}$$

by Lemma 5.2. We thus obtain $\sharp F^{+,\epsilon}_{k,p}(X) \cap F^+(X, m_i, p) \geq \left(\dfrac{9p}{16\pi^2(p^2-1)} - \varepsilon\right)X$ if $\varepsilon > 0$ and $X \gg_\varepsilon 0$ by Proposition 3.3. This yields

$$\sharp F^{+,\epsilon}_{k,p}(X) \geq \varpi_p \cdot \left(\frac{9p}{16\pi^2(p^2-1)} - \varepsilon\right)X.$$

This implies the assertion for $k = 6, 8$. Now suppose that $k \equiv 1 \pmod 3$, that is, $k = 10$. Let

$$E_4(z) := 1 - 240 \sum_{n \geq 1} \sigma_3(n) q^n \tag{5.21}$$

be the normalized Eisenstein series of weight 4 on $\mathrm{SL}_2(\mathbb{Z})$. Then $\delta_{6,p}(z)E_4(4z) \in S^+_{21/2}(\Gamma_0(4p), \chi_{4p})$ and we have $\delta_{6,p}(z)E_4(4z) \equiv \delta_{6,p}(z) \pmod 3$. We denote by $\beta_{10}(n)$ the n-th Fourier coefficient of $\delta_{6,p}(z)E_4(4z)$. We thus have for a fundamental discriminant $D > 1$ with $D \equiv -1 \pmod 3$,

$$\beta_{10}(D) \equiv \alpha_6(D) \equiv -h(-3D) \pmod 3 \tag{5.22}$$

by Lemma 5.2. Applying the argument of Lemma 5.3 to $\delta_{6,p}(z)E_4(4z)$, we see that there exists $\epsilon \in \{\pm 1\}$ such that for any $X > 0$, the cardinality of the set

$$\{D \in F_p^+(X) \mid \beta_{10}(D) \neq 0, \chi_D(p) = \epsilon\}, \tag{5.23}$$

is less than or equal to $\mathcal{N}^+_{10,p}(X)$. Since the set (5.23) contains $F^{+,\epsilon}_{6,p}(X)$ by (5.22), we have

$$\mathcal{N}^+_{10,p}(X) \geq \sharp F^{+,\epsilon}_{6,p}(X). \tag{5.24}$$

We have completed the proof.

5.2 Proof of Corollary 2.2

Let p be a prime with $p \equiv 1 \pmod 4$. Let $\varepsilon > 0$ and $X \gg_\varepsilon 0$. We denote by S the subset of $F^+(X)$ consisting of D such that there exists a newform $f \in S^{\mathrm{new}}_{2k}(\Gamma_0(p))$ satisfying $L(k, f \otimes \chi_D) \neq 0$ so that we have $\sharp S = \mathcal{N}^+_{k,p}(X)$. We put $S_f := \{D \in F^+(X) \mid L(k, f \otimes \chi_D) \neq 0\}$ for a newform $f \in S^{\mathrm{new}}_{2k}(\Gamma_0(p))$. Then we have $S \subset \bigsqcup S_f$, where $\bigsqcup$ runs over all newforms $f \in S^{\mathrm{new}}_{2k}(\Gamma_0(p))$. Let $f_0 \in S^{\mathrm{new}}_{2k}(\Gamma_0(p))$ be a newform satisfying $\mathcal{N}^+_{k,f_0}(X) \geq \mathcal{N}^+_{k,f}(X)$ for any newform $f \in S^{\mathrm{new}}_{2k}(\Gamma_0(p))$. Then we have $\mathcal{N}^+_{k,p}(X) \leq \sum_f \mathcal{N}^+_{k,f}(X) \leq d_{k,p} \cdot \mathcal{N}^+_{k,f_0}(X)$. This completes the proof.

5.3 Proof of Corollary 2.3

Let D be a fundamental discriminant with $D > 0$. We define the algebraic part of the central L-value of $f \otimes \chi_D$ by

$$L^{\mathrm{alg}}(k, f \otimes \chi_D) := \frac{\sqrt{D}(k-1)! L(k, f \otimes \chi_D)}{(2\pi\sqrt{-1})^{k-1}\Omega_f^\epsilon}, \tag{5.25}$$

where Ω_f^ϵ denotes a canonical period of f, which has the property that for any $\sigma \in G_\mathbb{Q}$,

$$L^{\mathrm{alg}}(k, f \otimes \chi_D)^\sigma = L^{\mathrm{alg}}(k, f^\sigma \otimes \chi_D) \tag{5.26}$$

and $\epsilon = \mathrm{sgn}((-1)^{k-1}\chi_D(-1))$ (see [Shi77, Theorem 1]). Then for any $\sigma \in G_{\mathbb{Q}}$,

$$\begin{aligned} L(k, f \otimes \chi_D) \neq 0 &\Leftrightarrow L^{\mathrm{alg}}(k, f \otimes \chi_D) \neq 0 \\ &\Leftrightarrow L^{\mathrm{alg}}(k, f \otimes \chi_D)^{\sigma} \neq 0 \\ &\Leftrightarrow L(k, f^{\sigma} \otimes \chi_D) \neq 0, \end{aligned}$$

where the last equivalence is deduced from (5.26). This competes the proof.

6 Proof of the vanishing

We use the following lemma to prove Theorem 2.4.

Lemma 6.1 (*see [Sak08, Remark 5]*) *Let $k \geq 2$ be an integer and N a positive odd integer such that $\mathrm{ord}_p(N)$ is odd for any prime divisor p of N. Let $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ be a newform. Let D be a fundamental discriminant with $(-1)^k D > 0$ and $(D, N) = 1$. If $\chi_D(p) = -w_p(f)$ for some $p \mid N$, then*

$$L(k, f \otimes \chi_D) = 0,$$

where we recall that $w_p(f)$ is defined just before Theorem 4.1.

We put $\epsilon := \mathrm{sgn}((-1)^k)$. Then each $D \in F_N^{\epsilon}(X)$ satisfies $(-1)^k D > 0$. Let p be a prime divisor of N. We set

$$S_p := \{D \in F_N^{\epsilon}(X) \mid \chi_D(p) = -w_p(f), \chi_D(\ell) = w_\ell(f) \text{ for any prime divisors } \ell \neq p \text{ of } N\}$$

and $S := \bigsqcup_{p \mid N:\mathrm{prime}} S_p$. By Lemma 6.1, we have

$$S \subset \{D \in F_N^{\pm}(X) \mid L(k, f \otimes \chi_D) = 0\}. \tag{6.1}$$

We put $\varpi_p := (p-1)/2$. Let $\{m_{p,i}\}_{i=1}^{\varpi_p}$ (respectively $\{m'_{p,i}\}_{i=1}^{\varpi_p}$) be a set of representatives of residue classes in $(\mathbb{Z}/p\mathbb{Z})^{\times}$ satisfying $\chi_{m_i}(p) = -w_p(f)$ (respectively $\chi_{m_i}(p) = w_p(f)$). We define the subsets M_p and M'_p of $(\mathbb{Z}/p\mathbb{Z})^{\times}$ by $M_p := \{m_{p,i} \bmod p\}_{i=1}^{\varpi_p}$ and $M'_p := \{m'_{p,i} \bmod p\}_{i=1}^{\varpi_p}$. Let $N_0 := \prod_{p \mid N:\mathrm{prime}} p$ be the square-free part of N. We put $\varpi := \prod_{p \mid N:\mathrm{prime}} \varpi_p = \varphi(N_0)/2^{\nu(N)}$. We take a set of representatives $\{m_i^{(p)}\}_{i=1}^{\varpi}$ of the image of

$$M_p \times \prod_{\substack{\ell \mid N:\mathrm{prime} \\ \ell \neq p}} M'_\ell$$

in $(\mathbb{Z}/N_0\mathbb{Z})^{\times}$ by Chinese Remainder Theorem $\prod_{p \mid N:\mathrm{prime}} (\mathbb{Z}/p\mathbb{Z})^{\times} \xrightarrow{\sim} (\mathbb{Z}/N_0\mathbb{Z})^{\times}$. We then have

$$S_p = \bigsqcup_{i=1}^{\varpi} F^{\epsilon}(X, m_i^{(p)}, N_0).$$

Combining this with (6.1) and taking their cardinalities, we have

$$\mathcal{V}_{k,f}^{\epsilon}(X) \geq \sum_{p|N:\text{prime}} \sum_{i=1}^{\varpi} N_2^{\epsilon}(X, m_i^{(p)}, N_0). \tag{6.2}$$

By Lemma 3.2 and Lemma 3.1 (1), we have

$$N_2^{\epsilon}(X, m_i^{(p)}, N_0) \geq \left(\frac{3}{\varphi(N_0)\pi^2} \prod_{p|N:\text{prime}} \frac{p}{p+1} - \varepsilon \right) X$$

for $\varepsilon > 0$ and $X \gg_{\varepsilon} 0$. By (6.2) together with this inequality, we have

$$\mathcal{V}_{k,f}^{\epsilon}(X) \geq \nu(N)\varpi \left(\frac{3}{\varphi(N_0)\pi^2} \prod_{p|N:\text{prime}} \frac{p}{p+1} - \varepsilon \right) X$$

This implies that for $\varepsilon > 0$ and $X \gg_{\varepsilon} 0$, we have

$$\mathcal{V}_{k,f}^{\epsilon}(X) \geq \left(\frac{3\nu(N)}{2^{\nu(N)}\pi^2} \prod_{p|N:\text{prime}} \frac{p}{p+1} - \varepsilon \right) X.$$

We have coompleted the proof of Theorem 2.4.

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