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Affine and sphere Schwarz maps for the hypergeometric differential equation

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Abstract. For an ordinary differential equation $u'' - qu = 0$, where q is holomorphic, the *sphere Schwarz map* is defined. For the hypergeometric equations with polyhedral monodromy groups, the image surfaces of the sphere Schwarz map are studied.

Contents

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Introduction

Consider an ordinary differential equation

$$u'' - q(x)u = 0, \quad x \in X,$$

where q is holomorphic in a domain $X \subset \mathbf{C}$. For two linearly independent solutions u and v, the **affine** Schwarz map is defined in [NKSY] as

$$\mathcal{S}^{\mathrm{aff}} : X \ni x \longmapsto (u(x), v(x)) \in (\mathbf{C}^2, 0),$$

where $(\mathbf{C}^2, 0)$ is the centro-affine plane, whose motion group is $GL_2(\mathbf{C})$. The **original** Schwarz map (cf. [Yo]) is defined by taking ratio of the affine one:

$$\mathcal{S}^{\mathrm{ori}} : X \ni x \longmapsto u(x) : v(x) \in \mathbf{P}^1_{\mathbf{C}}.$$

By taking *ratio*, the original one loses much information that the affine one has. To compensate this loss, we defined the **hyperbolic** and the **de Sitter** Schwarz maps in [SYY, FNSYY], and studied their behavior. In this paper, we propose another way.

If we recall that taking ratio of two complex numbers means taking quotient by the multiplicative group $\mathbf{C}^\times$, and the isomorphism

$$\mathbf{C}^\times \cong \mathbf{R}_{>0} \times S^1 \cong \mathbf{R}^\times \times \mathbf{P}^1_{\mathbf{R}},$$

we are led to the **sphere** Schwarz map

$$\mathcal{S}^{\mathrm{sph}} : X \ni x \longmapsto u_1(x) : u_2(x) : v_1(x) : v_2(x) \in \mathbf{P}^3_{\mathbf{R}}$$

where $u = u_1 + iu_2, v = v_1 + iv_2$. Since the 3-sphere S^3 is just the double cover of the real projective 3-space $\mathbf{P}^3_{\mathbf{R}}$, we sometimes regard S^3 the target of this map. They are related through the projections:

$$\mathbf{C}^2 - \{0\} \cong \mathbf{R}^4 - \{0\} \longrightarrow S^3 \longrightarrow \mathbf{P}^3_{\mathbf{R}} \longrightarrow \mathbf{P}^1_{\mathbf{C}} \cong S^2.$$

Though the natural motion groups of the real centro-afiine 4-space $(\mathbf{R}^4, 0)$ and $\mathbf{P}^3_{\mathbf{R}}$ are $GL_4(\mathbf{R})$ and $PGL_4(\mathbf{R})$, respectively, throughout this paper we fix the identification of $(\mathbf{C}^2, 0)$ and $(\mathbf{R}^4, 0)$ by

$$(u, v) \longleftrightarrow (u_1, u_2, v_1, v_2),$$

and accordingly fix the embedding $GL_2(\mathbf{C}) \to GL_4(\mathbf{R})$, and use the image group, say G, as the motion group of $(\mathbf{R}^4, 0)$, and the image PG of G under the projection $GL_4(\mathbf{R}) \to PGL_4(\mathbf{R})$ as the motion group of $\mathbf{P}^3_{\mathbf{R}}$.

The image of the affine Schwarz map is a complex plane curve in the complex centro-affine plane, while that of the sphere Schwarz map is a surface in $\mathbf{P}^3_{\mathbf{R}}$. A complex curve is a real surface in the 4-space, which is not easy to *see*. So we stand at the origin and look around the space and project the surface to the 3-dimensional screen.

If we take another pair of solutions u' and v', and define the three kinds of Schwarz maps: $\mathcal{S}'^{\text{aff}}, \mathcal{S}'^{\text{sph}}$ and $\mathcal{S}'^{\text{ori}}$, then they are related to the former ones via the motion groups $GL_2(\mathbf{C}), PG$ and $PGL_2(\mathbf{C})$ of $(\mathbf{C}^2, 0), \mathbf{P}^3_{\mathbf{R}}$ and $\mathbf{P}^1_{\mathbf{C}}$, respectively.

A natural problem is to find invariants for surfaces in $(\mathbf{P}^3_{\mathbf{R}}, PG)$ to identify those coming from complex curves in $(\mathbf{C}^2, 0)$ (cf. [An]). But we leave this problem for a future study.

In this paper we study the image surfaces under the sphere Schwarz map of the hypergeometric differential equation having a polyhedral group as its monodromy. For the dihedral monodromy groups, sphere Schwarz image surfaces are described more in detail.

1 Projections

1.1 Projection of algebraic curves

Let a curve C in $(\mathbf{C}^2, 0)$ be defined by a polynomial $F(u, v) = 0$. The defining equation R of the image surface $S' \subset \mathbf{P}^3_{\mathbf{R}}$ of the curve C under the projection

$$\pi : (\mathbf{C}^2, 0) \cong (\mathbf{R}^4, 0) \longrightarrow \mathbf{P}^3_{\mathbf{R}}$$

is given as follows. The inverse image $\pi^{-1}(S')$ of S' is a cone in $(\mathbf{R}^4, 0)$ weaved by the curves

$$C_k = \{(u, v) \mid F(k) = 0\}, \quad k \in \mathbf{R}^\times,$$

where $F(k) = F(ku, kv)$. The equation of this cone should be R. Substitute

$$u = u_1 + iu_2, \quad v = v_1 + iv_2$$

into $F(u, v)$ and write

$$F(u_1 + iu_2, v_1 + iv_2) = G(u_1, u_2, v_1, v_2) + iH(u_1, u_2, v_1, v_2),$$

where G and H are polynomials in u_1, u_2, v_1, v_2 with real coefficients. Set

$$G(k) = G(ku_1, ku_2, kv_1, kv_2), \quad H(k) = H(ku_1, ku_2, kv_1, kv_2).$$

Then the curve C_k defined by $F(k)$ in $(\mathbf{C}^2, 0)$ is defined by $G(k)$ and $H(k)$ in $(\mathbf{R}^4, 0)$. Thus we have

Proposition 1 *The homogeneous polynomial $R = R(u_1, u_2, v_1, v_2)$ defining the cone $\pi^{-1}(S')$ is obtained from $G(k)$ and $H(k)$ by eliminating k.*

The elimination can be done by making resultant of $G(k)$ and $H(k)$; this process does not care about the reality of k. So S' maybe part of the closed surface

$$S = \{R = 0\} \subset \mathbf{P}^3_{\mathbf{R}}.$$

If F is of degree n, so are G and H. Generically, the resultant R of the two polynomials $G(k)$ and $H(k)$ in k is of homogeneous degree n^2 in u_1, u_2, v_1, v_2. If F has no constant term, then R is at most of degree $n^2 - 1$. If F has no term of degree less than j, then R is at most of degree $n^2 - j^2$. In some special cases the degree of R can be lower. For example when F is of degree 4, generically, R is of degree $4^2 = 16$; if F has no linear terms, then of degree $4^2 - 2^2 = 12$, if moreover F has no cubic terms then $G(k)/k^2$ and $H(k)/k^2$ are of degree 1 in k^2, and so R is of degree $4 + 2 = 6$ (cf. §6.2).

Remark 1 *We assume that F itself is not homogeneous. If so, S is not a surface but a curve. If F is not homogeneous but if G or H is homogeneous, we must be careful: For example, consider*

$$F = u^2 + v^2 - 1.$$

The polynomial H is homogeneous so that the result of the elimination is just $R = H$ with some extra inequality due to the equation $0 = G(k) = k^2(u_1^2 - u_2^2 + v_1^2 - v_2^2) - 1$, so that $u_1^2 - u_2^2 + v_1^2 - v_2^2 \geq 0$.

If F is not a polynomial, the author has no idea of getting R. Inverse problem is also difficult:

Open problem: Characterize surfaces $S \subset \mathbf{P}^3_{\mathbf{R}}$ (or equations R) which are projections of complex curves in $(\mathbf{C}^2, 0)$.

If the surface is algebraic of degree 1 or 2, then it is easy. When S is a cubic surface, it is already non-trivial (cf. §7).

1.2 Hopf map

The Hopf map is

$$H : \mathbf{C}^2 \supset S^3 = \{(u, v) \mid |u|^2 + |v|^2 = 1\} \ni (u, v) \longmapsto u : v \in \mathbf{P}^1_{\mathbf{C}}.$$

A Hopf fiber is the inverse image of a point $u : v$ given as

$$\left\{ \frac{1}{\sqrt{|u|^2 + |v|^2}} (e^{it}u, e^{it}v) \mid 0 \leq t < 2\pi \right\},$$

a circle. The inverse image of a circle $|u/v| = r$:

$$z = u/v = re^{is}, \quad 0 \leq s < 2\pi$$

is a torus

$$T_r = \left\{ \frac{1}{\sqrt{1 + r^2}} \left(re^{i(s+t)}, e^{it} \right) \right\} = \left\{ |u|^2 = \frac{r^2}{1 + r^2} \right\}, \quad 0 \leq r \leq \infty$$

which degenerates to cores of the torus when $r = 0, \infty$.

The projective Hopf map $pH : \mathbf{P}^3_{\mathbf{R}} \to \mathbf{P}^1_{\mathbf{C}}$ is defined by an obvious manner. The inverse image of the circle $|v/u| = r$ is again a torus

$$T_r/\pm = \left\{ \left(re^{i(s+t/2)} : e^{it/2} \right) \right\}, \quad 0 \leq s, t < 2\pi.$$

2 Simplest case – cusps

Consider a cusp

$$\mathbf{C} \ni z \longmapsto (u, v) = (z^p, z^q) \in (\mathbf{C}^2, 0), \quad (p, q) = 1.$$

This can be considered as the affine Schwarz map of a very degenerate hypergeometric equation

$$u'' - \frac{\alpha(\alpha - 1)}{x^2} u = 0.$$

Indeed, this has solutions

$$u = x^\alpha, \quad v = x^{1-\alpha}.$$

If we set

$$\alpha = p/r, \quad r - p = q, \quad z = u^r,$$

then the affine Schwarz map $x \mapsto (u, v)$ gives the above map.

2.1 Cusp of type $(2, 3)$

For simplicity, we consider the simplest case $(p, q) = (2, 3)$, so

$$F = u^3 - v^2.$$

Since the cusp map followed by the projection

$$\begin{array}{ccccc} \mathbf{C} - \{0\} & \longrightarrow & (\mathbf{C}^2, 0) & \longrightarrow & \mathbf{P}^1_{\mathbf{C}} \\ z & \longmapsto & (z^2, z^3) & \longmapsto & z \end{array}$$

is the identity, you might think that the surface $S \subset \mathbf{P}^3_{\mathbf{R}}$ should be like a sphere. Let us see. Substituting $u = u_1 + iu_2, v = v_1 + iv_2$ into F, we have

$$u^3 - v^2 = u_1^3 - 3u_1u_2^2 + v_1^2 - v_2^2 + i(3u_1^2u_2 - u_2^3 + 2v_1v_2).$$

Thus the curve $F = 0$ in $(\mathbf{C}^2, 0)$ is defined by the system

$$G = u_1^3 - 3u_1u_2^2 + v_1^2 - v_2^2 = 0, \quad H = 3u_1^2u_2 - u_2^3 + 2v_1v_2 = 0$$

in $(\mathbf{R}^4, 0)$. Since each is the sum of terms of degree 3 and 2, it is easy to eliminate k from $G(k)$ and $H(k)$:

$$R = (u_1^3 - 3u_1u_2^2)2v_1v_2 - (3u_1^2u_2 - u_2^3)(v_1^2 - v_2^2).$$

Thus we have

Proposition 2 *The surface S is defined by the quintic polynomial R, and is singular along two (non-intersecting but linked) lines*

$$u_1 = u_2 = 0 \quad \text{and} \quad v_1 = v_2 = 0,$$

which are projective Hopf fibers of $z = 0$ and ∞, respectively.

- *Along the line $u_1 = u_2 = 0$, three leaves of S cross normally and turn by $2\pi/3$.*
- *Along the line $v_1 = v_2 = 0$, two leaves of S cross normally and turn by $3\pi/2$.*

The latter assertions can be seen as follows: The intersection of S and the plane v_1/v_2 =constant is the union of the three lines in the (u_1, u_2)-plane intersecting at the origin $(u_1, u_2) = (0, 0)$. So the surface S is weaved by trefoil knots on the tori $T_r/\pm$ added by the two cores of the tori.

2.2 A family of trefoil knots

Projecting the affine Schwarz map to the sphere, we have

$$re^{it} \longmapsto \frac{1}{\sqrt{1+r^2}}(\cos 2t, \sin 2t, r\cos 3t, r\sin 3t).$$

For a fixed $r > 0$, the image of the circle is a trefoil knot (knot of type $(2, 3)$) on the torus T_r (see Figure 1 left). As r tends to 0, three leaves come together to a core of the torus winding twice; and as r tends to ∞, two leaves come together to the other core of the torus winding three times.

Projecting further to the projective space, the image curve is a $(4, 3)$-curve on the torus $T_r/\pm$ (see Figure 1 right).

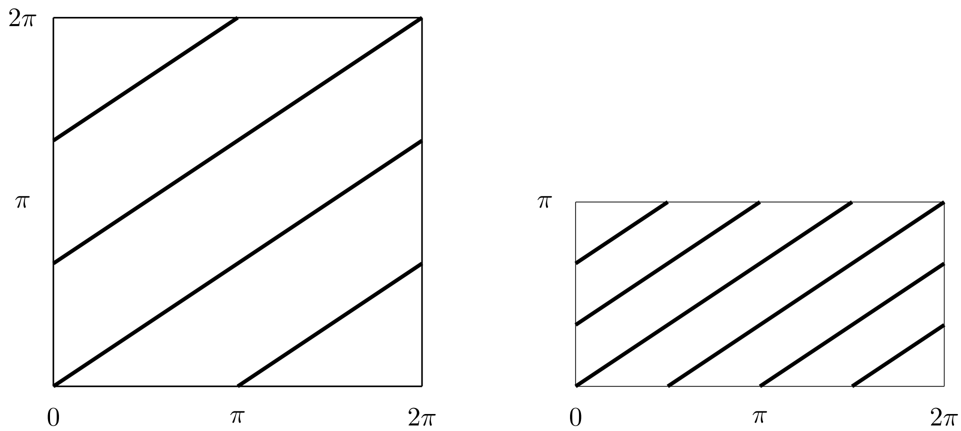

Figure 1: $(2, 3)$-curve on T_r and $(4, 3)$-curve on $T_r/\pm$

2.3 Cusps in general

The above statements for $(2,3)$-cusp are word to word valid for a cusp

$$F = u^p - v^q, \quad (p,q) = 1,$$

if we read p for 3, and q for 2. The polynomial R is of degree $p+q$.

3 Hypergeometric equations with polyhedral monodromy groups

When the hypergeometric equation has polyhedral monodromy group G and the inverse of the original Schwarz map

$$\mathcal{S}^{\mathrm{ori}} : X = \mathbf{C} - \{0,1\} \ni x \mapsto z = u(x)/v(x) \in \mathbf{P}^1_{\mathbf{C}}$$

is single valued, following holds (see for example [SYY]): The inverse is given by

$$x = A_0\ \frac{f_0(z)^{k_0}}{f_\infty(z)^{k_\infty}},$$

which leads to

$$1 - x = A_1\ \frac{f_1(z)^{k_1}}{f_\infty(z)^{k_\infty}}, \quad \frac{dx}{dz} = A\frac{f_0(z)^{k_0-1}f_1(z)^{k_1-1}}{f_\infty(z)^{k_\infty+1}}.$$

We write the hypergeometric equation in the SL-form:

$$u'' - q(x)u = 0, \quad q = \frac{\text{a quadratic polynomial in } x}{x^2(1-x)^2}.$$

Then the affine Schwarz map with z as variable is given by

$$z \longmapsto (u,v) = (z\sqrt{dx/dz}, \sqrt{dx/dz}).$$

Notation used above is given as follows:

Dihedral $(k_0, k_1, k_\infty) = (2, 2, n),\ N = 2n,$

$$\begin{array}{ll} A_0 & = \frac{1}{4}, \quad A_1 = -\frac{1}{4}, \quad A = n/4, \\ f_0 & = z^n + 1, \quad f_1 = z^n - 1, \quad f_\infty = z. \end{array}$$

Tetrahedral $(k_0, k_1, k_\infty) = (2, 3, 3), \quad N = 12,$

$$\begin{array}{ll} A_0 & = -12\sqrt{3}, \quad A_1 = 1, \quad A = 24\sqrt{3}, \\ f_0 & = z(z^4+1), \\ f_1 & = z^4 + 2\sqrt{3}z^2 - 1 = (z^2 - 2 + \sqrt{3})(z^2 + 2 + \sqrt{3}), \\ f_\infty & = z^4 - 2\sqrt{3}z^2 - 1 = (z^2 - 2 - \sqrt{3})(z^2 + 2 - \sqrt{3}). \end{array}$$

Octahedral $(k_0, k_1, k_\infty) = (3, 2, 4)$, $N = 24$,

$$\begin{aligned}
A_0 &= 1/108, \quad A_1 = -1/108, \quad A = 1/27,\\
f_0 &= z^8 + 14z^4 + 1 = (z^4 + 2z^3 + 2z^2 - 2z + 1)(z^4 - 2z^3 + 2z^2 + 2z + 1),\\
f_1 &= z^{12} - 33z^8 - 33z^4 + 1 = (z^4 + 1)(z^2 + 2z - 1)(z^2 - 2z - 1)(z^4 + 6z^2 + 1),\\
f_\infty &= z(z^4 - 1) = z(z^2 + 1)(z^2 - 1).
\end{aligned}$$

Icosahedral $(k_0, k_1, k_\infty) = (3, 2, 5)$, $N = 60$,

$$\begin{aligned}
A_0 &= -1/1728, \quad A_1 = 1/1728, \quad A = -5/1728,\\
f_0 &= z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1\\
&= (z^4 - 3z^3 - z^2 + 3z + 1)(z^8 - z^7 + 7z^6 + 7z^5 - 7z^3 + 7z^2 + z + 1)\\
&\quad \times (z^8 + 4z^7 + 7z^6 + 2z^5 + 15z^4 - 2z^3 + 7z^2 - 4z + 1),\\
f_1 &= z^{30} + 522z^{25} - 10005z^{20} - 10005z^{10} - 522z^5 + 1\\
&= (z^2 + 1)(z^8 - z^6 + z^4 - z^2 + 1)(z^4 + 2z^3 - 6z^2 - 2z + 1)\\
&\quad \times (z^8 + 4z^7 + 17z^6 + 22z^5 + 5z^4 - 22z^3 + 17z^2 - 4z + 1)\\
&\quad \times (z^8 - 6z^7 + 17z^6 - 18z^5 + 25z^4 + 18z^3 + 17z^2 + 6z + 1),\\
f_\infty &= z(z^{10} + 11z^5 - 1)\\
&= z(z^2 + z - 1)(z^4 + 2z^3 + 4z^2 + 3z + 1)(z^4 - 3z^3 + 4z^2 - 2z + 1).
\end{aligned}$$

4 Affine and sphere Schwarz map

Recall that the affine Schwarz map with variable $z \in \mathbf{P}^1_{\mathbf{C}}$ is given by

$$\mathcal{S}^{\mathrm{aff}} : z \longmapsto (u, v) = \left(z\sqrt{\frac{dx}{dz}}, \sqrt{\frac{dx}{dz}} \right), \quad \frac{dx}{dz} = \frac{f_0(z)^{k_0-1} f_1(z)^{k_1-1}}{f_\infty(z)^{k_\infty+1}},$$

and

group G	k_0	k_1	k_∞	deg f_0	deg f_1	deg f_∞
Dih $D_{2\cdot n}$	2	2	n	n	n	1
Tetrah	2	3	3	5	4	4
Octah	3	2	4	8	12	5
Icosah	3	2	5	20	30	11

Substituting $z = u/v$ into

$$v^2 = \frac{f_0(z)^{k_0-1} f_1(z)^{k_1-1}}{f_\infty(z)^{k_\infty+1}},$$

we get a polynomial F in $\{u, v\}$ defining the affine Schwarz image curve C. F is the sum of two homogeneous polynomials F_p of degree p and F_{p+2} of $p+2$, where

$$p \;=\; 2n, \quad 14, \quad 28, \quad 72,$$

when G is Dihedral $D_{2\cdot n}$, tetrahedral, octahedral and icosahedral, respectively.

Set

$$F_p = G_p + iH_p, \quad F_{p+2} = G_{p+2} + iH_{p+2}.$$

Since

$$F(ku, kv) = k^p \left\{(G_p + k^2 G_{p+2}) + i(H_p + k^2 H_{p+2})\right\},$$

by eliminating k^2 from $G_p + k^2 G_{p+2}$ and $H_p + k^2 H_{p+2}$, we get the equation R of the sphere Schwarz image surface as

$$R = G_p H_{p+2} - G_{p+2} H_p,$$

which is of homogeneous degree $2p + 2$ in $\{u_1, u_2, v_1, v_2\}$. Since $k^2 > 0$, we have

Proposition 3 *The sphere Schwarz image S' is not the whole surface S defined by R but a subdomain of S determined by*

$$G_p(u_1, u_2, v_1, v_2) G_{p+2}(u_1, u_2, v_1, v_2) < 0.$$

In particular, when $G = D_{2\cdot n}$, the affine Schwarz image curve is defined by

$$F_n := u^{n+1} u^{n+1} - u^{2n} + v^{2n}.$$

Writing

$$\begin{array}{ll} u^{n+1} = U_1 + iU_2, & u^{2n} = U_3 + iU_4, \\ v^{n+1} = V_1 + iV_2, & v^{2n} = V_3 + iV_4, \end{array}$$

the equation of the sphere Schwarz surface is

$$R_n = (U_1 V_1 - U_2 V_2)(U_4 - V_4) - (U_1 V_2 + U_2 V_1)(U_3 - V_3).$$

5 Local behavior of the sphere Schwarz map around the vertices of Schwarz triangles

We consider the sphere Schwarz map with variable $z \in \mathbf{P}^1_{\mathbf{C}}$:

$$\mathcal{S}^{\rm sph}_z : \mathbf{P}^1_{\mathbf{C}} - V \ni z \longmapsto u_1(z) : u_2(z) : v_1(z) : v_2(z) \in \mathbf{P}^3_{\mathbf{R}},$$

where $V \subset \mathbf{P}^1_{\mathbf{C}}$ is the set of vertices of Schwarz triangles; V is the union of ∞ and set of the zeros of f_0, f_1 and f_∞. Let S' be the image of $\mathbf{P}^1_{\mathbf{C}} - V$ under $\mathcal{S}^{\rm sph}_z$, and S the closure of S' in $\mathbf{P}^3_{\mathbf{R}}$.

Theorem 1 *The sphere Schwarz map $\mathcal{S}^{\rm sph}_z$ is one-to-one from $\mathbf{P}^1_{\mathbf{C}} - V$ onto S'. Let z_0 be a vertex of a Schwarz triangle, and C_ϵ the circle with center z_0 of radius ϵ. Then the image curve $\mathcal{S}^{\rm sph}_z(C_\epsilon)$ tends to a circle, say $\mathcal{S}^{\rm sph}_z(z_0)$, which is the projective Hopf fiber of z_0. Through the circle $\mathcal{S}^{\rm sph}_z(z_0)$, several leaves of S pass. The number of leaves are given in the proof.*

Proof: Let z_0 be a root of f_j $(j = 0, 1)$. Put $z = z_0 + re^{it}$; unless G is Tetrahedral and $j = 0$, we have $z_0 \neq 0$. Then the affine Schwarz image of z around z_0 can be written as

$$\left(z\sqrt{(re^{it})^{k_j - 1} h(z)}, \sqrt{(re^{it})^{k_j - 1} h(z)} \right),$$

where h is a function holomorphic and non-vanishing around z_0. If $\sim$ stands for '*up to multiplication of real numbers*', we have

$$\sim \left(ze^{(k_j-1)it/2}\sqrt{h(z)}, e^{(k_j-1)it/2}\sqrt{h(z)}\right), \quad h(z_0) \neq 0, \infty.$$

Thus $k_j - 1$ leaves pass through $\mathcal{S}^{sph}(z_0)$. If G is Tetrahedral and $j = 0$, then since we have

$$\sim \left(re^{(k_j+1)it/2}\sqrt{h(z)}, e^{(k_j-1)it/2}\sqrt{h(z)}\right),$$

the result is the same as above.

0 is a root of f_∞, if G is not Tetrahedral. The affine Schwarz image of $z = re^{it}$ around 0 can be written as

$$\left(z\sqrt{z^{-k_\infty-1}h}, \sqrt{z^{-k_\infty-1}h}\right) \sim \left(z^{1-(k_\infty+1)/2}\sqrt{h}, z^{-(k_\infty+1)/2}\sqrt{h}\right),$$

and

$$\sim \left(re^{-(k_\infty-1)it/2}\sqrt{h}, e^{-(k_\infty+1)it/2}\sqrt{h}\right).$$

Thus $k_\infty + 1$ leaves pass through $\mathcal{S}^{sph}(0)$.

Let z_0 be a non-zero root of f_∞. Put $z = z_0 + re^{it}$. Then since the affine Schwarz image of z around z_0 can be written as

$$\left(ze^{-(k_\infty+1)it/2}\sqrt{h(z)}, e^{-(k_\infty+1)it/2}\sqrt{h(z)}\right),$$

the result is the same as above.

At $z_0 = \infty$, we change coordinate as $z = 1/\zeta$. The affine Schwarz image of $\zeta = re^{it}$ around 0 can be written as

$$\sim \left(\zeta^{-1}\sqrt{\zeta^d h}, \sqrt{\zeta^d h}\right) \sim \left(e^{(d+2)it/2}\sqrt{h}, re^{dit/2}\sqrt{h}\right),$$

where

$$d = (k_\infty + 1)\deg f_\infty - (k_0 - 1)\deg f_0 - (k_1 - 1)\deg f_1.$$

Thus $d + 2$ leaves pass through $\mathcal{S}^{sph}(\infty)$.

6 Sphere Schwarz surface when $G = D_{2\cdot n}$

We study the surface S_n defined by R_n.

6.1 Symmetry

Recall that the affine Schwarz image curve C_n is defined by

$$F_n = u^{n+1}v^{n+1} - u^{2n} + v^{2n}.$$

The curve is invariant under complex linear transformations

$$(u, v) \longmapsto (iv, iu), \quad (\zeta u, v/\zeta), \qquad \zeta^{2n} = 1,$$

and under $(u, v) \mapsto (-v, u)$ when $n = 4k$, and under $(u, v) \mapsto (-u, v)$ when n is odd. Then the surface S_n is invariant under these transformations, of course.

Moreover, the surface S_n is invariant under complex conjugation:

$$(u_1, u_2, v_1, v_2) \longmapsto (u_1, -u_2, v_1, -v_2).$$

The transformation $(u, v) \mapsto (-u, v)$ does not keep C_n, but when n is even, since

$$(U_1, U_2, U_3, U_4) \longmapsto (-U_1, -U_2, U_3, U_4),$$

it keeps S_n

6.2 Simplest case $n = 1$

When $n = 1$, we have

$$R_1 = (U_1V_1 - U_2V_2)(U_2 - V_2) - (U_1V_2 + U_2V_1)(U_1 - V_1),$$

where

$$U_1 = u_1^2 - u_2^2, \quad U_2 = 2u_1u_2, \quad V_1 = v_1^2 - v_2^2, \quad V_2 = 2v_1v_2,$$

and so,

$$R_1 = (u_1^2 + u_2^2)^2 v_1v_2 - (v_1^2 + v_2^2)^2 u_1u_2.$$

6.2.1 Symmetry and quarter parts

The space $\mathbf{P}^3_{\mathbf{R}}$ is divided into eight tetrahedra by the four planes $u_1 = 0, u_2 = 0, v_1 = 0, v_2 = 0$. Though the six edges $u_1 = v_1 = 0, \ldots$ are in the surface S_1, it does not have intersection with the four open chambers satisfying $u_1u_2v_1v_2 < 0$. Since the surface S_1 is invariant under the transformations

$$(u, v) \mapsto (-u, v), \quad (-u_1, u_2, -v_1, v_2),$$

the four parts of S_1 in the four chambers $u_1u_2v_1v_2 > 0$ are projectively isomorphic, and since S_1 is invariant also under

$$(u, v) \mapsto (v, u), \quad (u_2, u_1, v),$$

the quarter part

$$S_1/4 := S_1 \cap \{u_1 : u_2 : v_1 : v_2 \mid u_1, u_2, v_1, v_2 \geq 0\}$$

still admits $D_{2\cdot 4}$-symmetry. The surface $S_1 \subset \mathbf{P}^3_{\mathbf{R}}$ is singular only along two lines $u_1 = u_2 = 0$ and $v_1 = v_2 = 0$, through each line two leaves pass.

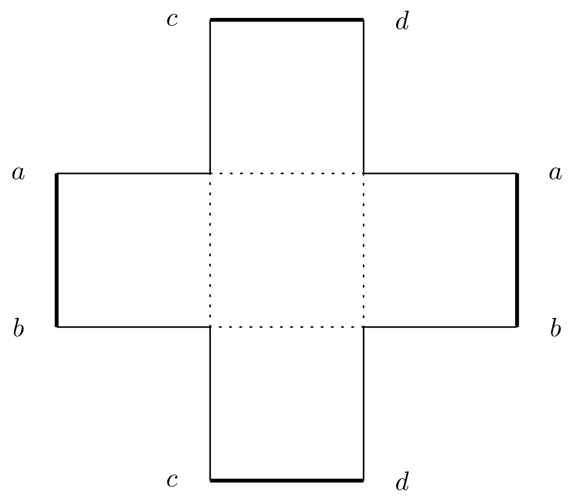

Figure 2: A combinatorial model of a quarter part

6.2.2 Models of a quarter part and the whole surface

A topological model of the quarter part $S_1/4$ is shown in Figure 2: cut out the cross, glue two sides $\overline{ab}$ (representing half of the line $u_1 = u_2 = 0$) above the paper, and glue two sides $\overline{cd}$ (representing half of the line $v_1 = v_2 = 0$) below the paper. The remaining sides $\overline{ac}, \overline{da}, \overline{bd}, \overline{cb}$ represents halves of the lines

$$u_1 = v_1 = 0, \quad u_2 = v_1 = 0, \quad u_2 = v_2 = 0, \quad u_1 = v_2 = 0,$$

respectively. The center of the cross is a saddle point. A picture drawn by Maple is shown in Figure 3.

To feel whole S_1, regard $v_1 = 0$ is the horizontal plane, $v_2 = 0$ is the plane at infinity, so the line $v_1 = v_2 = 0$ is the line at infinity on the horizontal plane, the line $u_1 = u_2 = 0$ is a vertical line. Then arrange four copies of $S_1/4$ in the even octants. A Maple picture Figure 4 shows the surface around the origin $u_1 = u_2 = v_1 = 0$.

6.2.3 Section with the planes $v_1 =$ constant ($v_2 = 1$)

Recall a curve in the real (x, y)-plane defined by

$$(x^2 + y^2)^2 - kxy = 0$$

is called a **lemniscate**; it has crossing at the origin, and has axes $x \pm y = 0$. Set $v_2 = 1$, the intersection of the surface S_1 and the plane $v_1 =$ constant is a lemniscate. In this sense, this surface should be called a *double lemniscate surface*, if it has no name yet. See Figure 5.

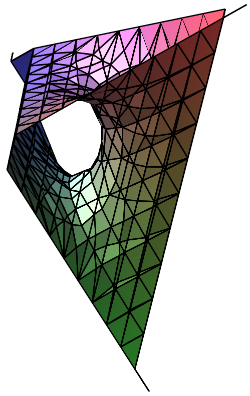

Figure 3: The surface in the quadrant $u_1, u_2, v_1, v_2 \geq 0$

6.2.4 Relation between the original, affine and sphere Schwarz images

Let $\tilde{X}$ be the monodromy cover (in this case, just the double cover) of $X = \mathbf{C} - \{0, 1\}$. Then the composed map (the original Schwarz map)

$$\begin{array}{ccccccc} \tilde{X} & \to & (\mathbf{C}^2, 0) & \to & \mathbf{P}^3_{\mathbf{R}} & \to & \mathbf{P}^1_{\mathbf{C}} \\ z & \longmapsto & (u, v) & \longmapsto & u_1 : u_2 : v_1 : v_2 & \longmapsto & z = u/v \end{array}$$

is a one-to-one map from $\tilde{X}$ onto $Z = \mathbf{P}^1_C - \{0, \pm 1, \infty\}$. On the other hand, the projection $pH : S_1 \to \mathbf{P}^1_{\mathbf{C}}$ is generically two-to-one. This somewhat contradictional fact can be explained as follows: For $z \in Z$, there are two points in S_1 which project to z. If we write one of them as (u, v), then the other is (iu, iv). Only one of them comes from the affine Schwarz curve through the identification $(\mathbf{R}^4, 0) = (\mathbf{C}^2, 0)$; that is, for one of them, no real multiple of it is on the affine Schwarz curve.

This is what Proposition 3 says: the part S_1' of S_1 coming from the affine Schwarz curve is given by $(U_1V_1 - U_2V_2)(U_3 - V_3) > 0$, when $n = 1$ it is

$$(u_2v_1 + u_1v_1 + u_1v_2 - u_2v_2)(u_1v_1 - u_2v_1 - u_2v_2 - u_1v_2)(u_1^2 - u_2^2 - v_1^2 + v_2^2) > 0.$$

6.2.5 Image of a Schwarz triangle

As a Schwarz triangle, choose the upper half part of the unit disc in the complex z-plane; the three vertices are $\{\bar{1}, 0, 1\}$ $(\bar{1} = -1)$.

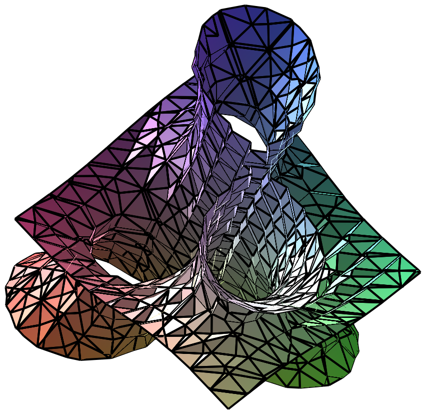

Figure 4: S_1 around the origin $u_1 = u_2 = v_1 = 0$

Proposition 4 *The corresponding part of the sphere Schwarz surface S_1 is given as:*

side	$(0,1)$	$\longrightarrow$	segment $u_1 = v_1 = 0$	$0:0:0:1 \to 0:1:0:1$
vertex	$\{1\}$	$\longrightarrow$	Hopf fiber $u_1 = v_1, u_2 = v_2$	$0:1:0:1 \to 1:1:1:1$
arc	$1 \to \bar{1}$	$\longrightarrow$	segment $u_1 = v_2, u_2 = v_1$	$1:1:1:1 \to \bar{1}:1:1:\bar{1}$
vertex	$\{\bar{1}\}$	$\longrightarrow$	Hopf fiber $u_1 = -v_1, u_2 = -v_2$	$\bar{1}:1:1:\bar{1} \to 0:0:0:1$
side	$(\bar{1},0)$	$\longrightarrow$	segment $u_1 = v_1 = 0$	$0:\bar{1}:0:1 \to 0:0:0:1$
vertex	$\{0\}$	$\longrightarrow$	Hopf fiber $u_1 = u_2 = 0$	$0:0:0:1 \to 0:0:0:1$

Sketch of a proof: Since

$$(u,v) = (\sqrt{z^2-1}, \sqrt{z^2-1}/z),$$

when z is real, we have only to note $|z| < 1$. On the unit circle, set $z = \cos t + i \sin t$. Then we have

$$\begin{aligned} z^2 - 1 &= \cos^2 t - \sin^2 t + 2i \sin t \cos t - 1 \\ &= 2i \sin t(\cos t + i \sin t), \\ \sqrt{z^2-1} &= (1+i)\sqrt{\sin t}(\cos t/2 + i \sin t/2), \\ u : v &\sim (1+i)(\cos t/2 + i \sin t/2) : (1+i)(\cos t/2 - i \sin t/2). \end{aligned}$$

The image of the origin $z = 0$ is not a point but a hemi-circle. When z comes near to the origin from above, setting $z = e^{i\theta}s$ $(0 \le \theta \le \pi)$, and let $s \to 0$, and we have

$$u : v \sim s\sqrt{z^2-1} : \sqrt{z^2-1}e^{-i\theta} \longrightarrow 0 : ie^{-i\theta} = 0 : 0 : \sin\theta : \cos\theta.$$

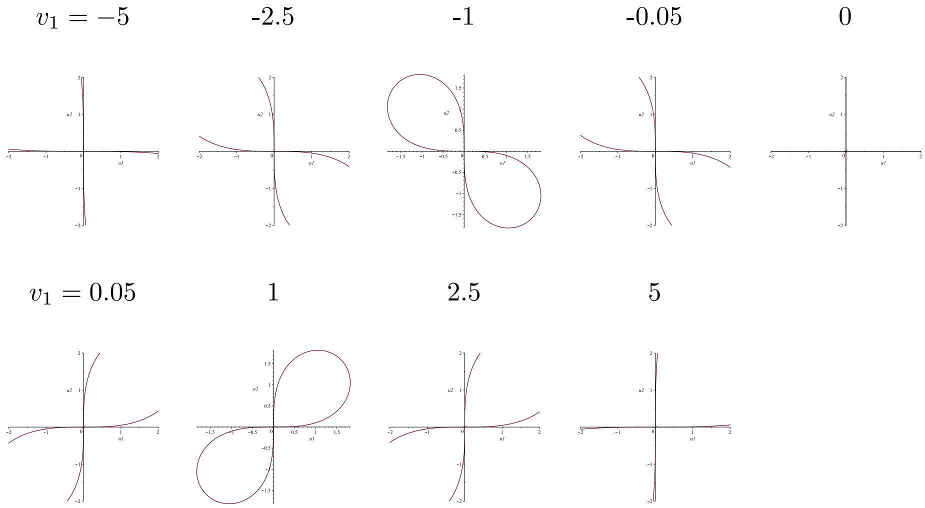

Figure 5: $n = 1$ Sections $v_1 =$ constant

6.3 Intersection with the torus T_r, and the singular locus of S_n

Parameterize the torus T_r by setting

$$u_1 = r\cos s, \quad u_2 = r\sin s, \qquad v_1 = \cos t, \quad v_2 = \sin t.$$

Since

$$(u_1 + iu_2)^{n+1} = r^{n+1}\cos(n+1)s + ir^{n+1}\sin(n+1)s, \dots,$$

and so

$$U_1 = r^{n+1}\cos(n+1)s, \quad U_2 = r^{n+1}\sin(n+1)s, \dots,$$

we have

$$\begin{aligned} R_n &= \begin{vmatrix} \cos(s+t) & r^{2n}\cos(2ns) - \cos(2nt) \\ \sin(s+t) & r^{2n}\sin(2ns) - \sin(2nt) \end{vmatrix} \\ &= r^{n+1}\left\{r^{2n}\sin\left(2ns - (n+1)(s+t)\right) - \sin\left(2nt - (n+1)(s+t)\right)\right\} \\ &= r^{n+1}\left\{r^{2n}\sin\left((n-1)s - (n+1)t\right) - \sin\left((n-1)t - (n+1)s\right)\right\}. \end{aligned}$$

When $r = 0$, we have $R_n = 0$ if and only if $(n-1)t - (n+1)s = \pi k$ $(k \in \mathbf{Z})$, that is,

$$\left.\begin{array}{ll} s = \dfrac{\pi k}{2} & n = 1, \\ t = \dfrac{n+1}{n-1}s + \dfrac{k}{n-1}\pi & n \geq 2 \end{array}\right. \quad k \in \mathbf{Z}.$$

This implies that on the torus T_r, where r is very small, $S_n \cap T_r$ can be approximated by

$$\begin{array}{ll} \text{four curves of type } \left(\frac{n+1}{2}, \frac{n-1}{2}\right) & n: \text{ odd}, \\ \text{two curves of type } (n+1, n-1) & n: \text{ even}. \end{array}$$

When $r = 1$, $R_n = 0$ if and only if

$$\begin{aligned} (n-1)s - (n+1)t &= (n-1)t - (n+1)s + 2\pi k, \quad \text{or} \\ (n-1)s - (n+1)t &= -(n-1)t + (n+1)s + \pi + 2\pi k, \end{aligned}$$

equivalently

$$t = s + k\pi/n \quad \text{or} \quad t = -s + \pi k + \pi/2.$$

If $0 < r \neq 1$, then $S_n \cap T_r$ consists of zigzag non-singular curves (see Figures 6, 7, 8). In fact, differentiating

$$R' = R_n/r^{n+1} = r^{2n} \sin(n_- s - n_+ t) - \sin(n_- t - n_+ s),$$

where $n_- = n - 1, n_+ = n + 1$, we have

$$\begin{aligned} R'_s &= r^{2n} n_- \cos(n_- s - n_+ t) + n_+ \cos(n_- t - n_+ s), \\ R'_t &= -r^{2n} n_+ \cos(n_- s - n_+ t) - n_- \cos(n_- t - n_+ s). \end{aligned}$$

Since $n_-^2 - n_+^2$ never vanishes, $R'_s = R'_t = 0$ implies

$$\cos(n_- s - n_+ t) = \cos(n_- t - n_+ s) = 0,$$

and so $R' = r^{2n} - 1$. Thus if $r \neq 1$, the curve $S_n \cap T_r$ is non-singular.

This also implies that the crossing points on $S_n \cap T_1$ are saddles. Therefore we get

Theorem 2 *the surface S_n has singularities only along the two lines $u_1 = u_2 = 0$ and $v_1 = v_2 = 0$.*

Several pictures of the intersections $S_n \cap T_r$ are shown in Figures 6 - 8.

$r = 0.5$ 0.9 1 1.1 1.5

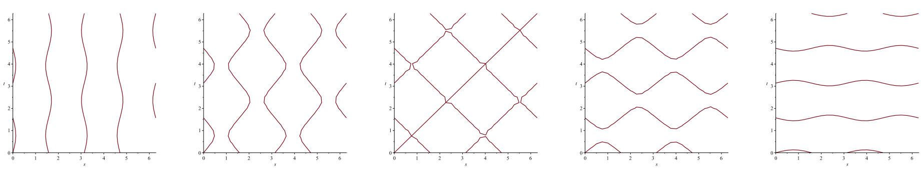

Figure 6: $n = 1$ Intersection with the tori T_r : $0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi$

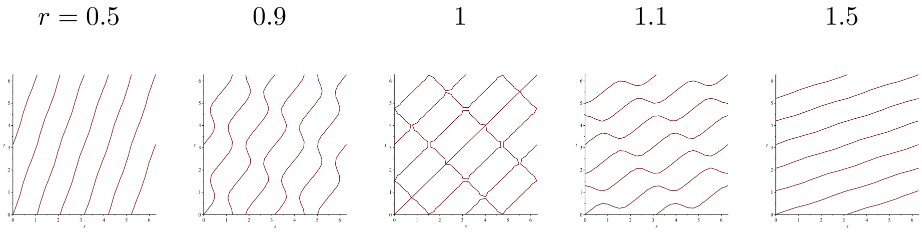

Figure 7: $n = 2$ Intersection with the tori $T_r : \ 0 \le s \le 2\pi, 0 \le t \le 2\pi$

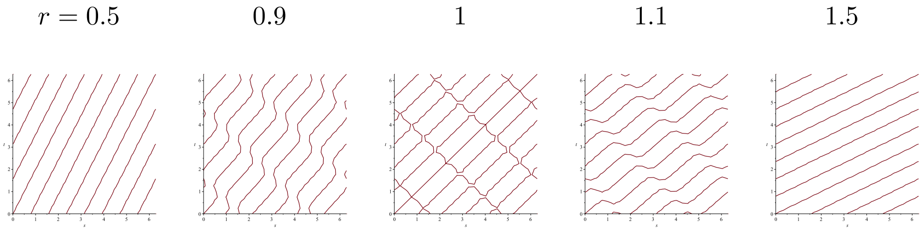

Figure 8: $n = 3$ Intersection with the tori $T_r : \ 0 \le s \le 2\pi, 0 \le t \le 2\pi$

6.4 Section with the planes $v_1 =$ constant $(v_2 = 1)$

We study the surface S_n around the singular line $u_1 = u_2 = 0$.

Proposition 5 *Set* $v_2 = 1$.

- *The intersection of* S_n *and the plane* $v_1 = 0$*: through the origin* $u_1 = u_2 = 0$, $n + 1$ *curves pass, their tangents are given by* $U_1(u_1, u_2) = 0$ $(n$ *even*$)$, $U_2(u_1, u_2) = 0$ $(n$ *odd*$)$.

- *The intersection of* S_n *and the plane* $v_1 = M$*: When* M *tends to* $\pm\infty$*, it tends to the union of* $n + 1$ *lines* $U_2(u_1, u_2) = 0$.

Proof: Recall the equation of the sphere Schwarz surface S_n:

$$R_n = (U_1V_1 - U_2V_2)(U_4 - V_4) - (U_1V_2 + U_2V_1)(U_3 - V_3),$$

where $u^{n+1} = U_1 + iU_2, u^{2n} = U_3 + iU_4, v^{n+1} = \cdots$. Since, when n is even or odd,

$$V_2|_{v_1=0,v_2=1} = V_4|_{v_1=0,v_2=1} = 0, \quad V_1|_{v_1=0,v_2=1} = V_3|_{v_1=0,v_2=1} = 0,$$

respectively, the least degree term of $R_n|_{v_1=0,v_2=1}$ is a constant multiple of U_1 or U_2, respectively.
Set $v_2 = 1$. Then V_1V_3 includes the maximal degree term v_1^{3n+1} in v_1. Thus the principal term of R_n when v_1 tends to infinity is a constant multiple of U_2.

6.4.1 $n = 2$

Set $v_2 = 1$. We show sections of the surface S_2 with the plane $v_1 =$ constant in the (u_1, u_2)-plane (Figure 9). In the picture $v_1 = 0$, we name the six branches: 1 to the vertical one to the top, and number the others $2, 3, 4, 5, 6$ anti-clockwise. Six branches intersects always transversely, so the order does not change. When two branches j and $j + 1$ form a loop, we write $(j, j + 1)$; for example, in cane $v_1 = 0$ we express as 1(23)4(56), and between $v_1 = 0.4$ and 0.45 the very moment when the curve 1 touches the leaf (56), and the curve 4 touches the leaf (23), we express (234)(561). Then the deformation of the sections when v_1 changes from 0 to ∞ can be described as

$$1(23)4(56) \to (234)(561) \to 2(34)5(61) \to (345)(612) \to (12)3(45)6 \to 123456,$$

and that when v_1 changes from 0 to $-\infty$ can be described as

$$1(23)4(56) \to (123)(456) \to (12)3(45)6 \to (345)(612) \to 2(34)5(61) \to 123456.$$

We abbreviate these, without loosing information, as

$$(23)(56) \longrightarrow (34)(61) \longrightarrow (45)(12) \longrightarrow 123456.$$

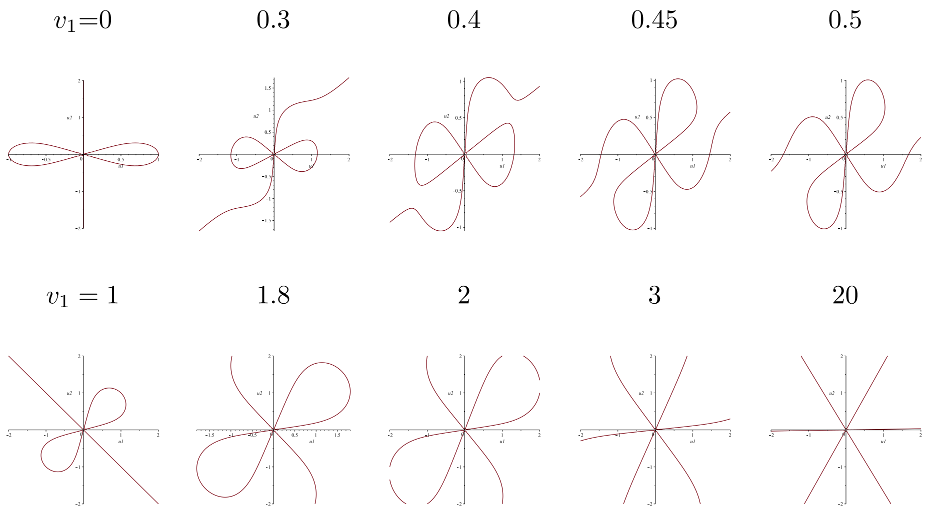

Figure 9: $n = 2$ Sections $v_1 =$ constant

6.4.2 $n \geq 3$

As we did in the previous section when $n = 2$, we describe the deformation of $n + 2$ branches when v_1 changes from o to ∞. When $n = 3$, there are eight rays, and $v_1 = 0$ is the very moment that changes from (23)(67) to (34)(78), so write (234)(678). Then, we have

$$(234)(678) \longrightarrow (34)(78) \longrightarrow (45)(81) \longrightarrow (56)(12) \longrightarrow 12345678.$$

When $n = 4$, name the ten rays as $1, \dots, 9, j$:

$$(34)(89) \to (45)(9j) \to (56)(j1) \to (67)(12) \to (78)(23) \to 123456789j$$

When $n = 5$, name the twelve rays as $1, \dots, 9, j, q, k$:

$$(345)(9jq) \to (45)(jq) \to (56)(qk) \to (67)(k1) \to (78)(89) \to (89)(23) \to 123456789jqk.$$

7 Appendix: cubic surfaces coming from quadratic curves

Let a quadratic curve C is defined by the sum F of a quadratic form F_2 and a linear form F_1. Generically, the roots of F_2 and that of F_1 are different; so we assume F is of the form

$$uv - a(u - v), \quad a \in \mathbf{C}.$$

The projection S is a cubic surface defined by

$$R = (y_1y_3 - y_2y_4)(a_1(y_2 - y_4) + a_2(y_1 - y_3)) - (y_2y_3 + y_1y_4)(a_1(y_1 - y_3) - a_2(y_2 - y_4)),$$

where we set $u = y_1 + iy_2,\ v = y_3 + iy_4,\ a = a_1 + ia_2$. Though this real cubic surface has no singularity at the real valued points, it is singular at four points

$$0 : 0 : 1 : \pm i, \quad 1 : \pm i : 0 : 0.$$

On the surface there are five real lines:

- $y_1 = y_2 = 0, \quad y_3 = y_4 = 0,$
- $y_1 - y_3 = y_2 - y_4 = 0, \quad a_1y_2 - a_2y_1 = a_1y_4 - a_2y_3 = 0,$
- $y_3 = \{(a_2^2 - a_1^2)y_1 - 2a_1a_2y_2\}/|a|^2,\ y_4 = \{-2a_1a_2y_1 + (a_1^2 - a_2^2)y_2\}/|a|^2,$

and the four imaginary ones:

$$y_1 = \pm iy_2,\ y_3 = \pm iy_4.$$

By the way, Cayley's nodal cubic is

$$z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 = 0;$$

it has four nodal singular points at $0 : 0 : 0 : 1$ and its permutations. When $a_1 = 1, a_2 = 0$,

$$R = (y_3^2 + y_4^2)y_2 - (y_1^2 + y_2^2)y_4.$$

So, this surface is isomorphic to the Cayley's nodal cubic if we admit an imaginary transformation: $z_1 = y_2 + iy_1, z_2 = y_2 - iy_1, z_3 = -y_4 + iy_3, z_4 = -y_4 - iy_3$.

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