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Twisted cohomology of a punctured Riemann surface

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Abstract. We investigate the structure of, especially non-vanishing, twisted cohomology groups with locally constant sheaf coefficients of a punctured Riemann surface with the aid of the theory of Deligne [5].

§0 Introduction

The hypergeometric function of one variable has an integral representation on a one-dimensional complex torus invented by Wirtinger [14] (see also [11]). This integral representation is understood as the pairing of a twisted homology class and a twisted cohomology class on a one-dimensional complex torus minus four points. There are several directions to obtain generalizations or analogues of this integral representation. One direction is to study twisted cohomology of a one-dimensional complex torus minus n points (where $n \geq 2$), which we discussed in [8]. Another direction is to study twisted cohomology of an abelian variety minus several theta divisors. In [13] we investigated the structure of twisted cohomology of an abelian surface minus theta divisors with normal crossings.

In this paper we treat the third direction. Namely, we study here the structure of twisted cohomology of a compact Riemann surface of genus $g(\geq 1)$ minus n points with the aid of the theory of Deligne [5]. Let X be a compact Riemann surface of genus g, and $p_1, \ldots, p_n$ be $n(\geq 2)$ distinct points on X. We set $X^* = X - \{p_1, \ldots, p_n\}$. A complex one-dimensional representation of the fundamental group of X^* determines a locally constant sheaf $\mathcal{L}$ on X^*, which is the coefficient sheaf of our twisted cohomology of X^*. Thanks to Deligne's theory [5], the twisted cohomology of X^* relates to hypercohomology of logarithmic Čech-de Rham complex (§1). To study the structure of the non-vanishing cohomology group $H^1(X^*, \mathcal{L})$, we calculate the spectral sequence associated to the hypercohomology (§2). To investigate its structure further, under the assumption $n > \max\{1, 2g-2\}$, we consider a short exact sequence of sheaves (cf. Proof of

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Lemma 3.1) introduced by Deligne [5], and we arrive at the main theorem (Theorem 4.1), where we give a basis of $H^1(X^*, \mathcal{L})$ as a set of meromorphic 1-forms on X. In [6] Ito gives a basis of $H^1(X^*, \mathcal{L})$ as a set of Čech cocyles. Finally in §5 we treat the genus one case, where a corollary to Theorem 4.1, equivalent to Theorem 2.7 in [8], is given. Finally we add that this paper is a generalized version of a previous paper [12] where only the genus one case is treated.

§1 Cohomology with coefficients in $\mathcal{L}$

Let X be a compact Riemann surface of genus $g(\geq 1)$. Let $p_1, \ldots, p_n$ be $n(\geq 2)$ distinct points on X. Let $m_1, \ldots, m_n$ be elements of $\boldsymbol{C} - \boldsymbol{Z}$ such that $\sum_{k=1}^{n} m_k = 0$. We set $X^* = X - \{p_1, \ldots, p_n\}$. Then it is easy to see that the Euler number $\chi(X^*)$ of X^* is $\chi(X^*) = 2 - 2g - n$. Let p_0 be a fixed base point of X^*. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be $2g$ simple closed curves on X with base point p_0 which generate the fundamental group $\pi_1(X, p_0)$ and correspond to a canonical homology basis of $H_1(X, \boldsymbol{Z})$ (cf. [2]). We have the single relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\ldots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1$ in $\pi_1(X, p_0)$. Without loss of generality we may assume that the n points $p_1, \ldots, p_n$ are contained in the interior of the $4g$-gon edged by $4g$ sides $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1}$. Let γ_k $(k = 1, \ldots, n)$ be a small circle about p_k with anti-clockwise direction such that the n circles $\gamma_1, \ldots, \gamma_n$ are mutually disjoint, and are contained in the interior of the $4g$-gon. The $2g+n$ closed curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1 \ldots, \gamma_n$ generate the fundamental group $\pi_1(X^*, p_0)$, and satisfy the single relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\ldots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = \gamma_1\ldots\gamma_n$ in $\pi_1(X^*, p_0)$. Let ρ be a group homomorphism of $\pi_1(X^*, p_0)$ into the multiplicative group $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ such that $\rho(\gamma_k) = e^{2\pi i m_k}$. Then, as is well-known (e.g. [2], see also [7]), there exists a multivalued meromorphic multiplicative function $T(u)$ on X^* such that for any $\gamma \in \pi_1(X^*, p_0)$ the relation $T^\gamma(u) = \rho(\gamma)T(u)$ holds where T^γ denotes the analytic continuation of T along γ. Let $\omega_{k,l}$ $(k \neq l)$ be a 1-form holomorphic on $X - \{p_k, p_l\}$ and having poles of order 1 at p_k and p_l with residues $+1$ and -1 respectively. Then it is easy to see that there exists a multivalued holomorphic multiplicative function $T_1(u)$ on X^*, unique up to constants, such that the relation $T_1^{\gamma_k}(u) = \rho(\gamma_k)T_1(u)$ holds for $k = 1, \ldots, n$ and the equation $d\log T_1 = \sum_{k=1}^{n-1}(m_1 + \cdots + m_k)\omega_{k,k+1}$ is satisfied. Then $T_2(u) = T(u)/T_1(u)$ is a multivalued meromorphic multiplicative function on X such that $T_2^{\gamma_k}(u) = T_2(u)$ holds for $k = 1, \ldots, n$. Let $\mathcal{L}$ be the locally constant sheaf on X^* generated by branches of the multivalued function $T(u)^{-1}$. Let P be the holomorphic line bundle on X having $T_2(u)^{-1}$ as a meromorphic global section. We have $c_1(P) = 0$. Let L (resp. L_1) be the holomorphic line bundle on X^* defined by the 1-cocycle determined by the branches of $T(u)^{-1}$ (resp. $T_1(u)^{-1}$). Then we have $L = L_1 \otimes P|X^*$, where $P|X^*$ denotes the restriction of P to X^*. Let $\mathcal{O}_{X^*}(L)$ be the sheaf of modules over the structure sheaf $\mathcal{O}_{X^*}$ generated by local sections of L. Then we have a natural sheaf isomorphism $\mathcal{O}_{X^*}(L) \cong \mathcal{O}_{X^*} \otimes_{\boldsymbol{C}} \mathcal{L}$, which is given by the correspondence $\Gamma(U, \mathcal{O}_{X^*}) \ni \varphi \longleftrightarrow \varphi \otimes h \in \Gamma(U, \mathcal{O}_{X^*} \otimes \mathcal{L})$ for any sufficiently small open set $U \subset X^*$ and a branch h of $T(u)^{-1}$ over U. Let us consider the short exact sequence of sheaves on X^*:

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{O}_{X^*} \xrightarrow{d} \Omega^1_{X^*} \longrightarrow 0,$$

where $\Omega^1_{X^*}$ denotes the sheaf of holomorphic 1-forms on X^*. Tensoring $\mathcal{L}$ from the right on this sequence, we have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{X^*} \otimes_{\boldsymbol{C}} \mathcal{L} \xrightarrow{d\otimes 1} \Omega^1_{X^*} \otimes_{\boldsymbol{C}} \mathcal{L} \longrightarrow 0. \tag{1}$$

Here the operator $d \otimes 1$ is a canonical connection in the sense of [5, I, Prop. 2.16]. We set $\Omega^1_{X^*}(L) = \Omega^1_{X^*} \otimes_{\mathcal{O}_{X^*}} \mathcal{O}_{X^*}(L)$. Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}_{X^*} \otimes_{\boldsymbol{C}} \mathcal{L} & \xrightarrow{d\otimes 1} & \Omega^1_{X^*} \otimes_{\boldsymbol{C}} \mathcal{L} \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \mathcal{O}_{X^*}(L) & \xrightarrow[d]{} & \Omega^1_{X^*}(L) \end{array} \qquad \begin{array}{ccc} \varphi \otimes h & \overset{d\otimes 1}{\longmapsto} & d\varphi \otimes h \\ \updownarrow & & \updownarrow \\ \varphi & \underset{d}{\longmapsto} & d\varphi \end{array}$$

Therefore (1) is equivalent to the following exact sequence:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{X^*}(L) \xrightarrow{d} \Omega^1_{X^*}(L) \longrightarrow 0. \tag{2}$$

For an open set $U \subset X^*$ and a section $\varphi \in \Gamma(U, \Omega^p_{X^*}(L))$ (where p is 0 or 1, and $\Omega^0_{X^*} = \mathcal{O}_{X^*}$), there is a section $\psi \in \Gamma(U, \Omega^p_{X^*}(P|X^*))$ such that $\varphi = \psi \cdot T_1|U$, where $T_1|U$ is a branch of T_1 defined on U. Then we have a sheaf isomorphism

$$\Omega^p_{X^*}(P|X^*) \xrightarrow{\sim} \Omega^p_{X^*}(L) \qquad (p = 0, 1) \qquad\qquad \psi \longmapsto \varphi = \psi \cdot T_1$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_{X^*}(P|X^*) & \xrightarrow{\nabla} & \Omega^1_{X^*}(P|X^*) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \mathcal{O}_{X^*}(L) & \xrightarrow[d]{} & \Omega^1_{X^*}(L) \end{array}$$

where $\nabla\varphi = d\varphi + \varphi\, d(\log T_1)$ and $\nabla^2 = 0$. Note that $\nabla h_1 = 0$ for any branch h_1 of $T_1(u)^{-1}$. Then the exact sequence (2) is equivalent to the following:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{X^*}(P|X^*) \xrightarrow{\nabla} \Omega^1_{X^*}(P|X^*) \longrightarrow 0,$$

Since, as is well-known, $H^q(X^*, \Omega^p_{X^*}(P|X^*)) = 0$ for $p \geq 0$ and $q > 0$, it follows that $H^p(X^*, \mathcal{L}) \cong H^p_{\mathrm{DR}}(\Omega^\bullet_{X^*}(P|X^*), \nabla)$. Then we have

Lemma 1.1. $H^p(X^*, \mathcal{L}) = 0$ if $p \neq 1$. Therefore $\dim H^1(X^*, \mathcal{L}) = n + 2g - 2$.

Proof. Obviously, $H^p(X^*, \mathcal{L}) \cong H^p_{\mathrm{DR}}(\Omega^\bullet_{X^*}(P|X^*), \nabla) = 0$ if $p \geq 2$. Note that $H^0(X^*, \mathcal{L}) \cong \{f \in \Gamma(X^*, \mathcal{O}_{X^*}(P|X^*)) \mid \nabla f = 0\}$. It is also obvious that, if a section $f \in \Gamma(X^*, \mathcal{O}_{X^*}(P|X^*))$ satisfies $\nabla f = 0$, then $f = 0$.

Let $\mathcal{E}^p_{X^*}$ be the sheaf of complex-valued C^∞ forms of total degree p on X^*. Replacement of the sheaf $\Omega^p_{X^*}$ by $\mathcal{E}^p_{X^*}$ in the above argument is valid, and gives us the following exact sequence on X^*:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}^0_{X^*}(P|X^*) \xrightarrow{\nabla} \mathcal{E}^1_{X^*}(P|X^*) \longrightarrow 0,$$

where $\mathcal{E}^p_{X^*}(P|X^*) = \mathcal{E}^p_{X^*} \otimes_{\mathcal{E}^0_{X^*}} \mathcal{E}^0_{X^*}(P|X^*)$. Let $D = \sum_{k=1}^n p_k$ be a reduced divisor on X. Let $\Omega^p_X\langle D\rangle$ be the sheaf of p-forms over X with logarithmic pole along D ([10]). Since X is one-dimensional, we have by definition $\Omega^0_X\langle D\rangle = \mathcal{O}_X$ and $\Omega^1_X\langle D\rangle = \Omega^1_X(D)$. We have inclusion of sheaves over X: $\Omega^p_X\langle D\rangle \subset j_*\Omega^p_{X^*} \subset j_*\mathcal{E}^p_{X^*}$, where j denotes a natural inclusion mapping of X^* into X. We set $\Omega^p_X\langle D\rangle(P) = \Omega^p_X\langle D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(P)$. Since $j_*\Omega^p_{X^*} \otimes_{\mathcal{O}_X} \mathcal{O}_X(P) \cong j_*\Omega^p_{X^*}(P|X^*)$ and $j_*\mathcal{E}^p_{X^*} \otimes_{\mathcal{E}^0_X} \mathcal{E}^0_X(P) \cong j_*\mathcal{E}^p_{X^*}(P|X^*)$, we have inclusion of sheaves over X: $\Omega^p_X\langle D\rangle(P) \subset j_*\Omega^p_{X^*}(P|X^*) \subset j_*\mathcal{E}^p_{X^*}(P|X^*)$. Let us consider a complex of sheaves of logarithmic forms:

$$(\Omega^\bullet_X\langle D\rangle(P), \nabla) \quad : \quad \mathcal{O}_X(P) \xrightarrow{\nabla} \Omega^1_X(D)(P) \longrightarrow 0.$$

Then the next lemma follows immediately from [5, II, Cor. 3.14] (for an elementary, analytical, direct proof of it, see [13]).

Lemma 1.2. Two complexes of sheaves over X, $(\Omega^\bullet_X\langle D\rangle(P), \nabla)$ and $(j_*\mathcal{E}^\bullet_{X^*}(P|X^*), \nabla)$, are quasi-isomorphic to each other.

By taking global section functor we have

Corollary 1.3. $H^p(X^*, \mathcal{L}) \cong \mathbf{H}^p(X, \Omega^\bullet_X\langle D\rangle(P), \nabla)$, where $\mathbf{H}$ denotes a hypercohomology.

§2 Logarithmic Čech-de Rham complex and spectral sequence

The rest of this paper is devoted to investigating the structure of the non-vanishing cohomology group $H^1(X^*, \mathcal{L}) \cong \mathbf{H}^1(X, \Omega^\bullet_X\langle D\rangle(P), \nabla)$. To this end we study the spectral sequence associated to the hypercohomology. Note that $\Omega^0_X\langle D\rangle = \mathcal{O}_X$ and $\Omega^1_X\langle D\rangle = \Omega^1_X(D)$. Let $\mathcal{U} = \{U_i\}_i$ be an open covering of X. Let us consider the following double complex:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^2(\mathcal{U}, \mathcal{O}_X(P)) & \xrightarrow{\nabla} & C^2(\mathcal{U}, \Omega^1_X(D)(P)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^1(\mathcal{U}, \mathcal{O}_X(P)) & \xrightarrow{\nabla} & C^1(\mathcal{U}, \Omega^1_X(D)(P)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^0(\mathcal{U}, \mathcal{O}_X(P)) & \xrightarrow{\nabla} & C^0(\mathcal{U}, \Omega^1_X(D)(P)) & \longrightarrow & 0,
\end{array}$$

where δ denotes the coboundary operator satisfying $\delta\nabla = \nabla\delta$. The total differentiation operator Δ is defined to be $\Delta = \delta + (-1)^p\nabla$ on $C^p(\mathcal{U}, \mathcal{O}_X(P))$. For $p \geq 0$ we set $K^p = C^p(\mathcal{U}, \mathcal{O}_X(P)) \oplus C^{p-1}(\mathcal{U}, \Omega^1_X(D)(P))$, where $C^{-1} = 0$. We set $K(\mathcal{U}) = \oplus_{p=0}^\infty K^p$. Since $\Delta(K^p) \subset K^{p+1}$ and $\Delta^2 = 0$, the pair $(K(\mathcal{U}), \Delta)$ is a complex. By the definition of hypercohomology (e.g. [3]), we have

$$\mathbf{H}^p = \mathbf{H}^p(X, \Omega^\bullet_X\langle D\rangle(P), \nabla) = \varinjlim_{\mathcal{U}} H^p(K(\mathcal{U}), \Delta),$$

where $H^p(K(\mathcal{U}), \Delta) = \mathrm{Ker}[K^p \xrightarrow{\Delta} K^{p+1}]/\mathrm{Im}[K^{p-1} \xrightarrow{\Delta} K^p]$ and $\varinjlim_{\mathcal{U}}$ means the inductive limit taken with respect to refinements of open coverings of X. By setting $K_0 = K(\mathcal{U})$ and $K_1 = \oplus_{p=0}^\infty C^p(\mathcal{U}, \Omega^1_X(D)(P))$, we introduce a filtration of $K(\mathcal{U})$: $K(\mathcal{U}) = K_0 \supset K_1 \supset 0$. The spectral sequence $E_r(\mathcal{U})$ associated to the filtered modul $K(\mathcal{U})$ is given as follows: $E_1^p(\mathcal{U}) = H(K_p/K_{p+1}) = H_\delta(\oplus_{q=0}^\infty C^q(\mathcal{U}, \Omega^p_X\langle D\rangle(P)) = \oplus_{q=0}^\infty H^q(\mathcal{U}, \Omega^p_X\langle D\rangle(P))$ and the other $E_r^p(\mathcal{U})$ $(r \geq 2)$ are given inductively. The limit $E_r = \varinjlim_{\mathcal{U}} E_r(\mathcal{U})$ is also a spectral sequence and abuts to the hypercohomology $\mathbf{H}^p$. We have $E_1^p = \oplus_{q=0}^\infty H^q(X, \Omega^p_X\langle D\rangle(P))$ and $E_1^{pq} = H^q(X, \Omega^p_X\langle D\rangle(P))$. Note that $E_1^{pq} = 0 \quad (p > 1)$.

Lemma 2.1. Assume that the line bundle P has the first Chern class $c_1(P) = 0$, but is not holomorphically trivial. Then $E_1 = E_\infty$. Namely, we have $E_\infty^{01} = H^1(X, \mathcal{O}_X(P))$, $E_\infty^{10} = H^0(X, \Omega^1_X(D)(P))$, and $E_\infty^{pq} = 0$ if $(p, q) \neq (1, 0), (0, 1)$. Moreover we have $\dim E_\infty^{01} = g - 1$ and $\dim E_\infty^{10} = n + g - 1$.

Proof. Obviously $E_1^{0q} = H^q(X, \mathcal{O}_X(P)) = 0$ if $q \neq 1$ (cf. [4], p.74). By Riemann-Roch theorem we have $\dim E_1^{01} = \dim H^1(X, \mathcal{O}_X(P)) = g-1$. Since the divisor D is effective, by Kodaira vanishing theorem, we have $E_1^{1q} = H^q(X, \Omega^1_X(D)(P)) = 0$ if $q \geq 1$. Note that $\Omega^1_X(D)(P) \cong \mathcal{O}(K + [D])(P)$, where K is the canonical class and $[D]$ denotes the line bundle associated to the divisor D. Since $c_1(K + [D]) = 2g - 2 + n$, it follows ([4], p.111) that $\dim E_1^{10} = \dim H^0(X, \Omega^1_X(D)(P)) = (2g - 2 + n) - (g - 1) = g - 1 + n$. Therefore $E_1 = E_\infty$.

Lemma 2.2. Assume that the line bundle P is holomorphically trivial: $P = 1$. Then $E_2 = E_\infty$. Namely, we have $E_\infty^{01} = H^1(X, \mathcal{O}_X)$, $E_\infty^{10} = H^0(X, \Omega^1_X(D))/\boldsymbol{C} \cdot \nabla(1)$, and $E_\infty^{pq} = 0$ if $(p, q) \neq (1, 0), (0, 1)$. Moreover we have $\dim E_\infty^{01} = g$ and $\dim E_\infty^{10} = n + g - 2$.

Proof. Let us consider the complex $E_1^{0q} \xrightarrow{\nabla} E_1^{1q}$. If $q = 0$, this is turned to $H^0(X, \mathcal{O}_X) \xrightarrow{\nabla} H^0(X, \Omega^1_X(D))$. We have $H^0(X, \mathcal{O}_X) = \boldsymbol{C}$. If $f \in \boldsymbol{C}$ satisfies $\nabla f = 0$, then $f = 0$. So we have $E_2^{00} = 0$. Moreover, obviously we have $E_2^{10} = H^0(X, \Omega^1_X(D))/\boldsymbol{C} \cdot \nabla(1)$. If $q = 1$, by Kodaira vanishing theorem, we have $E_1^{11} = H^1(X, \Omega^1_X(D)) = 0$. So the complex above is turned to a trivial one: $E_1^{01} = H^1(X, \mathcal{O}_X) \xrightarrow{\nabla} 0$, from which it follows that $E_2^{01} = H^1(X, \mathcal{O}_X)$ and $E_2^{11} = 0$. If $q \geq 2$, we have $E_1^{0q} = E_1^{1q} = 0$ similarly by the vanishing theorem, from which it follows that $E_2^{0q} = E_2^{1q} = 0$. Consequently we have $E_2 = E_\infty$.

§3 Deligne's resolution

From now on we assume that $n > \max\{1, 2g-2\}$. Let us investigate the vector space E_∞^{01} further. We prove the following

Lemma 3.1. We have the isomorphism

$$E_\infty^{01} = H^1(X, \mathcal{O}_X(P)) \cong \frac{H^0(X, \Omega_X^1(2D)(P))}{\nabla\, H^0(X, \mathcal{O}_X(D)(P)) + H^0(X, \Omega_X^1(D)(P))}.$$

Proof. Let us consider the resolution of the sheaf $\mathcal{O}_X(P)$ by Deligne [5, II, Prop. 3.13]:

$$0 \longrightarrow \mathcal{O}_X(P) \longrightarrow \mathcal{O}_X(D)(P) \xrightarrow{\nabla'} \frac{\Omega_X^1(2D)(P)}{\Omega_X^1(D)(P)} \xrightarrow{\nabla'} 0,$$

where ∇' is the mapping induced by ∇. Since $n > 2g-2$, it follows by Serre duality that $H^1(X, \mathcal{O}_X(D)(P)) = 0$. Therefore the short exact sequence above implies the exact sequence of cohomology groups:

$$\begin{aligned} 0 &\longrightarrow H^0(X, \mathcal{O}_X(P)) \longrightarrow H^0(X, \mathcal{O}_X(D)(P)) \\ &\xrightarrow{\nabla'} H^0\left(X, \frac{\Omega_X^1(2D)(P)}{\Omega_X^1(D)(P)}\right) \longrightarrow H^1(X, \mathcal{O}_X(P)) \longrightarrow 0, \end{aligned}$$

from which we have the isomorphism

$$H^1(X, \mathcal{O}_X(P)) \cong \frac{H^0\left(X, \dfrac{\Omega_X^1(2D)(P)}{\Omega_X^1(D)(P)}\right)}{\nabla'\, H^0(X, \mathcal{O}_X(D)(P))}. \tag{3}$$

Since $H^1(X, \Omega_X^1(D)(P)) = 0$ by Kodaira's vanishing theorem, we have

$$H^0\left(X, \frac{\Omega_X^1(2D)(P)}{\Omega_X^1(D)(P)}\right) \cong \frac{H^0(X, \Omega_X^1(2D)(P))}{H^0(X, \Omega_X^1(D)(P))}. \tag{4}$$

Moreover, by this isomorphism, we may identify the mapping

$$\nabla' : H^0(X, \mathcal{O}_X(D)(P)) \longrightarrow H^0\left(X, \frac{\Omega_X^1(2D)(P)}{\Omega_X^1(D)(P)}\right)$$

with the composition

$$H^0(X, \mathcal{O}_X(D)(P)) \xrightarrow{\nabla} H^0\left(X, \Omega_X^1(2D)(P)\right) \longrightarrow \frac{H^0(X, \Omega_X^1(2D)(P))}{H^0(X, \Omega_X^1(D)(P))}.$$

Then we have

$$\begin{aligned} \nabla' H^0(X, \mathcal{O}_X(D)(P)) &\cong \frac{\nabla H^0(X, \mathcal{O}_X(D)(P))}{\nabla H^0(X, \mathcal{O}_X(D)(P)) \cap H^0(X, \Omega_X^1(D)(P))} \\ &\cong \frac{\nabla H^0(X, \mathcal{O}_X(D)(P)) + H^0(X, \Omega_X^1(D)(P))}{H^0(X, \Omega_X^1(D)(P))}. \end{aligned} \tag{5}$$

Substitution of (4) and (5) into (3) gives us the desired formula, which proves Lemma 3.1.

§4 Basis of cohomology group

For each p the hypercohomology group $\mathbf{H}^p$ inherits a filtration from the corresponding filtered module K^p. Especially in the case where $p = 1$ we have $\mathbf{H}^1 = \mathbf{H}^1_0 \supset \mathbf{H}^1_1 \supset 0$. According to the general theory of spectral sequences we have $G_p(\mathbf{H}^1) \cong E_\infty^{p,1-p}$, that is, we have $\mathbf{H}^1_0/\mathbf{H}^1_1 = G_0(\mathbf{H}^1) = E_\infty^{01}$, $\mathbf{H}^1_1 = G_1(\mathbf{H}^1) = E_\infty^{10}$. Combining the results of Lemmas 2.1, 2.2 and 3.1, we arrive at the following theorem:

Theorem 4.1. We have the isomorphism $H^1(X^*, \mathcal{L}) \cong E_\infty^{01} \oplus E_\infty^{10}$, where,
(i) if P is topologically trivial but not holomorphically, then

$$E_\infty^{10} = H^0(X, \Omega_X^1(D)(P)),$$
$$E_\infty^{01} = \frac{H^0(X, \Omega_X^1(2D)(P))}{\nabla H^0(X, \mathcal{O}_X(D)(P)) + H^0(X, \Omega_X^1(D)(P))};$$

(ii) if P is holomorphically trivial, then

$$E_\infty^{10} = H^0(X, \Omega_X^1(D))/\boldsymbol{C} \cdot \nabla(1),$$
$$E_\infty^{01} = \frac{H^0(X, \Omega_X^1(2D))}{\nabla H^0(X, \mathcal{O}_X(D)) + H^0(X, \Omega_X^1(D))}.$$

According to this theorem, we can choose a basis of $H^1(X^*, \mathcal{L})$ as follows:

Example 1. Assume that P is topologically trivial but not holomorphically. We have $H^0(X, \Omega_X^1(P)) \subset H^0(X, \Omega_X^1(D)(P))$. By Riemann-Roch theorem and Serre duality we have $\dim H^0(X, \Omega_X^1(P)) = g-1$. Let $\omega_1, \ldots, \omega_{g-1}$ be 1-forms holomorphic on X with values in P which form a basis of $H^0(X, \Omega_X^1(P))$. We denote by σ_k $(1 \le k \le n)$ a 1-form holomorphic on $X - \{p_k\}$ with values in P which has a unique pole of order 1 at p_k. Then $n+g-1$ 1-forms $\omega_1, \ldots, \omega_{g-1}, \sigma_1, \ldots, \sigma_n$ form a basis of E_∞^{10}. Let τ_k $(1 \le k \le n)$ be a 1-form holomorphic on $X - \{p_k\}$ with values in P which has a unique pole of order 2 at p_k. Since $\dim E_\infty^{01} = g-1 < n$, $g-1$ 1-forms $\tau_1, \ldots, \tau_{g-1}$ which are not in $\nabla H^0(X, \mathcal{O}_X(D)(P)) + H^0(X, \Omega_X^1(D)(P))$ define a basis of E_∞^{01}. Therefore $n+2g-2$ 1-forms $\omega_1, \ldots, \omega_{g-1}, \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_{g-1}$ define a basis of $H^1(X^*, \mathcal{L})$.

Example 2. Assume that P is holomorphically trivial. We have $H^0(X, \Omega_X^1) \subset H^0(X, \Omega_X^1(D))$ and $\dim H^0(X, \Omega_X^1) = g$. Let $\omega_1, \ldots, \omega_g$ be 1-forms holomorphic on X which form a basis of $H^0(X, \Omega_X^1)$. Let $\omega_{k,l}$ be a 1-form holomorphic on $X - \{p_k, p_l\}$ $(k \ne l)$ and having poles of order 1 at p_k and p_l with residues $+1$ and -1 respectively. Then $n+g-1$ 1-forms $\omega_1, \ldots, \omega_g, \omega_{12}, \omega_{23}, \omega_{34}, \ldots, \omega_{n-1,n}$ form a basis of $H^0(X, \Omega_X^1(D))$. Since $\nabla(1) = \sum_{k=1}^{n-1}(m_1 + \cdots + m_k)\omega_{k,k+1}$, $n+g-2$ 1-forms $\omega_1, \ldots, \omega_g, \omega_{12}, \omega_{23}, \omega_{34}, \ldots, \omega_{n-2,n-1}$ define a basis of E_∞^{10}. Let τ_k

$(1 \le k \le n)$ be a 1-form holomorphic on $X - \{p_k\}$ which has a unique pole of order 2 at p_k. Since $\dim E_\infty^{01} = g \le n$, g 1-forms $\tau_1, \ldots, \tau_g$ which are not in $\nabla H^0(X, \mathcal{O}_X(D)) + H^0(X, \Omega_X^1(D))$ define a basis of E_∞^{01}. Therefore $n + 2g - 2$ 1-forms $\omega_1, \ldots, \omega_g, \omega_{12}, \omega_{23}, \omega_{34}, \ldots, \omega_{n-2,n-1}, \tau_1, \ldots, \tau_g$ define a basis of $H^1(X^*, \mathcal{L})$.

§5 Genus one case

In this section we consider the case where X is a one-dimensional complex torus: $X = \boldsymbol{C}/(\boldsymbol{Z} + \tau \boldsymbol{Z})$. Let n, n' be integers such that $2 \le n \le n'$. Let a_k, b_k $(1 \le k \le n')$ be real numbers, c_k $(1 \le k \le n)$ complex numbers but not integers, and c_k $(n+1 \le k \le n')$ non-zero integers. We assume that $\sum_{k=1}^{n} c_k = \sum_{k=n+1}^{n'} c_k = 0$. We also assume that (a_k, b_k) $(1 \le k \le n)$, regarded as elements of $\boldsymbol{R}^2/\boldsymbol{Z}^2$, are different from each other. Let t_k $(1 \le k \le n)$ be the point defined by $-a_k\tau - b_k$ on X. Then we have $t_k \ne t_l$ if $k \ne l$. We set $D = \{t_1, \ldots, t_n\}$, and $X^* = X - D$. Note that the point t_k is a unique zero of the theta function $\theta_{a_k,b_k}(u,\tau)$ on X ([9]). We set $T_1(u) = \prod_{k=1}^{n} \theta_{a_k,b_k}(u,\tau)^{c_k}$, $T_2(u) = \prod_{k=n+1}^{n'} \theta_{a_k,b_k}(u,\tau)^{c_k}$, and $T(u) = T_1(u)T_2(u)$. Then we have

$$\begin{aligned} T_1(u+1) &= e^{2\pi i(a_1c_1 + \cdots + a_nc_n)} T_1(u), \\ T_1(u+\tau) &= e^{-2\pi i(b_1c_1 + \cdots + b_nc_n)} T_1(u), \\ T_2(u+1) &= e^{2\pi i(a_{n+1}c_{n+1} + \cdots + a_{n'}c_{n'})} T_2(u), \\ T_2(u+\tau) &= e^{-2\pi i(b_{n+1}c_{n+1} + \cdots + b_{n'}c_{n'})} T_2(u). \end{aligned}$$

Let $\mathcal{L}$ be the locally constant sheaf on X^* defined by the one-dimensional representation of the fundamental group $\pi_1(X^*, *)$ by the multivalued function $T(u)^{-1}$. Let P be the holomorphic line bundle on X having $T_2(u)^{-1}$ as a meromorphic global section. We have $c_1(P) = 0$. Then Theorem 4.1 is reduced to the following

Corollary 5.1. (i) If P is not holomorphically trivial, then $H^1(X^*, \mathcal{L}) \cong H^0(X, \Omega_X^1(D)(P))$, and $H^p(X^*, \mathcal{L}) = 0$ $(p \ne 1)$. We have $\dim H^0(X, \Omega_X^1(D)(P)) = n$.
(ii) If P is holomorphically trivial (i.e., $P = 1$), then $H^1(X^*, \mathcal{L}) \cong \big(H^0(X, \Omega_X^1(D))/\boldsymbol{C} \cdot \nabla(1)\big) \oplus H^1(X, \mathcal{O}_X)$, and $H^p(X^*, \mathcal{L}) = 0$ $(p \ne 1)$, where ∇ is defined by $\nabla\varphi = d\varphi + \varphi\, d(\log T_1)$ for any differential form φ. We have $\dim H^0(X, \Omega_X^1(D))/\boldsymbol{C} \cdot \nabla(1) = n - 1$ and $\dim H^1(X, \mathcal{O}_X) = 1$.

Remark. This corollary is already proved in [8] by applying Mittag-Leffler theorem in the complex analytical theory on Riemann surfaces.

Let $\wp(u)$ be a meromorphic function with a unique pole at $t_1 \in D$ of order 2. Then the non-zero cohomology class $[\wp(u)du]$ in $H^1(X, \mathcal{O}_X)$ defined by the 1-form $\wp(u)du \in \Gamma(X^*, \Omega_{X^*}^1)$ forms a basis of $H^1(X, \mathcal{O}_X)$. Therefore the cohomology classes forming a basis of $H^0(X, \Omega_X^1(D))/\boldsymbol{C} \cdot \nabla(1)$ and the cohomology class $[\![\wp(u)du]\!]$ in $H^1(X^*, \mathcal{L})$ defined by $\wp(u)du$ form a basis of $H^1(X^*, \mathcal{L})$.

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