

Symmetries of the Darboux Equation

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(Received June 15, 2016)

(Accepted May 16, 2017)

Abstract. This paper establishes the symmetries of Darboux's equations (1882) on tori. We extend Ince's work (1940) by developing new infinite series expansions in terms of Jacobi elliptic functions around each of the four regular singular points of the Darboux equation which are located at the four half-periods of the torus. The symmetry group of the Darboux equation is given by the Coxeter group $B_4 \cong G_I \rtimes_{\Gamma} G_{II}$ where the actions of G_I correspond to the sign changes of the parameters in the solutions of the equations, which have no effective changes on the equation itself, while the actions of G_{II} permute the four half-periods of the torus, which are described by a short-exact sequence. We are able to clarify the symmetries of the equation when ordered bases of the underlying torus change. Our results show that it is much more transparent to consider the symmetries of the Darboux equation on a torus than on the Riemann sphere \mathbb{CP}^1 as was usually considered by earlier researchers such as Maier (2007). We list the symmetry tables of the Darboux equation in both the Jacobian form and Weierstrass form. A consolidated list of 192 solutions of the Darboux equation is also given. We then consider the symmetries of several classical equations (e.g. Lamé equation) all of which are special cases of the Darboux equation. Those terminating solutions of the Darboux equation generalize the classical Lamé polynomials.

1 Introduction

1.1 The Darboux equation

The symmetries of the classical Lamé equation [30] (1837)

$$\frac{d^2y}{du^2} + (h - \nu(\nu + 1)k^2 \operatorname{sn}^2 u)y = 0, \quad (1.1)$$

Mathematical Subject Classification (2010): Primary 33E10; Secondary 34M3

Key words: Darboux equation, Lamé equation

*The first and third authors are partially supported by Hong Kong Research Grant Council project no. 601111 and 16300814

was first discussed by Arscott and Reid [2, (1971)]¹ despite the long history of the equation. In this case the symmetry group is isomorphic to S_3 (see §9.1 for more detail).

We characterize the symmetries of the Darboux equation [9] (1882)²

$$\frac{d^2y}{du^2} + \left[h - \xi(\xi + 1) \operatorname{ns}^2(u, k) - \eta(\eta + 1) \operatorname{dc}^2(u, k) - \mu(\mu + 1)k^2 \operatorname{cd}^2(u, k) - \nu(\nu + 1)k^2 \operatorname{sn}^2(u, k) \right] y = 0, \quad (1.2)$$

which includes the Lamé equation as a special case, in this paper to be isomorphic to the *hyperoctahedral (Coxeter) group* B_4 , where $\operatorname{ns}(u, k)$, $\operatorname{dc}(u, k)$, $\operatorname{cd}(u, k)$, $\operatorname{sn}(u, k)$ are Jacobi elliptic functions with elliptic modulus k ($k^2 \neq 0, \pm 1$). Notice that $\operatorname{ns}^2(u, k)$, $\operatorname{dc}^2(u, k)$, $\operatorname{cd}^2(u, k)$, $\operatorname{sn}^2(u, k)$ are doubly periodic functions with periods $\omega_1 = 2K(k)$, $\omega_2 = 2iK'(k)$ and have a double pole at $u = 0, K(k), K(k) + iK'(k)$ and $iK'(k)$ respectively on the fundamental parallelogram $P(\omega_1, \omega_2) = \{s\omega_1 + t\omega_2 : s, t \in [0, 1)\}$. The Darboux equation (1.2) is defined on a torus of \mathbb{C} modulo the lattice $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, which can be specified by the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{cccccc} 0 & K(k) & K(k) + iK'(k) & iK'(k) & & \\ \xi + 1 & \eta + 1 & \mu + 1 & \nu + 1 & u; h & \\ -\xi & -\eta & -\mu & -\nu & & \end{array} \right\},$$

where the entries on the top row represent the locations of the regular singularities and the entries of the succeeding two rows under the corresponding singularities represent the two exponents of the local solutions there, and u, h are the independent variable and the accessory parameter respectively. We have obtained local series solutions about the singularities on a torus by developing an infinite series expansion theory in terms of Jacobi elliptic functions. For example, the series expansion about the singular point $u = 0$ with the exponent $\xi + 1$ is given by

$$\operatorname{sn}(u, k)^{\xi+1} \operatorname{cn}(u, k)^{\eta+1} \operatorname{dn}(u, k)^{\mu+1} \sum_{m=0}^{\infty} C_m \operatorname{sn}(u, k)^{2m},$$

where the coefficients C_m satisfy a certain three-term recursion relation and the series converges in certain region on the torus (see §7). The idea of developing such a series expansion solution goes back to Ince [26, (1940)] in a study of the well-known Lamé equation which has only one regular singular point on a torus. We have adopted this idea for the Darboux differential equation which has four elliptic coefficients and each of which contributes a regular singular point within its fundamental region. We generate a total of 192 local series solutions in terms of Jacobi elliptic functions by the symmetry group of the Darboux equation to be explored in this paper.

¹Although the discussion of geometric reasoning was absent, namely the ordered bases of the underlying torus. See §3.

²Also known as Darboux-Treibich-Verdier equation [47].

Maier ([32, Theorem 4.3], [33, Theorem 3.1]) has recently clarified that the automorphism group of the second-order Fuchsian type differential equation (FDE) in asymmetric form with $n \geq 3$ regular singular points $0, 1, \infty, a_1, \dots, a_{n-3}$ in \mathbb{CP}^1 is the Coxeter group $B_n \cong (\mathbb{Z}_2)^n \rtimes_{\Gamma} S_n$ (or $D_n \cong (\mathbb{Z}_2)^{n-1} \rtimes_{\Gamma} S_n$ if the transformation $[\infty]_-$ interchanging the exponents α, β at ∞ is excluded). While for FDE in symmetric form, the action of the part $(\mathbb{Z}_2)^n$ on the equation is trivial. When $n = 3$, a FDE in asymmetric form is the Gaussian hypergeometric equation, which has three regular singular points $\{0, 1, \infty\}$ on \mathbb{CP}^1 . Its symmetry group is well-known to be isomorphic to $S_4 \cong (\mathbb{Z}_2)^2 \rtimes_{\Gamma} S_3$ (if $[\infty]_-$ is excluded), and its 24 local solutions were worked out by Kummer (see for example, Whittaker and Watson [48, pp. 284-285]). In the case of the Heun equation [23, (1889)], which corresponds to the FDE in asymmetric form with four regular singular points $\{0, 1, a, \infty\}$, has its symmetry group isomorphic to $(\mathbb{Z}_2)^4 \rtimes_{\Gamma} S_4$ (if $[\infty]_-$ is included). Maier also provided a complete list of the 192 local solutions and the corresponding accessory parameters. We exhibit in the case of $n = 4$ that it is more natural, as far as the symmetry is concerned, to consider Fuchsian differential equations on a torus \mathbb{C}/Λ and to place the four regular singular points at the corresponding four half-periods.

The Heun equation

$$\frac{d^2 y}{dt^2} + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-a} \right) \frac{dy}{dt} + \frac{\alpha\beta t - q}{t(t-1)(t-a)} y = 0, \quad (1.3)$$

where the parameters satisfy the Fuchsian constraint $\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0$ that Maier considered has the Riemann P -scheme

$$P_{\mathbb{CP}^1} \left\{ \begin{array}{cccc} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{array} \right\} t; q.$$

Although the Heun equation is connected to the Darboux equation by a simple change of dependent and independent variables, we can replace the symmetry group $(\mathbb{Z}_2)^4 \rtimes_{\Gamma} S_4$ for the Heun equation, found by Maier, by $G_{\text{I}} \rtimes_{\Gamma} G_{\text{II}}$ for the Darboux equation, where the group $G_{\text{I}} \cong (\mathbb{Z}_2)^4$ whose action is analogous to that of the Heun equation, while the group G_{II} satisfies the short-exact sequence

$$0 \rightarrow K \rightarrow G_{\text{II}} \rightarrow \text{anh} \rightarrow 0.$$

Here the group K denotes the *Klein four-group* which plays the role of translations of the half-periods (and so on the regular singular points) on the torus, while the anh denotes the *anharmonic group* which is isomorphic to $\text{Aut}(X(2))$, where $X(2) = \mathbb{H}/\Gamma(2)$ and \mathbb{H} is the upper half-plane of \mathbb{C} (see [31, p. 9]). The group $\text{Aut}(X(2))$ is parametrized by the quotient of the *full-modular group* $\Gamma = \text{SL}(2, \mathbb{C})$ by $\Gamma(2)$, the principal congruence subgroup of level 2, which plays the role of permutations of the half-periods of the torus. The group G_{II} thus describes the combined actions of the half-periods of K and anh . This paradigm allows us to see the symmetry group G_{II} acting on the half-period of the torus in a natural way. Hence it is most natural to consider Fuchsian equations with four regular singular

points on a torus. Although it is intrinsic in Maier's work that one can consider the six equivalent classes of arrangements of the four regular singular points in \mathbb{CP}^1 , with each of these classes characterized by the cross-ratio of the four singular points, it is much more subtle and less natural compared to putting the four points at the four half-period of a torus. We also exhibit that Maier's hyper-Kummer group \mathfrak{K} is isomorphic to our group G_{II} .

Moreover, the stumbling block (see Maier [32, p. 812]) of describing the accessory parameters (eigenvalues) corresponding to the 192 local solutions of the Heun equation is reduced to only six classes (Table 4) when considered in the Jacobian form (1.2) of the Darboux equation, and to a mere of three classes (Table 5) when considered in the Weierstrass form (1.4). This is because the Heun equation is written in *asymmetric form* (as in the case of the hypergeometric equation), while the Darboux equation (1.2) that is commonly referred to in the literature is written in the *symmetric form*. The part $(\mathbb{Z}_2)^{n-1}$ of its symmetry group $D_n \cong (\mathbb{Z}_2)^{n-1} \rtimes_{\Gamma} S_n$ of a Fuchsian equation on \mathbb{CP}^1 in the symmetric form does not affect its accessory parameter. Although the Darboux equation in both the Jacobian form and Weierstrass form are in symmetric forms, these reductions are still substantially more intuitive compared to the symmetric form of the Heun equation as described in [32, Eqn(2.2)]. We shall consider both the Jacobian and Weierstrass forms of the Darboux equation (in the symmetric form) in this paper. We have listed the corresponding asymmetric form in Jacobian form of the Darboux equation in Appendix B. Another equation named after Sparre (1883) is related to the Darboux equation in Jacobian form is parallel to the Papperitz form of the hypergeometric equation (see Whittaker and Watson [48, p. 206] or Poole [40, p. 86]) will also be discussed there. Both the symmetric and the asymmetric forms of the Darboux equation are special cases of the Sparre equation.

The subgroup G_{II} acts on Weierstrass form in a simple manner while its action is more subtle on the Jacobian form. However, the study of the solutions of the Darboux equation in Jacobian form is preferred. It is because formulas of Jacobi elliptic functions are better developed (see e.g. [48, XXII], [6]), which make practical computation easier. A subtle point in considering the symmetries of the Darboux equation arises is that the group actions from the G_{II} would permute the half-periods the torus \mathbb{C}/Λ and hence causing changes of the *apparent* underlying space \mathbb{C}/Λ on which the equation is defined and simultaneously while they act on the equation itself. Therefore, we address how we could make this ambiguity precise by constructing a proper underlying space on which the equations live in §3 when discussing the symmetries of the Darboux equation in general. Such ambiguity does not arise when considering the symmetries of the Heun equation over \mathbb{CP}^1 .

The Darboux equation had been forgotten by the general mathematics community until it was rediscovered for over a century later, by Treibich and Verdier [45, 46] (without knowing Darboux's work) in the context of the finite-gap theory and algebraic geometry. Indeed, there is a simple conversion between the Jacobi elliptic functions and the Weierstrass elliptic function $\wp(u, \tau)$. The Darboux

equation becomes

$$\frac{d^2 y}{du^2} + \left[h - \xi(\xi + 1)\wp(u; \tau) - \eta(\eta + 1)\wp(u + \omega_1; \tau) - \mu(\mu + 1)\wp(u + \omega_2; \tau) - \nu(\nu + 1)\wp(u + \omega_3; \tau) \right] y = 0, \quad (1.4)$$

where $\omega_1, \omega_2, \omega_3$ are the half-periods of the corresponding torus. It is in this context that the Darboux equation was rediscovered by Treibich and Verdier [45, 46] that the elliptic potential in (1.4) is a finite-gap potential if and only if all four parameters ξ, η, μ, ν are integers. We refer to [16, §1] for a brief history.

Let us consider a family of elliptic curves over the upper half-plane \mathbb{H} whose fibre at τ (together with the point at infinity) is the zero-locus of

$$F(x, y) = y^2 - (4x^3 - g_2(\tau)x - g_3(\tau)).$$

Explicitly, we have a family \mathbb{E} (together with the point at infinity)

$$\mathbb{E} = \{(x, y; \tau) \in \mathbb{C}^2 \times \mathbb{X}(2) : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\}$$

on \mathbb{CP}^2 . The Weierstrass elliptic function \wp can now be regarded as a function defined on \mathbb{E} , which is just the projection to the first coordinate. On the one hand, this illustrates that both the independent variable and (half-)periods of the (1.4) are changed when acted upon by its symmetry group. This aspect is less apparent from the Jacobian form (1.2) compared with the Weierstrass form (1.4) of the Darboux equation (as the half-periods for the Jacobian elliptic functions are signified by the elliptic moduli k, k'). It turns out that the Jacobian form of the Darboux equation is preferred when we are dealing with local series expansion because more formulae for computation are available as mentioned earlier and their analogy with trigonometric functions, while the Weierstrass form is more advantageous when dealing with symmetry properties. As we shall see later that our theory of the series expansions, written as the sums in terms of Jacobi elliptic functions has an old origin, which can be traced back to the work of Ince [26, (1940)] for solving a special case of the Darboux equation, namely the Lamé equation. As a result, we will present our main results for both the Jacobian form and Weierstrass form equations, but we present the argument leading to these results for the Jacobian form (1.2) only in this paper.

Indeed, amide the pioneering work of Treibich and Verdier, the equation has been under intense study from different perspectives by recent researchers such as Gesztesy and Weikard [16, (1996)], Matveev and Smirnov [34, (2006)], Takemura [42, 43, 44, (2003, 2008, 2009)], Veselov [47, (2011)] etc. On the other hand, the Lamé equation, which is a special case of the Darboux equation, appears in many different contexts such as boundary value problems (Courant-Hilbert) [8], finite-gap problem (integrable systems) [10, 36, 27, 15], algebraic geometry [3, 29, 7], Cologero-Moser-Sutherland models [42], etc.

This paper is organised as follows. We split the discussion of the group action G_I of transposition of local solutions at each singularity in §2 and G_{II} for the group action of permutations of half-periods of a torus §3 and §4. More specifically,

the group actions G_{II} on the Weierstrass form are given in §3 and those on the Jacobian form are given in §4. The joint actions of G_{I} and G_{II} on both forms of the Darboux equation are discussed in §5, while the symmetric group action on the accessory parameter h is discussed in §6. In §7, we give the convergence criteria of local series expansions about the regular singularities on the torus. Then we discuss the criteria that lead to the termination of these infinite expansions. These finite sum solutions which we call Darboux polynomials are analogous to the classical Lamé polynomials. Section 8 enumerates the 192 local series expansions. We apply our main results to well-known special cases of the Darboux equation in Section 9.

2 Group actions: transpositions of local solutions

As we have mentioned in the Introduction that the symmetry group of the Darboux equation is given by a semi-direct product $G_{\text{I}} \rtimes_{\Gamma} G_{\text{II}}$. In this section, we introduce G_{I} which consists of *transpositions* of two local solutions at each of the four regular singularities. As one can easily see below, these actions are automorphisms of the Darboux equations which leave the equation unchanged. Specifically, these are given by the transformations on the parameters ξ, η, μ, ν , which represent the local monodromies, in the Darboux equations in both the Jacobian form (1.2) and in the Weierstrass form (1.4).

Let $\gamma \in \{\xi, \eta, \mu, \nu\}$. Denote $\gamma^+ = \gamma$ and $\gamma^- = -\gamma - 1$. Since

$$\gamma^+(\gamma^+ + 1) = \gamma^-(\gamma^- + 1) = \gamma(\gamma + 1),$$

both equations (1.2) and (1.4) remain unchanged under the following transformations

$$\xi \mapsto \xi^{\pm}, \eta \mapsto \eta^{\pm}, \mu \mapsto \mu^{\pm}, \nu \mapsto \nu^{\pm}.$$

Such transformations give the following

Definition 2.1. Let $G_{\text{I}} = \{+, -\}^4$ be the group of 16 automorphisms of the set of parameters in the Darboux equation defined as above. This group can also be identified as the (additive) group of mappings from $\{0, 1, 2, 3\}$ to \mathbb{Z}_2 .

It is easy to see that G_{I} forms a group which is isomorphic to $(\mathbb{Z}_2)^4$.

3 Group actions on half-periods: Weierstrass elliptic form

In this section, we consider G_{II} in the semi-direct product $G_{\text{I}} \rtimes_{\Gamma} G_{\text{II}}$ which consists of translations and permutations of the half-periods of the underlying torus of the Darboux equation in the Weierstrass form. The consideration of Darboux equation

in the Jacobian form will be discussed in the next section. So we discuss the transformations of Weierstrass elliptic functions, and hence they give transformations from a Darboux operator to another Darboux operator.

In general, if $w : X \rightarrow \mathbb{C}$ is a function and $g \in \text{Aut}(X)$, then g acts on w naturally by composition. In the study of Darboux equations, the elliptic functions like $\wp(u, \tau)$ or $\text{sn}^2(u, k)$ are involved. The first task is to clarify the domain of these elliptic functions so that the automorphisms of their domain act on these elliptic functions by composition (or pulling back). The natural candidate is the “universal family” of elliptic curves. However, it is standard that this universal family does not exist [19, p. 35]. Therefore, we will settle with a “six-to-one” family.

The first step in our general formulation is to parametrize “lattices with an ordered basis”. Recall that if a given lattice Λ in \mathbb{C} is equipped with an ordered base $(2\omega_1, 2\omega_3)$. Its λ -invariant is defined by

$$\lambda(\omega_1, \omega_3) = \frac{\wp(\omega_3) - \wp(\omega_1 + \omega_3)}{\wp(\omega_1) - \wp(\omega_1 + \omega_3)}.$$

It can be shown that this λ -invariant is invariant under the action of $\Gamma(2) = \{M \in SL(2, \mathbb{Z}) : M \equiv I \pmod{2}\}$ (see [31, p. 9] and [1, p. 278]). Therefore, “lattices with an ordered base” are parametrized by the well-known quotient $\mathbb{H}/\Gamma(2) = X(2)$ (see [1, p. 278]).

Our next step is to parametrize all the lattices. We know that two “lattices with an ordered basis” $(2\omega_1, 2\omega_3)$ and $(2\omega'_1, 2\omega'_3)$ have isomorphic underlying lattices when there exists $M \in SL(2, \mathbb{C})$ which takes ω_1 to ω'_1 and takes ω_3 to ω'_3 . This \mathbb{Z} -linear map from $\text{span}(2\omega_1, 2\omega_3)$ to $\text{span}(2M\omega_1, 2M\omega_3)$ induces an action on the λ -invariants.

It is well-known that the orbit of the λ -invariant under the full modular group $\Gamma = SL(2, \mathbb{Z})$ is

$$\left\{ \lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, 1-\lambda \right\}.$$

These are in fact the Möbius transformations which are permutations of $\{0, 1, \infty\}$. These Möbius transformations form a group known as the *anharmonic group* anh [1, p. 278]. Indeed, we have

$$\text{anh} = \{M \in SL(2, \mathbb{Z}) : M : \mathbb{CP}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{CP}^1 \setminus \{0, 1, \infty\}\}.$$

Since $\Gamma(2)$ is a normal subgroup of Γ [31, p. 9], so anh is isomorphic to the quotient $\Gamma/\Gamma(2)$. Thus lattices are parametrized by $X(2)$ modulo the action of anh .

The corresponding transformations of lattices with ordered basis are named, following the classical convention [13, p. 369], as $X = I, A, B, C, D, E$. They also correspond to the action of permutations ρ_X of $e_i = \wp(\omega_i)$ on the λ -invariant. These are summarized in the Table 1 below.

Now we consider a family of elliptic curves whose fibre at τ (together with the point at infinity) is cut out (i.e., zero-locus) by

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

Table 1:

X	Six cross-ratios	Matrix representative	ρ_X
I	λ	$M_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$(e_1)(e_2)(e_3)$
A	$\frac{\lambda}{\lambda-1}$	$M_A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$(e_1)(e_2e_3)$
B	$\frac{1}{\lambda}$	$M_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$(e_1e_3)(e_2)$
C	$1-\lambda$	$M_C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$(e_1e_2)(e_3)$
D	$\frac{\lambda-1}{\lambda}$	$M_D = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$(e_1e_2e_3)$
E	$\frac{1}{1-\lambda}$	$M_E = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	$(e_1e_3e_2)$

Explicitly, we have a family \mathbb{E} (together with the point at infinity, i.e., in $\mathbb{C}\mathbb{P}^2$)

$$\mathbb{E} = \{(x, y; \tau) \in \mathbb{C}^2 \times \mathbb{X}(2) : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\} \xrightarrow{\lambda \circ \pi_3} \mathbb{C} \setminus \{0, 1\},$$

where the map π_3 denotes the projection to the third coordinate. The Weierstrass elliptic function \wp can now be regarded as a function defined on \mathbb{E} , which is just the projection to the first coordinate.

In fact, each of such six transformations $I, A, B, C, D, E : \mathbb{E} \rightarrow \mathbb{E}$ can be described explicitly by

$$\begin{aligned}
I(x, y; \tau) &= (x, y; \tau); \\
A(x, y; \tau) &= ((\tau-1)^2x, (\tau-1)^3y; \tau/(\tau-1)); \\
B(x, y; \tau) &= (\tau^2x, \tau^3y; 1/\tau); \\
C(x, y; \tau) &= (x, y; 1-\tau); \\
D(x, y; \tau) &= (\tau^2x, \tau^3y; (\tau-1)/\tau); \\
E(x, y; \tau) &= ((1-\tau)^2x, (1-\tau)^3y; 1/(1-\tau)).
\end{aligned} \tag{3.1}$$

It is easy to verify that they are well-defined maps from \mathbb{E} to \mathbb{E} , since g_2 and g_3 are modular forms of weights 4 and 6 respectively. On the other hand, given a torus with ordered base $(2\omega_1, 2\omega_3)$, the subgroup of the order 2 points of the corresponding torus would be

$$T = \{0, \omega_1, \omega_3, \omega_2 = \omega_1 + \omega_3\}.$$

For each $j \in \{1, 2, 3\}$, let I_j be an action on T which is translation by ω_j . Obviously, $K = \{I_0, I_1, I_2, I_3\}$ is isomorphic to the Klein four-group.

Finally we are able to describe the group G_{II} :

Definition 3.1. The group G_{II} consists of commutative diagrams of the form

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & \mathbb{E} \\ \lambda \circ \pi_3 \downarrow & & \downarrow \lambda \circ \pi_3 \\ \mathbb{C} \setminus \{0, 1\} & \longrightarrow & \mathbb{C} \setminus \{0, 1\} \end{array} \quad (3.2)$$

so that both of the horizontal maps are biholomorphic. The group multiplication between two such commutative diagrams is defined by obvious concatenation of two such diagrams.

Theorem 3.2. *Let K , G_{II} and anh be the groups defined above. Then we have the short exact sequence*

$$0 \longrightarrow K \longrightarrow G_{\text{II}} \longrightarrow \text{anh} \longrightarrow 0.$$

Proof. Here the surjective map $G_{\text{II}} \longrightarrow \text{anh}$ is canonical in the sense that we apply the lower homomorphism of the above commutative diagram (3.2). We have established the right half of the short exact sequence. To establish the left-half of the short exact sequence, we define the map $K \rightarrow G_{\text{II}}$ as follows. Recall that the cubic curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ is an Abelian group (with the usual group addition “+” of two points of the elliptic curve is defined by the inverse point of a third point obtained from the intersection of the straight line passing through the two points and the elliptic curve) with the point at infinity chosen as the identity. Let x_1, x_2, x_3 be the zeros of $\wp'(\cdot; \tau)$. Then the map $K \rightarrow G_{\text{II}}$ given by

$$\begin{aligned} I_0 &\longmapsto [(x, y; \tau) \mapsto (x, y; \tau)], \\ I_1 &\longmapsto [(x, y; \tau) \mapsto (x, y; \tau) + (\wp(x_1), 0; \tau)], \\ I_2 &\longmapsto [(x, y; \tau) \mapsto (x, y; \tau) + (\wp(x_2), 0; \tau)], \\ I_3 &\longmapsto [(x, y; \tau) \mapsto (x, y; \tau) + (\wp(x_3), 0; \tau)] \end{aligned} \quad (3.3)$$

is clearly injective.

Remark 3.3. We have seen that the anharmonic group anh corresponds to the permutation group of the half-periods $\{\omega_1, \omega_2, \omega_3\}$, while the Klein four group acts on a torus by translations by half-periods. Therefore they generate a group G_{II} which corresponds to the permutation group of the order-two points $\{0, \omega_1, \omega_2, \omega_3\}$.

3.1 Maier’s hyper-Kummer group \mathfrak{K} and G_{II}

The analysis of G_{II} can also be applied to the hyper-Kummer group mentioned in Maier [32, 33]. Recall that the hyper-Kummer group \mathfrak{K} (in the case of $\mathbb{C}\mathbb{P}^1$ together with four marked points) is the subgroup of $S_4 \times \text{PSL}(2, \mathbb{C})$ which acts on $(0, 1, \infty, \lambda)$ leaving each of the first three coordinates fixed. We define a map

$$\mathfrak{K} \longrightarrow \left\{ \lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{1}{\lambda}, \frac{\lambda}{\lambda-1}, 1-\lambda \right\} \quad (3.4)$$

so that $(\sigma, T) \in S_4 \times \mathrm{PSL}(2, \mathbb{C})$ is sent to the cross ratio $(\sigma(0), \sigma(1), \sigma(\infty), \sigma(\lambda))$. Note that the latter group of order six is identified to the anharmonic group anh . We remark that one of the $\sigma \in S_4$ or $T \in \mathrm{PSL}(2, \mathbb{C})$ is redundant in the above description since σ uniquely determines T (and vice versa). However, no such simplification is possible when the discussion is about a second order differential equation with n singular points where $n \geq 5$ as Maier considered [32].

Theorem 3.4. *We have the following commutative diagram consisting of two rows of short exact sequences:*

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \rightarrow & \mathfrak{K} & \rightarrow & \mathrm{anh} & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \parallel & & \\ 0 & \rightarrow & K & \rightarrow & G_{\mathrm{II}} & \rightarrow & \mathrm{anh} & \rightarrow & 0. \end{array} \quad (3.5)$$

In particular, the hyper-Kummer group \mathfrak{K} is isomorphic to G_{II} .

Proof. We have already established the lower short exact sequence in the Theorem 3.2. To establish the upper short exact sequence

$$0 \longrightarrow K \longrightarrow \mathfrak{K} \longrightarrow \mathrm{anh} \longrightarrow 0,$$

we note that the kernel of the map $\mathfrak{K} \rightarrow \mathrm{anh}$ as defined in (3.4) above consists of two disjoint pairs of transpositions of the points in the cross-ratios $(\sigma(0), \sigma(1), \sigma(\infty), \sigma(\lambda))$ which can easily be seen to be isomorphic to the Klein four-group K .

Now we would like to construct a map $\mathfrak{K} \rightarrow G_{\mathrm{II}}$. First of all, we note that the family $\mathbb{E} \rightarrow \mathbb{C} \setminus \{0, 1\}$ is written explicitly as

$$\mathbb{E} = \{(x, y; \tau) \in \mathbb{C}^2 \times \mathrm{X}(2) : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\}.$$

Then $(\sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in S_4 \times \mathrm{PSL}(2, \mathbb{C})$ induces the commutative diagram

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathbb{C} \setminus \{0, 1\} & \longrightarrow & \mathbb{C} \setminus \{0, 1\} \end{array}$$

where the map in the top row is given by

$$(x, y; \tau) \mapsto \left((c\tau + d)^2 x, (c\tau + d)^3 y; \frac{a\tau + b}{c\tau + d} \right).$$

The other maps in the diagram (3.5) are natural.

The commutative diagram (3.5) established above describes that the hyper-Kummer group \mathfrak{K} of Heun equation has the same kind of semi-direct product decomposition as G_{II} .

3.2 The action of G_{II} on Darboux operator in Weierstrass form

Given an elliptic function $\wp : \mathbb{E} \rightarrow \mathbb{CP}^1$, the group G_{II} acts on \wp by pulling back. This action by anh can be found in (3.1).

For each τ in \mathbb{H} , let $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ be the zeros of

$$4x^3 - g_2(\tau)x - g_3(\tau).$$

Definition 3.5. The Darboux operator is the differential operator on \mathbb{E}

$$\begin{aligned} D_{(\tau; \xi, \eta, \mu, \nu)} = & \left(y \frac{d}{dx} \right)^2 - \left[\xi(\xi + 1)x \right. \\ & + \eta(\eta + 1) \frac{(x - e_2(\tau))(x - e_3(\tau)) - (x - e_1(\tau))^2}{x - e_1(\tau)} \\ & + \mu(\mu + 1) \frac{(x - e_3(\tau))(x - e_1(\tau)) - (x - e_2(\tau))^2}{x - e_2(\tau)} \\ & \left. + \nu(\nu + 1) \frac{(x - e_1(\tau))(x - e_2(\tau)) - (x - e_3(\tau))^2}{x - e_3(\tau)} \right]. \end{aligned}$$

Note that $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ are formally modular forms of weight 2, in particular, for $j = 1, 2, 3$ and $X = I, A, B, C, D, E$,

$$e_j \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \rho_X(e_j)(\tau), \quad (3.6)$$

where $M_X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and ρ_X are given in Table 1. The actions of anh (and hence G_{II}) on the Darboux operator are easily obtained as the following theorem by direct computation and (3.6).

Theorem 3.6. For each $M_X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{anh}$, the pull-back of the Darboux operator $D_{(\tau; \xi, \rho_X(\eta), \rho_X(\mu), \rho_X(\nu))}$ by M_X is

$$(c\tau + d)^{-2} D_{\left(\frac{a\tau+b}{c\tau+d}; \xi, \eta, \mu, \nu\right)}.$$

Now we look at the action of anh (and hence G_{II}) on Darboux equation (1.4) instead of the Darboux operator. For example, we consider the map A from (3.1) acting on the Darboux equation (1.4) with $\omega_1 = \tau/2$, $\omega_2 = 1/2 + \tau/2$ and $\omega_3 = 1/2$. Notice that

$$\begin{aligned} A(\wp(u; \tau), \wp'(u; \tau); \tau) &= ((\tau - 1)^2 \wp(u; \tau), (\tau - 1)^3 \wp'(u; \tau); \frac{\tau}{\tau - 1}) \\ &= (\wp(\tilde{u}; \tilde{\tau}), \wp'(\tilde{u}; \tilde{\tau}); \tilde{\tau}), \end{aligned}$$

where $\tilde{u} = \frac{u}{\tau - 1}$ and $\tilde{\tau} = \frac{\tau}{\tau - 1}$. So the Darboux equation (1.4) becomes

$$\begin{aligned} (\tau - 1)^{-2} \frac{d^2 y}{d\tilde{u}^2} + \left[h - (\tau - 1)^{-2} (\xi(\xi + 1) \wp(\tilde{u}; \tilde{\tau}) - \eta(\eta + 1) \wp(\tilde{u} + \frac{\tilde{\tau}}{2}; \tilde{\tau}) \right. \\ \left. - \nu(\nu + 1) \wp(\tilde{u} + \frac{\tilde{\tau}}{2} + \frac{1}{2}; \tilde{\tau}) - \mu(\mu + 1) \wp(\tilde{u} + \frac{1}{2}; \tilde{\tau}) \right] y = 0, \end{aligned} \quad (3.7)$$

which is equivalent to

$$\frac{d^2y}{d\tilde{u}^2} + \left[\tilde{h} - \xi(\xi + 1)\wp(\tilde{u}; \tilde{\tau}) - \eta(\eta + 1)\wp(\tilde{u} + \frac{\tilde{\tau}}{2}; \tilde{\tau}) - \nu(\nu + 1)\wp(\tilde{u} + \frac{\tilde{\tau}}{2} + \frac{1}{2}; \tilde{\tau}) - \mu(\mu + 1)\wp(\tilde{u} + \frac{1}{2}; \tilde{\tau}) \right] y = 0,$$

where $\tilde{h} = (\tau - 1)^2 h$. Similarly, one can write down the group action of G_{II} on the Darboux equation in Weierstrass form by other elements B, C, D, E , while the action of the Klein four-group on the Darboux equation consists simply of shifting the half-periods in the independent variable u .

4 Group actions on half-periods: Jacobi elliptic form

Since the Jacobi elliptic functions are much more convenient to compute, so historically, most of these classical differential equations with doubly periodic coefficients, such as the Darboux equation, and those listed in the Appendix B are written in Jacobian forms [6]. However, as we have explained earlier in the last section that the action of the group G_{II} on Jacobi elliptic functions is much more complicated compared with that of the Weierstrass elliptic functions.

This is due to the parametrization of the elliptic curves using Jacobi elliptic functions. For example, the $\text{sn}(u, k)$ depends on the independent variable u and modulus k , and the half-periods are implicit in the notation. One can convert the half-periods $\omega_1, \omega_2, \omega_3$ in terms of the modulus k and the complementary modulus $k' = \sqrt{1 - k^2}$. We note that k^2 is the λ -invariant of the torus. We refer the reader to the Appendix C for the relationships. The lower short exact sequence of (3.5) in Theorem 3.4 gives rise to symmetry group G_{II} whose actions on the Jacobi elliptic functions $\text{sn}, \text{cn}, \text{dn}$ can also be split into two parts, that are, the actions by the Klein four-group K and the actions of anharmonic group anh .

4.1 The action of anh on Jacobi elliptic functions

In this section, we discuss the transformations $(u, k) \mapsto (\tau(u, k), \kappa(k))$ of Jacobi elliptic functions into other Jacobi elliptic functions by the anharmonic group anh . The six permutations I, A, B, C, D, E written in the Table 2 on the Jacobi elliptic functions are different from those given in (3.1) and are more complicated. Indeed, they had been already given in [13, p. 369] which we reproduce in Table 2.

Table 2: Transformations of Jacobi elliptic functions

X	τ_X	κ_X	κ'_X	$K(\kappa_X)$	$K'(\kappa_X)$	$\operatorname{sn}(\tau_X, \kappa_X)$	$\operatorname{cn}(\tau_X, \kappa_X)$	$\operatorname{dn}(\tau_X, \kappa_X)$
I	u	k	k'	$K(k)$	$K'(k)$	$\operatorname{sn}(u, k)$	$\operatorname{cn}(u, k)$	$\operatorname{dn}(u, k)$
A	$k'u$	$ik'k'^{-1}$	k'^{-1}	$k'K(k)$	$k'[K'(k) - iK(k)]$	$k' \operatorname{sd}(u, k)$	$\operatorname{cd}(u, k)$	$\operatorname{nd}(u, k)$
B	$-iu$	k'	k	$K'(k)$	$K(k)$	$i \operatorname{sc}(u, k)$	$\operatorname{nc}(u, k)$	$\operatorname{dc}(u, k)$
C	ku	k^{-1}	$-ik'k'^{-1}$	$k[K(k) + iK'(k)]$	$kK'(k)$	$k \operatorname{sn}(u, k)$	$\operatorname{dn}(u, k)$	$\operatorname{cn}(u, k)$
D	$-ik'u$	k'^{-1}	$-ik'k'^{-1}$	$k'[K'(k) + iK(k)]$	$k'K(k)$	$-ik' \operatorname{sc}(u, k)$	$\operatorname{dc}(u, k)$	$\operatorname{nc}(u, k)$
E	$-iku$	$ik'k'^{-1}$	k^{-1}	$kK'(k)$	$k[K(k) + iK'(k)]$	$-ik \operatorname{sd}(u, k)$	$\operatorname{nd}(u, k)$	$\operatorname{cd}(u, k)$

Table 3:

X_i	$\mathcal{T}X_i$	κ_X	κ'_X	$\text{sn}(\mathcal{T}X_i, \kappa_X)$	$\text{cn}(\mathcal{T}X_i, \kappa_X)$	$\text{dn}(\mathcal{T}X_i, \kappa_X)$
I_0	u			$\text{sn}(u, k)$	$\text{cn}(u, k)$	$\text{dn}(u, k)$
I_1	$u + K(k)$			$\text{cd}(u, k)$	$-k' \text{sd}(u, k)$	$k' \text{nd}(u, k)$
I_2	$u + K(k) + iK'(k)$	k	k'	$k^{-1} \text{dc}(u, k)$	$ik'k^{-1} \text{nc}(u, k)$	$ik' \text{sc}(u, k)$
I_3	$u + iK'(k)$			$k^{-1} \text{ns}(u, k)$	$-ik^{-1} \text{ds}(u, k)$	$-i \text{cs}(u, k)$
A_0	$k'u$			$k' \text{sd}(u, k)$	$\text{cd}(u, k)$	$\text{nd}(u, k)$
A_1	$k'(u + K(k))$			$\text{cn}(u, k)$	$-\text{sn}(u, k)$	$k'^{-1} \text{dn}(u, k)$
A_2	$k'(u + K(k) + iK'(k))$	$ik'k'^{-1}$	k'^{-1}	$-ik'^{-1} \text{ds}(u, k)$	$k^{-1} \text{ns}(u, k)$	$-ik'^{-1} \text{cs}(u, k)$
A_3	$k'(u + iK'(k))$			$ik'k'^{-1} \text{nc}(u, k)$	$k^{-1} \text{dc}(u, k)$	$i \text{sc}(u, k)$
B_0	$-iu$			$i \text{sc}(u, k)$	$\text{nc}(u, k)$	$\text{dc}(u, k)$
B_1	$-i(u + K(k))$			$ik'^{-1} \text{cs}(u, k)$	$k'^{-1} \text{ds}(u, k)$	$-\text{ns}(u, k)$
B_2	$-i(u + K(k) + iK'(k))$	k'	k	$k'^{-1} \text{dn}(u, k)$	$ik'k'^{-1} \text{cn}(u, k)$	$k \text{sn}(u, k)$
B_3	$-i(u + iK'(k))$			$\text{nd}(u, k)$	$ik' \text{sd}(u, k)$	$k \text{cd}(u, k)$
C_0	ku			$k \text{sn}(u, k)$	$\text{dn}(u, k)$	$\text{cn}(u, k)$
C_1	$k(u + K(k))$			$k \text{cd}(u, k)$	$k' \text{nd}(u, k)$	$-k' \text{sd}(u, k)$
C_2	$k(u + K(k) + iK'(k))$	k^{-1}	$-ik'k^{-1}$	$\text{dc}(u, k)$	$ik' \text{sc}(u, k)$	$ik'k'^{-1} \text{nc}(u, k)$
C_3	$k(u + iK'(k))$			$\text{ns}(u, k)$	$-i \text{cs}(u, k)$	$-ik'^{-1} \text{dc}(u, k)$
D_0	$-ik'u$			$-ik' \text{sc}(u, k)$	$\text{dc}(u, k)$	$\text{nc}(u, k)$
D_1	$-ik'(u + K(k) + iK'(k))$			$-i \text{cs}(u, k)$	$-\text{ns}(u, k)$	$-k'^{-1} \text{ds}(u, k)$
D_2	$-ik'(u + K(k) + iK'(k))$	k'^{-1}	$-ik'k'^{-1}$	$\text{dn}(u, k)$	$k \text{sn}(u, k)$	$ik'k'^{-1} \text{cn}(u, k)$
D_3	$-ik'(u + iK'(k))$			$k' \text{nd}(u, k)$	$k \text{cd}(u, k)$	$ik' \text{sd}(u, k)$
E_0	$-ik'u$			$-ik' \text{sd}(u, k)$	$\text{nd}(u, k)$	$\text{cd}(u, k)$
E_1	$-ik(u + K(k))$			$-ik'k'^{-1} \text{cs}(u, k)$	$k'^{-1} \text{dn}(u, k)$	$-\text{sn}(u, k)$
E_2	$-ik(u + K(k) + iK'(k))$	$-ik'k'^{-1}$	k^{-1}	$-ik'^{-1} \text{ds}(u, k)$	$-ik'^{-1} \text{cs}(u, k)$	$k^{-1} \text{ns}(u, k)$
E_3	$-ik(u + iK'(k))$			$\text{nc}(u, k)$	$i \text{sc}(u, k)$	$k^{-1} \text{dc}(u, k)$

The second column τ_X denotes the change of the independent variable u under the actions of X . The second and third columns denote change in the modulus κ_X and the complementary modulus κ'_X , respectively, under the actions of X . The fourth and the fifth columns denote the change of the quarter-periods K and K' under the actions of X , while the last three columns describe the transformations on the Jacobi elliptic functions sn , cn , dn . We note that, as far as the Darboux equation in the Jacobian form is concerned, the periodic coefficients, written in terms of sn^2 , cn^2 , dn^2 , have half-periods K and iK' as denoted in the fourth and fifth columns in the Table 2.

4.2 The action of the Klein four-group K on Jacobi elliptic functions

Similarly, the action of the Klein four-group K on the Jacobi elliptic functions sn , cn , dn and their moduli are given by the first section (i.e., I_0, I_1, I_2, I_3) of the Table 4.1. These actions correspond to the *shifting* of the half-periods of sn^2 , cn^2 , dn^2 of the Jacobian form of Darboux equation (1.2). We will apply these transformations when we discuss the joint group actions of the Darboux equations in the next section (§4.3). We remark that as one can see that the transformations are much more complicated compared with those for the Weierstrass form of the Darboux equation.

4.3 The joint actions of the Klein four-group K and anh

Finally, we come to describing the joint group actions of the Klein four-group K and the anharmonic group anh . The combined actions act on the Jacobi elliptic functions are given by the Table 4.1 which consists of six types of permutations of quarter-periods (one period of sn doubles that of sn^2) each of which leaves the origin of the torus unchanged, and four types of translations of the quarter-periods. This is reflected in the Table 4.1 having six horizontal sections labelled by type I, A, B, C, D, E which corresponds to actions of anh (permutations of quarter-periods), and each of which is further subdivided into four translations. These four translations represent the Klein four-group actions. For example, the map B_3 represents the transformation by the translation I_3 acting after the action of B .

5 Joint actions of G_I and G_{II} automorphisms

Definition 5.1. Let G be the symmetry group of the Darboux equation generated by the groups G_I (i.e., the transpositions of local solutions) and G_{II} (i.e., actions on half-periods).

Theorem 5.2. *The group G is a semi-direct product*³

$$G \cong G_I \rtimes_{\Gamma} G_{II},$$

³See the Definition D.1 in Appendix D for the semi-direct product.

for some action $\Gamma : G_{\text{II}} \rightarrow \text{Aut}(G_I)$.

Proof. Identify G_I as the group of mappings $\{0, 1, 2, 3\} \rightarrow \mathbb{Z}_2$, and identify G_{II} as the group of permutations of half-periods. That is, the permutations group of $\{0, 1, 2, 3\}$ (see Remark 3.3). Then G_{II} acts on G_I from the left and we call this action $\Gamma : G_{\text{II}} \rightarrow \text{Aut}(G_I)$. It remains to show that for each $s \in G_I$ and $X \in G_{\text{II}}$,

$$\Gamma(X)(s) = X s X^{-1}.$$

But this verification is routine.

Remark 5.3. Since G_{II} is a permutation group, the group G above is indeed the group of *signed permutations*. It can also be described as the Coxeter group B_4 .

6 Group actions on accessory parameters

Under the combined actions of $G \cong G_I \rtimes_{\Gamma} G_{\text{II}}$, we have

Theorem 6.1. *For each $X = I, A, B, C, D, E ; i = 0, 1, 2, 3, X_i$ transforms the Darboux equation (1.2) to the form*

$$\begin{aligned} \frac{d^2 y}{dw^2} + \left[h_X - \sigma_{X_i}(\xi)(\sigma_{X_i}(\xi) + 1) \text{ns}^2(w, \kappa_X) - \sigma_{X_i}(\eta)(\sigma_{X_i}(\eta) + 1) \text{dc}^2(w, \kappa_X) \right. \\ \left. - \sigma_{X_i}(\mu)(\sigma_{X_i}(\mu) + 1) \kappa_X^2 \text{cd}^2(w, \kappa_X) - \sigma_{X_i}(\nu)(\sigma_{X_i}(\nu) + 1) \kappa_X^2 \text{sn}^2(w, \kappa_X) \right] y = 0, \end{aligned} \quad (6.1)$$

with the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{cccc} 0 & K(\kappa_X) & K(\kappa_X) + iK'(\kappa_X) & iK'(\kappa_X) \\ \sigma_{X_i}(\xi) + 1 & \sigma_{X_i}(\eta) + 1 & \sigma_{X_i}(\mu) + 1 & \sigma_{X_i}(\nu) + 1 \\ -\sigma_{X_i}(\xi) & -\sigma_{X_i}(\eta) & -\sigma_{X_i}(\mu) & -\sigma_{X_i}(\nu) \end{array} \quad w; h_X \right\},$$

for some h_X , where $w := \tau_{X_i}(u, k)$ and κ_X are listed in Table 4.1, and σ_{X_i} is a permutation of $\{\xi, \eta, \mu, \nu\}$ as listed in Table 4.

Remark 6.2. The group G_I leaves the Darboux equation unchanged.

Remark 6.3. The actions of Klein four-group K correspond to shifting of the half-periods of the torus, hence resulting in permuting the four Jacobi elliptic functions in the Darboux potential, it obviously leaves the accessory parameter h_{X_i} unaltered, so we can denote the accessory parameter by h_X , instead of h_{X_i} .

The description of h_X for each transformation X_i can be found in Table 4. One can see from Table 4 that these new accessory parameters can be divided into six types.

Table 4:

X	h_X	σ_{X_0}	σ_{X_1}	σ_{X_2}	σ_{X_3}
I	h	$(\xi)(\eta)(\mu)(\nu)$	$(\xi\eta)(\mu\nu)$	$(\xi\mu)(\eta\nu)$	$(\xi\nu)(\eta\mu)$
A	$k'^{-2}[h - k^2(\xi(\xi + 1) + \eta(\eta + 1) + \mu(\mu + 1) + \nu(\nu + 1))]$	$(\xi)(\eta)(\mu\nu)$	$(\xi\eta)(\mu)(\nu)$	$(\xi\mu\eta\nu)$	$(\xi\nu\eta\mu)$
B	$-h + \xi(\xi + 1) + \eta(\eta + 1) + \mu(\mu + 1) + \nu(\nu + 1)$	$(\xi)(\mu)(\eta\nu)$	$(\xi\eta\mu\nu)$	$(\xi\mu)(\eta)(\nu)$	$(\xi\nu\mu\eta)$
C	hk^{-2}	$(\xi)(\nu)(\eta\mu)$	$(\xi\eta\nu\mu)$	$(\xi\mu\nu\eta)$	$(\xi\nu)(\mu)(\eta)$
D	$hk'^{-2}[-h + (\xi(\xi + 1) + \eta(\eta + 1) + \mu(\mu + 1) + \nu(\nu + 1))]$	$(\xi)(\eta\mu\nu)$	$(\xi\eta\nu)(\mu)$	$(\xi\mu\eta)(\nu)$	$(\xi\nu\mu)(\eta)$
E	$-hk^{-2} + \xi(\xi + 1) + \eta(\eta + 1) + \mu(\mu + 1) + \nu(\nu + 1)$	$(\xi)(\eta\nu\mu)$	$(\xi\eta\mu)(\nu)$	$(\xi\mu\nu)(\eta)$	$(\xi\nu\eta)(\mu)$

Table 5:

X	h_X
I, C	h
A, E	$(\tau - 1)^2 h$
B, D	$\tau^2 h$

On the other hand, the accessory parameters of the Darboux equation in the Weierstrass form after transformation are listed in Table 5. In this case, there are only three types of accessory parameters after the symmetry group transformations. Moreover, they are only constant multiples of the “original” accessory parameter. It is because the potential of the Darboux equation in the Weierstrass form has an extra constant multiple after the actions of G_{II} (see (3.7) in §3.2).

7 Expansion of local solutions: Darboux functions

Historically, it was Ince [26, (1940)] who first considered the existence of *Lamé functions* by developing infinite series expansions composed of Jacobian elliptic functions as solutions of the Lamé equation. Ince also had an alternative *Fourier-Jacobi expansions* approach to Lamé equations [25]. This puts the study of Lamé functions in an equal standing as that of the Lamé polynomials. Ince’s work turns out to be important in Novikov’s formulation of finite-gap potential of Lax pair representation of KdV with periodic boundary condition (see [10, 36, 27, 15] for instance). All these expansions are valid on certain domains lying on tori. We extend Ince’s ideas to represent local solutions of the Darboux equation of Jacobian form in terms of infinite expansions of Jacobi elliptic functions.

In this section, let us define one local solution at $u = 0$ with exponent $\xi + 1$ and call it the *local Darboux solution*, denoted by $Dl(\xi, \eta, \mu, \nu; h; u, k)$. The expansions of Dl at the other regular singular points $K, iK', K + iK'$ can be obtained after applying the symmetries of the Darboux equation considered in §2 (for G_{I}) and §4 (for G_{II}). If $\xi = -\frac{3}{2}, -\frac{5}{2}, \dots$, then $Dl(\xi, \eta, \mu, \nu; h; u, k)$ will generically be logarithmic and we do not discuss this degenerate case further in this paper.

Definition 7.1. Suppose that $\xi \neq -\frac{3}{2}, -\frac{5}{2}, \dots$. Let $Dl(\xi, \eta, \mu, \nu; h; u, k)$ be defined by the following series expansion

$$\operatorname{sn}(u, k)^{\xi+1} \operatorname{cn}(u, k)^{\eta+1} \operatorname{dn}(u, k)^{\mu+1} \sum_{m=0}^{\infty} C_m \operatorname{sn}(u, k)^{2m}, \quad (7.1)$$

where the coefficients $C_m(\xi, \eta, \mu, \nu; h; k)$ satisfy the relation

$$\begin{aligned} & (2m+2)(2m+2\xi+3)C_{m+1} \\ & + \{h - [2m+\eta+\xi+2]^2 - k^2[2m+\mu+\xi+2]^2 + (k^2+1)(\xi+1)^2\}C_m \\ & + k^2(2m+\xi+\eta+\mu+\nu+2)(2m+\xi+\eta+\mu-\nu+1)C_{m-1} = 0, \end{aligned} \quad (7.2)$$

where $m \geq 0$ and the initial conditions $C_{-1} = 0$, $C_0 = 1$.

Remark 7.2. The factor in front of the summation sign of (7.1) is multi-valued in general. Indeed if we expand this solution in terms of the Weierstrass elliptic form, then one would encounter one more ambiguity of having an extra square-root sign in addition to the complex power that appears in the first factor in front of the expansion (7.1).

For brevity, let

$$\begin{aligned} M_m(\xi) & := (2m+2)(2m+2\xi+3); \\ L_m(\xi, \eta, \mu; h; k) & := h - [2m+\eta+\xi+2]^2 - k^2[2m+\mu+\xi+2]^2 + (k^2+1)(\xi+1)^2; \\ K_m(\xi, \eta, \mu, \nu; k) & := k^2(2m+\xi+\eta+\mu+\nu+2)(2m+\xi+\eta+\mu-\nu+1) \end{aligned}$$

such that (7.2) can be written as

$$M_m C_{m+1} + L_m C_m + K_m C_{m-1} = 0, \quad m \geq 0. \quad (7.3)$$

We show that the series always converges for $|\operatorname{sn}(u, k)| < \min(1, |k|^{-1})$. Before that, we look at the conditions for the series to be terminating, which is the case we need not concern about the convergence.

Theorem 7.3. *If there exists a positive integer q such that either*

$$\xi + \eta + \mu + \nu = -2q - 4 \quad \text{or} \quad \xi + \eta + \mu - \nu = -2q - 3, \quad (7.4)$$

holds, then there exist $q+1$ values h_0, \dots, h_q of h such that the series expansion $Dl(\xi, \eta, \mu, \nu; h_j; u, k)$ ($j = 0, 1, \dots, q$) terminates.

Proof. It follows from (7.2) that the local Darboux solution becomes the finite series

$$\operatorname{sn}(u, k)^{\xi+1} \operatorname{cn}(u, k)^{\eta+1} \operatorname{dn}(u, k)^{\mu+1} \sum_{m=0}^q C_m \operatorname{sn}(u, k)^{2m} \quad \text{with } C_q \neq 0$$

if and only if

$$K_{q+1}(\xi, \eta, \mu, \nu; k) = 0 = C_{q+1}(\xi, \eta, \mu, \nu; h; k).$$

Notice that $C_{q+1}(\xi, \eta, \mu, \nu; h; k) = 0$ is equivalent to saying the vanishing of the finite continued-fraction

$$f(\xi, \eta, \mu, \nu; h; k) := L_0/M_0 - \frac{K_1/M_1}{L_1/M_1 - L_2/M_2} \dots \frac{K_q/M_q}{L_q/M_q} = 0. \quad (7.5)$$

If ξ, η, μ, ν are chosen such that $K_{q+1}(\xi, \eta, \mu, \nu; k) = 0$, that is, (7.4) holds, then there exist $q+1$ values h_0, \dots, h_q of h such that (7.5) holds, and hence the series terminates.

In this terminating case, we call the solution the *Darboux polynomial*, denoted by $Dp(\xi, \eta, \mu, \nu; h_j; u, k)$ ($j = 0, \dots, q$). Note that the Darboux polynomials are generalisation of the classical Lamé polynomials which are eigensolutions to the Lamé equation. The Lamé polynomials are orthogonal polynomials which have some remarkable properties. Some of them are discovered only recently (see e.g. Grosset and Veselov [17, 18] (2006, 2008), Borcea and Shapiro [4] (2008)).

Remark 7.4. The condition on ξ, η, μ, ν so that a solution of Darboux equation exists in the form of a Darboux polynomial is indeed the condition that the Picard-Vessiot extension of the given Darboux equation is a Liouvillian extension of the field of elliptic functions. This criterion can also be obtained by a standard application of Kovacic algorithm [11].

Now we discuss the convergence when the series is in general non-terminating.

Theorem 7.5. *Suppose that $Dl(\xi, \eta, \mu, \nu; h; u, k)$ is non-terminating, i.e., not a Darboux polynomial. Then it converges on the domain $\{|\operatorname{sn} u| < \max(1, |k|^{-1})\}$ ($|k| \neq 1$) if*

$$g(\xi, \eta, \mu, \nu; h; k) := L_0/M_0 - \frac{K_1/M_1}{L_1/M_1 - L_2/M_2} \frac{K_2/M_2}{L_2/M_2 - \dots} = 0 \quad (7.6)$$

holds. Otherwise, it converges only on the domain $\{|\operatorname{sn} u| < \min(1, |k|^{-1})\}$.

Proof. As

$$\lim_{m \rightarrow \infty} M_m/4m^2 = 1, \quad \lim_{m \rightarrow \infty} L_m/4m^2 = -1 - k^2 \quad \text{and} \quad \lim_{m \rightarrow \infty} K_m/4m^2 = k^2,$$

it follows from Theorem A.1 that $\lim_{m \rightarrow \infty} C_{m+1}/C_m$ exists, where the coefficient C_m is defined in (7.3), and is equal to one of the roots of the quadratic equation

$$t^2 - (1 + k^2)t + k^2 = 0,$$

that is, $\lim_{m \rightarrow \infty} C_{m+1}/C_m = 1$ or k^2 . In fact, the limit depends on whether the infinite continued fraction $g(\xi, \eta, \mu, \nu; h; k) = 0$ holds or not. By Theorem A.2,

$$\lim_{m \rightarrow \infty} |C_{m+1}/C_m| = \begin{cases} \min(1, |k|^2) & \text{if (7.6) holds} \\ \max(1, |k|^2) & \text{otherwise} \end{cases}.$$

Then by the ratio-test, the series converges for

$$|\operatorname{sn} u| < \begin{cases} \max(1, |k|^{-1}) & \text{if (7.6) holds} \\ \min(1, |k|^{-1}) & \text{otherwise} \end{cases}.$$

Definition 7.6. If $h = \hat{h}$ is chosen such that (7.6) holds, then $Dl(\xi, \eta, \mu, \nu; h; u, k)$ converges on the larger domain $\{|\operatorname{sn} u| < \max(1, |k|^{-1})\}$ and in this case we call the solution the *Darboux function*, denoted by $Df(\xi, \eta, \mu, \nu; \hat{h}; u, k)$.

Remark 7.7. When the parameters ξ, η, μ in the Dl become zero (or -1), we recover Ince's series expansions (see formulas (2.1), (3.1), (4.1), (4.3), (5.1), (5.3), (6.1), (6.3) in [26]).

8 The 192 solutions of the Darboux equation

In general, it is well-known that if a Fuchsian equation on \mathbb{CP}^1 has one of the exponent differences at a singular point is an integer, then one of the local solutions corresponding to the singular point is generically logarithmic. For the Darboux equation, it has a logarithmic solution only if at least one of ξ, η, μ, ν is in $\frac{2\mathbb{Z}+1}{2}$. One such exceptional case for $Dl(\xi, \eta, \mu, \nu; h; u, k)$ was already mentioned in the last section. We shall not consider these degenerate cases henceforth.

Now using the automorphisms in $G \cong G_I \rtimes_{\Gamma} G_{II}$, we can generate $2 \times 2 \times 2 \times 24 = 192$ solutions of Darboux equation (1.2) in the following form when $\xi, \eta, \mu, \nu \notin \frac{2\mathbb{Z}+1}{2}$:

$$Dl(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \tau_{X_i}(u, k), \kappa_X(k)),$$

where $s_\xi, s_\eta, s_\mu = +$ or $-$; $X = I, A, B, C, D, E$; $i = 0, 1, 2, 3$. Note that the transformation $\sigma_{X_i}(\nu) \mapsto \sigma_{X_i}(\nu)^-$ does not give a new solution since by definition,

$$\begin{aligned} & Dl(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \tau_{X_i}(u, k), \kappa_X(k)) \\ &= Dl(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu)^-; h_X; \tau_{X_i}(u, k), \kappa_X(k)). \end{aligned}$$

This is because the parameter ν does not present in the expansion of Dl . Therefore each of 192 solutions can be identified by an element of the Coxeter group $D_4 \cong (\mathbb{Z}_2)^3 \rtimes_{\Gamma} G_{II}$.

Moreover, the 192 solutions split into 8 sets of 24 formally distinct but equivalent expressions, where each set defining the local solution corresponding to one of the two exponents in the neighborhood of one of the four singular points. For each $s, t = +$ or $-$ and $X = I, A, B, C, D, E$:

1. the local solutions at $u = 0$ with exponent $\xi + 1$:

$$Dl(\xi, \sigma_{X_0}(\eta)^s, \sigma_{X_0}(\mu)^t, \sigma_{X_0}(\nu); h_X; \tau_{X_0}(u, k), \kappa_X(k)),$$

2. the local solutions at $u = 0$ with exponent $-\xi$:

$$Dl(-\xi - 1, \sigma_{X_0}(\eta)^s, \sigma_{X_0}(\mu)^t, \sigma_{X_0}(\nu); h_X; \tau_{X_0}(u, k), \kappa_X(k)),$$

3. the local solutions at $u = K(k)$ with exponent $\eta + 1$:

$$Dl(\eta, \sigma_{X_1}(\eta)^s, \sigma_{X_1}(\mu)^t, \sigma_{X_1}(\nu); h_X; \tau_{X_1}(u, k), \kappa_X(k)),$$

4. the local solutions at $u = K(k)$ with exponent $-\eta$:

$$Dl(-\eta - 1, \sigma_{X_1}(\eta)^s, \sigma_{X_1}(\mu)^t, \sigma_{X_1}(\nu); h_X; \tau_{X_1}(u, k), \kappa_X(k)),$$

5. the local solutions at $u = K(k) + iK'(k)$ with exponent $\mu + 1$:

$$Dl(\mu, \sigma_{X_2}(\eta)^s, \sigma_{X_2}(\mu)^t, \sigma_{X_2}(\nu); h_X; \tau_{X_2}(u, k), \kappa_X(k)),$$

6. the local solutions at $u = K(k) + iK'(k)$ with exponent $-\mu$:

$$Dl(-\mu - 1, \sigma_{X_2}(\eta)^s, \sigma_{X_2}(\mu)^t, \sigma_{X_2}(\nu); h_X; \tau_{X_2}(u, k), \kappa_X(k)),$$

7. the local solutions at $u = iK'(k)$ with exponent $\nu + 1$:

$$Dl(\nu, \sigma_{X_3}(\eta)^s, \sigma_{X_3}(\mu)^t, \sigma_{X_3}(\nu); h_X; \tau_{X_3}(u, k), \kappa_X(k)),$$

8. the local solutions at $u = iK'(k)$ with exponent $-\nu$:

$$Dl(-\nu - 1, \sigma_{X_3}(\eta)^s, \sigma_{X_3}(\mu)^t, \sigma_{X_3}(\nu); h_X; \tau_{X_3}(u, k), \kappa_X(k)).$$

The above list gives all the 192 local series expansions for the Darboux equation.

The terminating conditions for the above 192 series can easily be obtained by applying the symmetry group $G \cong G_I \rtimes_{\Gamma} G_{II}$ to that for the Dl in (7.4). We record this result in the following theorem.

Theorem 8.1. *If there exists a positive integer q such that either*

$$\sigma_{X_i}(\xi)^{s\xi} + \sigma_{X_i}(\eta)^{s\eta} + \sigma_{X_i}(\mu)^{s\mu} + \sigma_{X_i}(\nu) = -2q - 4$$

or

$$\sigma_{X_i}(\xi)^{s\xi} + \sigma_{X_i}(\eta)^{s\eta} + \sigma_{X_i}(\mu)^{s\mu} - \sigma_{X_i}(\nu) = -2q - 3,$$

holds, and there exists a value of h such that $h_X(\xi, \eta, \mu, \nu; h; k)$ satisfies the finite continued fraction

$$f(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \kappa_X(k)) = 0,$$

then the series $Dl(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \tau_{X_i}(u, k), \kappa_X(k))$ terminates.

We record the remaining case when the infinite continued-fraction in (7.6) vanishes as

Theorem 8.2. *If h is chosen such that $h_X(\xi, \eta, \mu, \nu; h; k)$ satisfies the infinite continued fraction*

$$g(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \kappa_X(k)) = 0,$$

then the series $Dl(\sigma_{X_i}(\xi)^{s\xi}, \sigma_{X_i}(\eta)^{s\eta}, \sigma_{X_i}(\mu)^{s\mu}, \sigma_{X_i}(\nu); h_X; \tau_{X_i}(u, k), \kappa_X(k))$ converges on the larger domain $\{| \operatorname{sn} u | < \max(1, |\kappa_X(k)|^{-1})\}$ for $|\kappa_X(k)|^{-1} \neq 1$.

9 Special cases

Here we consider the symmetries of a number of special cases of the Darboux equation we have found. These special cases of differential equations have been studied by mathematicians since the beginning of the 19th century. We first return to the Lamé equation which we encountered in the Introduction.

9.1 Lamé equation: (Eqn(1.2) for $\xi, \eta, \mu = 0$ or -1)

The *Lamé equation*

$$\frac{d^2 y}{du^2} + (h - \nu(\nu + 1)k^2 \operatorname{sn}^2 u)y = 0, \quad (9.1)$$

with the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{l} iK'(k) \\ \nu + 1 \\ -\nu \end{array} \quad u; h \right\},$$

was first derived by Lamé in 1837 [48, XXIII] after separation of variables of Laplace's equation in three-space under the *ellipsoidal coordinates*. Since then, the equation has been worked on by well-known mathematicians such as Heine [20, (1878)], Hermite [21, (1878)], Hobson [24, (1892)] in the 19th century, and Ince [26, 25, (1940)] and Erdélyi [12, (1940)] in the 20th century. The Lamé equation has recently appeared in Knot theory [28], PDEs and algebraic geometry [7], Integrable systems [15, 16] and finite-gap theory [36] which is already mentioned.

Clearly, the Lamé equation remains unchanged under the transformation $\nu \mapsto \nu^\pm$, which induces the subgroup of G_I isomorphic to \mathbb{Z}_2 .

Moreover, the transformation A sends the half-period $\omega_3 = iK'$ to $\omega_2 = K + iK'$. Hence the transformation A_1 sends $\omega_3 = iK'$ back to $\omega_2 + \omega_1 = \omega_3$. This transformation A_1 fixed the singularity ω_3 and hence the form of the Lamé equation. Similarly, one reads off from Table 4.1 that there exist only six transformations $X_i = I_0, A_1, B_2, C_0, D_2, E_1$ such that the Lamé equation preserves the form, that is, under such six transformations X_i , the equation (9.1) becomes

$$\frac{d^2 y}{dw^2} + \left(h_X - \nu(\nu + 1)\kappa_X^2 \operatorname{sn}^2(w, \kappa_X) \right) y = 0, \quad (9.2)$$

where $w = \tau_{X_i}(u, k)$, κ_X and h_X are given in Table 4.1 and 4 respectively. These six transformations induce a subgroup of G_{II} isomorphic to S_3 .

9.2 Associated Lamé equation: (Eqn(1.2) for $\xi, \eta = 0$ or -1)

The associated Lamé equation

$$\frac{d^2 y}{du^2} + \left(h - \mu(\mu + 1)k^2 \operatorname{cd}^2(u, k) - \nu(\nu + 1)k^2 \operatorname{sn}^2(u, k) \right) y = 0 \quad (9.3)$$

has the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{ccc} K(k) + iK'(k) & iK'(k) & \\ \mu + 1 & \nu + 1 & u; h \\ -\mu & -\nu & \end{array} \right\}.$$

Clearly, the associated Lamé equation remains unchanged under $\mu \mapsto \mu^\pm$ and $\nu \mapsto \nu^\pm$, which induce the subgroup of G_I isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Moreover, it follows from Table 4.1 that there exist only four transformations $X_i = I_0, I_1, A_0, A_1$ such that the associated Lamé equation preserves the form, that is, under such four transformations X_i , the equation (9.3) becomes

$$\begin{aligned} \frac{d^2y}{dw^2} + \left(h_X - \sigma_{X_i}(\mu)(\sigma_{X_i}(\mu) + 1)\kappa_X^2 \operatorname{cd}^2(u, \kappa_X) \right. \\ \left. - \sigma_{X_i}(\nu)(\sigma_{X_i}(\nu) + 1)\kappa_X^2 \operatorname{sn}^2(w, \kappa_X) \right) y = 0, \end{aligned} \quad (9.4)$$

where $w = \tau_{X_i}(u, k)$, κ_X and h_X, σ_{X_i} are given in Table 2 and 4 respectively. These four transformations induce a subgroup of G_{II} isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

10 Other transformations of the Darboux equation

The exponent parameters ξ, η, μ, ν are unrestricted in the transformation in G_{II} . In the section, we study the other kinds of the transformations of Darboux equation with one of the following restrictions: either $(\xi = \eta, \mu = \nu)$ or $(\xi = \eta = \mu = \nu)$.

For the case $\xi = \eta, \mu = \nu$, we consider the Landen transformation $L : (u, k) \mapsto ((1 + k')u, \frac{1-k'}{1+k'})$ and under the transformation, we obtain the following

$$\begin{aligned} \operatorname{sn} \left((1 + k')u, \frac{1 - k'}{1 + k'} \right) &= \frac{(1 + k') \operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)}; \\ \operatorname{cn} \left((1 + k')u, \frac{1 - k'}{1 + k'} \right) &= \frac{1 - (1 + k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}; \\ \operatorname{dn} \left((1 + k')u, \frac{1 - k'}{1 + k'} \right) &= \frac{1 - (1 - k') \operatorname{sn}^2(u, k)}{\operatorname{dn}(u, k)}, \end{aligned}$$

(see [13, p. 372]). Then we can derive

$$\begin{aligned} (1 + k') \operatorname{dn} \left((1 + k')u, \frac{1 - k'}{1 + k'} \right) &= \operatorname{dn}(u, k) + k' \operatorname{nd}(u, k); \\ (1 + k') \operatorname{cs} \left((1 + k')u, \frac{1 - k'}{1 + k'} \right) &= \operatorname{cs}(u, k) - k' \operatorname{sc}(u, k). \end{aligned}$$

Thus, let $w = (1 + k')u$ and $\kappa^* = \frac{1-k'}{1+k'}$, we have

$$(1 + k')^2 \kappa^{*2} \operatorname{sn}^2(w, \kappa^*) + k^2 = k^2 \operatorname{sn}^2(u, k) + k^2 \operatorname{cd}^2(u, k);$$

$$(1 + k')^2 \kappa^{*2} \operatorname{ns}^2(w, \kappa^*) + k^2 = \operatorname{ns}^2(u, k) + \operatorname{dc}^2(u, k).$$

So under the Landen transformation L , the Darboux equation

$$\frac{d^2 y}{du^2} + \left(h - \xi(\xi + 1) \operatorname{ns}^2(u, k) - \xi(\xi + 1) \operatorname{dc}^2(u, k) - \nu(\nu + 1) k^2 \operatorname{cd}^2(u, k) - \nu(\nu + 1) k^2 \operatorname{sn}^2(u, k) \right) y = 0$$

becomes

$$\frac{d^2 y}{dw^2} + \left(h^* - \xi(\xi + 1) \operatorname{ns}^2(w, \kappa^*) - \nu(\nu + 1) \kappa^{*2} \operatorname{sn}^2(w, \kappa^*) \right) y = 0,$$

where $h^* = [h - k^2 \xi(\xi + 1) - k^2 \nu(\nu + 1)] / (1 + k')^2$. Therefore, we obtain

Theorem 10.1 (Landen Transformation Formula). *For $\xi = -\frac{3}{2}, -\frac{5}{2}, \dots$, we have*

$$Dl\left(\xi, 0, 0, \nu; h^*; (1 + k')u, \frac{1 - k'}{1 + k'}\right) = (1 + k')^{\xi+1} Dl(\xi, \xi, \nu, \nu; h; u, k),$$

where $h^* = [h - k^2 \xi(\xi + 1) - k^2 \nu(\nu + 1)] / (1 + k')^2$.

For the case $\xi = \eta = \mu = \nu$, it follows from the duplication formula for Jacobi elliptic function

$$\operatorname{sn}^2(u, k) = \frac{1 - \operatorname{cn}(2u, k)}{1 + \operatorname{dn}(2u, k)}$$

that

$$\operatorname{ns}^2(u, k) + \operatorname{dc}^2(u, k) + k^2 \operatorname{cd}^2(u, k) + k^2 \operatorname{sn}^2(u, k) = 4 \operatorname{ns}^2(2u, k).$$

So the Darboux equation

$$\frac{d^2 y}{du^2} + \left(h - \xi(\xi + 1) \operatorname{ns}^2(u, k) - \xi(\xi + 1) \operatorname{dc}^2(u, k) - \xi(\xi + 1) k^2 \operatorname{cd}^2(u, k) - \xi(\xi + 1) k^2 \operatorname{sn}^2(u, k) \right) y = 0$$

becomes

$$\frac{d^2 y}{d\tilde{u}^2} + \left(h/4 - \xi(\xi + 1) \operatorname{ns}^2(\tilde{u}, k) \right) y = 0, \quad (10.1)$$

where $\tilde{u} = 2u$.

Remark 10.2. Note that this form (10.1) of Lamé equation has been discussed in Whittaker-Watson [48, p. 555].

Thus we obtain

Theorem 10.3 (Duplication Formula). *For $\xi = -\frac{3}{2}, -\frac{5}{2}, \dots$, we have*

$$Dl(\xi, 0, 0, 0; h/4; 2u, k) = 2^{\xi+1} Dl(\xi, \xi, \xi, \xi; h; u, k).$$

A Results on three-term recursion relations

In this section, we review some useful results about three-term recursion relations. We refer to the readers to Gautschi [14] for more details.

Given the three-term recursion

$$R_r C_{r+1} + S_r C_r + P_r C_{r-1} = 0 \quad (r = 0, 1, 2, \dots),$$

where $C_{-1} = 0$ and $R_r \neq 0$ for all $r = 0, 1, 2, \dots$. Assume that $\lim_{r \rightarrow \infty} P_r := P$, $\lim_{r \rightarrow \infty} S_r := S$ and $\lim_{r \rightarrow \infty} R_r := R$ exist. The limit of C_{r+1}/C_r ($r \rightarrow \infty$) can be determined by Poincaré's Theorem and Perron's Theorem.

Theorem A.1 (Poincaré's Theorem (see [39] or [35, p.527])). *The limit*

$$\lim_{r \rightarrow \infty} \frac{C_{r+1}}{C_r} = t_1 \quad \text{or} \quad t_2,$$

where t_1 and t_2 are the roots of the quadratic equation $Rt^2 + St + P$.

Theorem A.2 (Perron's Theorem (see [37, §57])). *Suppose that $|t_1| < |t_2|$. If the infinite continued fraction*

$$S_0/R_0 - \frac{P_1/R_1}{S_1/R_1 -} \frac{P_2/R_2}{S_2/R_2 -} \dots = 0$$

holds, then

$$\lim_{r \rightarrow \infty} \left| \frac{C_{r+1}}{C_r} \right| = t_1.$$

Otherwise,

$$\lim_{r \rightarrow \infty} \left| \frac{C_{r+1}}{C_r} \right| = t_2.$$

B Asymmetric form of Sparre equation

Sparre ([41]) in 1883 discovered the following related equation

$$\begin{aligned} \frac{d^2 \tilde{y}}{du^2} + \left[2\ell \frac{k^2 \operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} + 2\ell_1 \frac{\operatorname{sn}(u, k) \operatorname{dn}(u, k)}{\operatorname{cn}(u, k)} - 2\ell_2 \frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{\operatorname{sn}(u, k)} \right] \frac{d\tilde{y}}{du} \\ + \left[\tilde{h} - (\xi - \ell_2)(\xi + \ell_2 + 1) \operatorname{ns}^2(u, k) \right. \\ \left. - (\eta - \ell_1)(\eta + \ell_1 + 1) \operatorname{dc}^2(u, k) - (\mu - \ell)(\mu + \ell + 1) k^2 \operatorname{cd}^2(u, k) \right. \\ \left. - (\nu + \ell + \ell_1 + \ell_2)(\nu - \ell - \ell_1 - \ell_2 + 1) k^2 \operatorname{sn}^2(u, k) \right] \tilde{y} = 0, \end{aligned} \tag{B.1}$$

where the Riemann P -scheme

$$P \left\{ \begin{array}{cccc} 0 & K(k) & K(k) + iK'(k) & iK'(k) \\ \xi + 1 + \ell_2 & \eta + 1 + \ell_1 & \mu + 1 + \ell & \nu + 1 - \ell - \ell_1 - \ell_2 & u; \tilde{h} \\ -\xi + \ell_2 & -\eta + \ell_1 & -\mu + \ell & -\nu - \ell - \ell_1 - \ell_2 \end{array} \right\}.$$

The Sparre equation (B.1) can be normalized into the following two versions:

1. (Symmetric form) the Darboux equation (1.2) by using the transformation $\tilde{y} = (\operatorname{sn} u)^{\ell_2} (\operatorname{cn} u)^{\ell_1} (\operatorname{dn} u)^{\ell} y$ (see [34, §3]);
2. (Asymmetric form)

$$\begin{aligned} \frac{d^2 y}{du^2} + \left[2\mu \frac{k^2 \operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} + 2\eta \frac{\operatorname{sn}(u, k) \operatorname{dn}(u, k)}{\operatorname{cn}(u, k)} - 2\xi \frac{\operatorname{cn}(u, k) \operatorname{dn}(u, k)}{\operatorname{sn}(u, k)} \right] \frac{dy}{du} \\ + \left[h - (\nu + \xi + \eta + \mu)(\nu - \xi - \eta - \mu + 1)k^2 \operatorname{sn}^2(u, k) \right] y = 0, \end{aligned} \quad (\text{B.2})$$

where the Riemann P -scheme

$$P \left\{ \begin{array}{cccc} 0 & K(k) & K(k) + iK'(k) & iK'(k) \\ 0 & 0 & 0 & \nu - \xi - \eta - \mu + 1 & u; h \\ 2\xi + 1 & 2\eta + 1 & 2\mu + 1 & -\nu - \xi - \eta - \mu & \end{array} \right\},$$

by using the transformation $\tilde{y} = (\operatorname{sn} u)^{\ell_2 - \xi} (\operatorname{cn} u)^{\ell_1 - \eta} (\operatorname{dn} u)^{\ell - \mu} y$.

The automorphism group of the Sparre equation in asymmetric form is isomorphic to the Coxeter group $D_4 \cong (\mathbb{Z}_2)^3 \rtimes S_4$ which is induced by the transformations $w = \tau_{X_i}(u, k)$ (change of the independent variable from u to w) by the map X_i in Table 4.1, followed by the transformations $y(w) = (\operatorname{sn} w)^{n_1} (\operatorname{cn} w)^{n_2} (\operatorname{dn} w)^{n_3} z(w)$ (change of the dependent variable from $y(w)$ to $z(w)$) for two suitable values of n_1, n_2, n_3 such that the Sparre equation preserves the asymmetric form. (Here the transformation interchanging the exponents at $iK'(k)$ is excluded.)

B.1 Special cases for the asymmetric form

B.1.1 Picard equation [38]:(Eqn(B.2) for $\mu = -\nu = n/2$, $\eta = \xi = 0$, $\alpha = h$)

The Picard equation

$$\frac{d^2 y}{du^2} + nk^2 \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} \frac{dy}{du} + \alpha y = 0 \quad (\text{B.3})$$

has the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{ccc} K(k) + iK'(k) & iK'(k) & \\ 0 & 0 & u; \alpha \\ 1 + n & 1 - n & \end{array} \right\}.$$

It follows from Table 4.1 that there exist only four transformations $X_i = I_0, I_1, A_0, A_1$ such that the Picard equation preserves the form, that is, under such four transformations X_i , the equation (B.3) becomes

$$\frac{d^2 y}{dw^2} + n_{X_i} \kappa_X^2 \frac{\operatorname{sn}(w, \kappa_X) \operatorname{cn}(w, \kappa_X)}{\operatorname{dn}(w, \kappa_X)} \frac{dy}{dw} + \alpha_X y = 0, \quad (\text{B.4})$$

Table 6:

X_i	$n_{X_i}(n)$	$\alpha_X(\alpha, k)$
I_0	n	α
I_1	$-n$	
A_0	$-n$	$\alpha/(k')^2$
A_1	n	

where $w = \tau_{X_i}(u, k)$, κ_X are given in Table 4.1 and $n_{X_i}(n), \alpha_X(\alpha, k)$ are given in Table 6. These four transformations induce the automorphism group of the Picard equation isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark B.1. The transformations $y(w) = (\operatorname{sn} w)^{n_1} (\operatorname{cn} w)^{n_2} (\operatorname{dn} w)^{n_3} z(w)$ for any values n_1, n_2, n_3 do not preserve the form of Picard equation.

B.1.2 Hermite equation [22]:(Eqn(B.2) for $\mu = \lambda + 1, \nu = n, \eta = \xi = 0$)

The Hermite equation

$$\frac{d^2 y}{du^2} + 2(\lambda + 1)k^2 \frac{\operatorname{sn}(u, k) \operatorname{cn}(u, k)}{\operatorname{dn}(u, k)} \frac{dy}{du} - (h + (n - \lambda)(n + \lambda + 1)k^2 \operatorname{sn}^2(u, k))y = 0 \quad (\text{B.5})$$

has the Riemann P -scheme

$$P_{\mathbb{C}/\Lambda} \left\{ \begin{array}{ccc} K(k) + iK'(k) & iK'(k) & \\ 0 & n - \lambda & u; h \\ 2\lambda + 3 & -n - \lambda - 1 & \end{array} \right\}.$$

The automorphism group of the Hermite equation has three generators induced by the following transformations (preserving the form of the equation):

1. $w = \tau_{A_1}(u, k)$ (change of the independent variable from u to w) by the map A_1 in Table 4.1;
2. $y(u) = (\operatorname{dn} u)^{2\lambda+3} z(u)$ (change of the dependent variable from $y(u)$ to $z(u)$);
3. $w = \tau_{A_0}(u, k)$ (change of the independent variable from u to w) by the map A_0 given in Table 4.1, followed by $y(w) = (\operatorname{dn} w)^{-n-\lambda-1} z(w)$ (change of the dependent variable from $y(w)$ to $z(w)$).

Since each generator has order two, the automorphism group is isomorphic to $(\mathbb{Z}_2)^3$.

Remark B.2. Both Picard and Hermite equations can be normalized into the form of the associated Lamé equation (9.3) by a suitable transformation.

C Elliptic functions

Let $\wp(z) = \wp(z|\omega, \omega')$ be the Weierstrass elliptic function of periods $2\omega, 2\omega'$ with a double pole at $z = 0$, where $\text{Im}(\omega'/\omega) > 0$. Let

$$\omega_1 = \omega, \quad \omega_2 = -\omega - \omega', \quad \omega_3 = \omega',$$

$$e_i = \wp(\omega_i), \quad i = 1, 2, 3.$$

Now for the Jacobi elliptic function, the modulus k and the complementary modulus k' can be expressed in terms of e_1, e_2, e_3 :

$$k = \frac{(e_2 - e_3)^{1/2}}{(e_1 - e_3)^{1/2}}, \quad k' = \frac{(e_1 - e_2)^{1/2}}{(e_1 - e_3)^{1/2}}.$$

Let

$$u = (e_1 - e_3)^{1/2} z.$$

Then

$$\text{sn}(u, k) = \frac{(e_1 - e_3)^{1/2}}{[\wp(z) - e_3]^{1/2}}, \quad \text{cn}(u, k) = \frac{[\wp(z) - e_1]^{1/2}}{[\wp(z) - e_3]^{1/2}}, \quad \text{dn}(u, k) = \frac{[\wp(z) - e_2]^{1/2}}{[\wp(z) - e_3]^{1/2}}.$$

and the other nine Jacobi elliptic functions are defined to be the reciprocals of these three functions and the quotients of any two of them. Also, the quarter periods

$$K(k) = (e_1 - e_3)^{1/2} \omega, \quad iK'(k) = (e_1 - e_3)^{1/2} \omega'.$$

D Semi-direct products

Definition D.1. Given any two groups N and H and a group homomorphism $\Gamma : H \rightarrow \text{Aut}(N)$, we can construct a new group $N \rtimes_{\Gamma} H$, called the (outer) semidirect product of N and H with respect to Γ , defined as follows:

1. As a set, $N \rtimes_{\Gamma} H$ is the cartesian product $N \times H$;
2. Multiplication of elements in $N \rtimes_{\Gamma} H$ is determined by the homomorphism Γ . The operation is

$$* : (N \rtimes_{\Gamma} H) \times (N \rtimes_{\Gamma} H) \rightarrow N \rtimes_{\Gamma} H$$

defined by

$$(n_1, h_1) * (n_2, h_2) = (n_1 \Gamma_{h_1}(n_2), h_1 h_2)$$

for $n_1, n_2 \in N$ and $h_1, h_2 \in H$.

The idea of semi-direct product can be expressed in the language of exact sequence also, which is the next definition.

Definition D.2. Let L, M, N be groups. A short exact sequence is the collection of group homomorphisms

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

such that the image of each of these homomorphisms equals to the kernel of the next homomorphism.

Definition D.3. Two exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & M & \rightarrow & N \rightarrow 0 \\ 0 & \rightarrow & L & \rightarrow & M' & \rightarrow & N \rightarrow 0 \end{array}$$

are equivalent if there exists a homomorphism $M \rightarrow M'$ such that

$$\begin{array}{ccccccc} 0 & \rightarrow & L & \rightarrow & M & \rightarrow & N \rightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & L & \rightarrow & M' & \rightarrow & N \rightarrow 0 \end{array}$$

is a commutative diagram. It can be shown that equivalence of exact sequences is an equivalence relation.

Definition D.4. Given an exact sequence

$$0 \rightarrow L \rightarrow M \xrightarrow{\pi} N \rightarrow 0,$$

a choice of splitting (or a section) is a homomorphism $N \xrightarrow{\iota} M$ such that $\pi \circ \iota$ is the identity map of N .

Example D.5. Fix $n \in \mathbb{N}$ and let R be the group of rigid motions in \mathbb{R}^n (that is, the group of continuous maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ which preserve the Euclidean norm). Then there is an exact sequence

$$0 \rightarrow \mathbb{R}^n \rightarrow R \rightarrow O(n) \rightarrow 0.$$

This is just another way of saying that R is a semi-direct product of \mathbb{R}^n (identified as the group of translations) and $O(n)$. A choice of section is easy also, which is simply the inclusion $O(n) \rightarrow R$.

The languages of exact sequence (with splitting) and semi-direct product are equivalent, in the sense of the following identification.

Theorem D.6 (see [5, p.65-68]). *Given two groups N and H . There is a one-to-one correspondence between the isomorphism classes of semi-direct products $N \rtimes H$ and the equivalence classes of exact sequences $0 \rightarrow N \rightarrow \dots \rightarrow H \rightarrow 0$ with a section.*

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