Kumamoto J. Math.
Vol.**32** (2019), 1–8

Asymptotic expression for certain hypergeometric polynomials of type ${}_3F_1$

Shotaro Nakai and Hideshi Yamane

(Received January 22, 2019)
(Accepted June 21, 2019)

Abstract. We obtain an asymptotic expression for certain hypergeometric polynomials of type ${}_3F_1$ on a segment in the complex plane. This segment was excluded in the article by Neuschel.

In [5], Neuschel studied the asymptotic expansion of the polynomials $F_n(z)$ defined by

$$F_n(z) = {}_3F_1\left(\begin{matrix} -n & n & \alpha \\ & 1/2 & \end{matrix} \middle| \frac{z}{2n} \right),$$

where α is a positive integer and

$${}_3F_1\left(\begin{matrix} -n & n & \alpha \\ & 1/2 & \end{matrix} \middle| z \right) = \sum_{k=0}^{n} \frac{(-n)_k (n)_k (\alpha)_k}{(1/2)_k k!} z^k.$$

The polynomials $F_n(z)$ are concerned with what are called Ménage polynomials that appear in combinatorics ([5]).

Calculating the behavior of $F_n(z)$ as $n \to \infty$ is a part of the vast field of asymptotic analysis of hypergeometric quantities. One can find many formulas and references in [2]. Relatively little is known about ${}_3F_1$ and [5] is one of rare major results.

Set

$$[-i, i] = \{it | -1 \le t \le 1\}, \mathbb{D} = \{z \in \mathbb{C} | \, |z| < 1\}, \overline{\mathbb{D}}^c = \{z \in \mathbb{C} | \, |z| > 1\}.$$

Let the mapping

$$\mathbb{C} \setminus [-i, i] \to \overline{\mathbb{D}}^c, z \mapsto z + \sqrt{z^2 + 1} \tag{1}$$

be defined as the inverse mapping of the conformal mapping (a variant of the Joukowsky transformation)

$$\overline{\mathbb{D}}^c \to \mathbb{C} \setminus [-i, i], w \mapsto \frac{1}{2}\left(w - \frac{1}{w} \right).$$

Mathematical Subject Classification (2010): Primary 33C20; Secondary 34E10
Key words: hypergeometric polynomial, asymptotic expansion

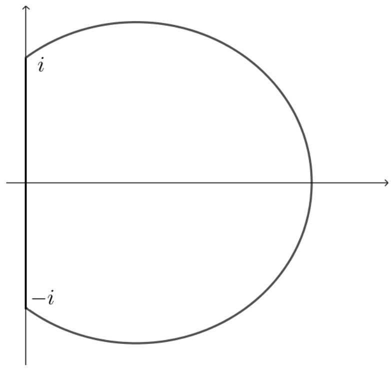

Figure 1: The curve $\mathcal{C}$

The mapping

$$\begin{aligned}\varphi(z) &= \left(z+\sqrt{z^2+1}\right)\exp\left(\frac{-2}{z+\sqrt{z^2+1}-1}-1\right)\\ &= \left(z+\sqrt{z^2+1}\right)\exp\left(-\frac{1}{z}-\frac{\sqrt{z^2+1}}{z}\right)\end{aligned}$$

is defined accordingly. We stipulate that the value of the mapping (1) for $z = iy \in [-i,i]\,(-1 \le y \le 1)$ is $iy+\sqrt{1-y^2}$, which is on the right half of the unit circle. The curve $\mathcal{C}$ is defined by $|\varphi(z)| = 1$ and it contains the line segment $[-i,i]$. Although $\mathcal{C}$ is continuous, it is not differentiable at $\pm i$.

Let $\mathcal{E}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ be the exterior and the interior respectively. Then the main result of [5] is the following. In $\mathcal{E}(\mathcal{C})$, one has

$$\begin{aligned}F_n(z) &= {}_3F_1\left(\begin{matrix}-n & n & \alpha\\ & 1/2 & \end{matrix}\,\middle|\,\frac{z}{2n}\right)\\ &\sim \frac{(-1)^n}{\Gamma(\alpha)}n^{\alpha-\frac{1}{2}}\sqrt{\frac{\pi}{2}}\left(\frac{1}{z}+\frac{\sqrt{z^2+1}}{z}\right)^{\alpha-1}\left(\frac{\sqrt{z^2+1}}{z}\right)^{-\frac{1}{2}}\varphi(z)^n\end{aligned}$$

as $n\to\infty$. Notice that $|\varphi(z)| > 1$ in $\mathcal{E}(\mathcal{C})$. On the other hand, in $\mathcal{I}(\mathcal{C})$ one has

$$F_n(z) = {}_3F_1\left(\begin{matrix}-n & n & \alpha\\ & 1/2 & \end{matrix}\,\middle|\,\frac{z}{2n}\right) \sim \left(\frac{2}{n}\right)^{\alpha}\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}}\left(-\frac{1}{z}\right)^{\alpha}.$$

The behavior on $\mathcal{C}$ remained to be an open problem. In the present paper, we give some information about the case of $\alpha = 1, z \in [-i,i] \subset \mathcal{C}$. Notice that we have $F_n(0) = 1$.

1 Finite Fourier transform of the Chebyshev polynomials

The Jacobi polynomials are defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{\alpha+n}{k}\binom{\beta+n}{n-k}(x-1)^{n-k}(x+1)^k \qquad (\alpha,\beta>-1). \end{aligned}$$

The Chebyshev polynomials are

$$T_n(x) = \frac{(-1)^n}{(2n-1)!!}(1-x^2)^{1/2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2} = \frac{n!}{\left(\frac{1}{2}\right)_n}P_n^{(-1/2,-1/2)}(x),$$

and we have

$$T_n(\cos\theta) = \cos n\theta.$$

According to [4], the finite Fourier transform of the Jacobi polynomials is given by

$$\begin{aligned} &\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)e^{i\lambda t}\,dt \\ &= \frac{(\beta+1)_n}{i\lambda n!}(-1)^{n+1}e^{-i\lambda}\,{}_3F_1\left(\begin{matrix} n+\alpha+\beta+1 & -n & 1 \\ & \beta+1 & \end{matrix}\middle|\frac{-1}{2i\lambda}\right) \\ &\quad + \frac{(\alpha+1)_n}{i\lambda n!}e^{i\lambda}\,{}_3F_1\left(\begin{matrix} n+\alpha+\beta+1 & -n & 1 \\ & \alpha+1 & \end{matrix}\middle|\frac{1}{2i\lambda}\right). \end{aligned}$$

Therefore by setting $\alpha=\beta=-1/2$, $\lambda=-n/y$, we get

$$S = -\frac{in}{y}\int_{-1}^{1} T_n(t)e^{-int/y}\,dt, \tag{2}$$

where

$$S := (-1)^{n+1}e^{in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\middle|\frac{-iy}{2n}\right) + e^{-in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\middle|\frac{iy}{2n}\right).$$

If y is real, we have

$$S = \begin{cases} 2i\mathrm{Im}\left\{e^{-in/y}{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\middle|\frac{iy}{2n}\right)\right\} & (n:\text{even}), \\ 2\mathrm{Re}\left\{e^{-in/y}{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\middle|\frac{iy}{2n}\right)\right\} & (n:\text{odd}). \end{cases} \tag{3}$$

Our aim is to calculate the asymptotic behavior of S by using (2). If $0<|y|\le 1$, the value of ${}_3F_1$ corresponds to the polynomial studied in [5] with $z=iy\in[-i,i]\setminus\{0\}$, $\alpha=1$. Notice that we have ${}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\middle|0\right)=1$.

2 Method of stationary phase

We recall some known facts about the method of stationary phase following [1]. More detailed accounts can be found in [3]. We consider the asymptotic behavior of the integral

$$I[n] = \int_a^b f(t)e^{in\phi(t)}dt \quad (a < b)$$

as $n \to \infty$. We assume that f and ϕ are sufficiently smooth on $[a, b]$ and ϕ is real-valued. In the typical case where $\phi(t) = t$, we get $I[n] = O(1/n)$ by integration by parts. This is a special case of the Riemann-Lebesgue Lemma.

If $\phi'(c) = 0$, then $t = c$ is called a *stationary point*. If there is no stationary point, we have the following variant of the Riemann-Lebesgue Lemma.

Proposition 1. *If $\phi'(t)$ never vanishes on $[a, b]$, then $I[n] = O(1/n)$.*

Proof. The function $\phi\colon [a, b] \to [\min(\phi(a), \phi(b)), \max(\phi(a), \phi(b))]$ is a bijection and its inverse ϕ^{-1} is smooth. Set $s = \phi(t)$. Then

$$I[n] = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(s))e^{ins}d\phi^{-1}(s) = \int_{\phi(a)}^{\phi(b)} \frac{f(\phi^{-1}(s))}{\phi'(\phi^{-1}(s))}e^{ins}ds.$$

Since f/ϕ' is smooth, we have $I[n] = O(1/n)$.

Next we consider the cases of one or more stationary points.

Proposition 2. *Assume $c \in]a, b[$ is the only stationary point of ϕ on $[a, b]$. Furthermore, assume $\phi''(c) \neq 0$ and $f(c) \neq 0$. Then*

$$I[n] \sim \sqrt{\pi}e^{\frac{i\pi}{4}\mu}f(c)e^{in\phi(c)}\left(\frac{2}{n|\phi''(c)|}\right)^{\frac{1}{2}},$$

where $\mu = \mathrm{sgn}\phi''(c)$.

Proof. This formula can be found in [1, Lemma 6.3.3]. Although f is assumed to vanish infinitely smoothly at the two endpoints $t = a, b$ in [1], this assumption can be removed by using a cut-off function. Let $\psi(t)$ be a smooth function such that $\psi(t) = 1$ on $|t - c| \leq \varepsilon$ and $\psi(t) = 0$ on $|t - c| \geq 2\varepsilon$, where $\varepsilon > 0$ is sufficiently small. Set

$$J[n] = \int_a^b \psi(t)f(t)e^{in\psi(t)}dt.$$

Since $\psi(t)f(t)$ vanishes infinitely smoothly at the two endpoints, we can calculate the asymptotic behavior of $J[n]$. What remains to be proved is that $I[n] - J[n]$ can be neglected. We have

$$\begin{aligned} I[n] - J[n] &= \int_a^b (1 - \psi(t))f(t)e^{in\phi(t)}\,dt \\ &= \int_a^{c-\varepsilon} (1 - \psi(t))f(t)e^{in\phi(t)}\,dt + \int_{c+\varepsilon}^b (1 - \psi(t))f(t)e^{in\phi(t)}\,dt. \end{aligned}$$

Each of these two integrals is of order $O(1/n)$ by Proposition 1.

If there are two stationary points c_1 and c_2, we have only to split $I[n]$ into a sum of two integrals: $I[n] = \int_a^{(c_1+c_2)/2} + \int_{(c_1+c_2)/2}^b$. Each of the two intervals $a \le t \le (c_1+c_2)/2$ and $(c_1+c_2)/2 \le t \le b$ contains only one stationary point and we can apply Proposition 2.

Next we consider the case of a stationary point of higher order. First we assume that it coincides with the left endpoint.

Proposition 3. *If $\phi'(a) = \cdots = \phi^{(p-1)}(a) = 0$, $\phi^{(p)}(a) \neq 0$, $f(a) \neq 0$ and $\phi'(t) \neq 0$ for all $t \in]a, b]$, then*

$$I[n] \sim \frac{\Gamma(1/p)}{p} e^{\frac{i\pi}{2p}\mu} f(a) e^{in\phi(a)} \left(\frac{p!}{n|\phi^{(p)}(a)|} \right)^{\frac{1}{p}}, \tag{4}$$

where $\mu = \mathrm{sgn}\phi^{(p)}(a)$.

Proof. See [1, Example 6.3.5].

We can derive an formula about the case of the right endpoint. Set $s = a+b-t$. It exchanges the left and right endpoints. Let $g(s) = f(t) = f(a+b-s)$, $\psi(s) = \phi(t) = \phi(a+b-s)$. Then $\psi'(b) = \cdots = \psi^{(p-1)}(b) = 0$, $\psi^{(p)}(b) \neq 0$ and $\psi'(t) \neq 0$ for all $t \in [a, b[$, and

$$I[n] = \int_a^b g(s) e^{in\psi(s)} ds.$$

We can rewrite (4) and get

$$\int_a^b g(s) e^{in\psi(s)} ds \sim \frac{\Gamma(1/p)}{p} e^{\frac{i\pi}{2p}\mu} g(b) e^{in\psi(b)} \left(\frac{p!}{n|\psi^{(p)}(b)|} \right)^{\frac{1}{p}}, \tag{5}$$

where $\mu = \mathrm{sgn}(-1)^p \psi^{(p)}(b)$. Since we will be interested in the case $p = 3$, the factor $(-1)^p$ must not be missed.

Finally we consider the case of a higher order stationary point in $]a, b[$. Let $c \in]a, b[$ be a higher order stationary point and assume that $\phi'(t)$ never vanishes elsewhere. Then we split $I[n]$ into the sum of $\int_a^c$ and $\int_c^b$. We can apply (4) to the second integral and (5) to the first.

3 Asymptotic expansion on $[-i, i]$

In view of (2), it is enough to calculate

$$I_n = \int_{-1}^1 T_n(t) \exp\left(-\frac{int}{y}\right) dt,$$

when $-1 \le y \le 1, y \neq 0$. Set $t = \cos\theta \, (0 \le \theta \le \pi)$. Then we have

$$I_n = \frac{1}{2}(I_n^+ + I_n^-), \quad I_n^{\pm} = \int_0^\pi \exp\left(in\left[-\frac{\cos\theta}{y} \pm \theta\right]\right) \sin\theta \, d\theta. \tag{6}$$

Now we apply the method of stationary phase as opposed to the saddle point method employed in [5]. Set

$$\varphi_{\pm}(\theta) = -\frac{\cos\theta}{y} \pm \theta,$$

then $\varphi'_{\pm}(\theta) = y^{-1}(\sin\theta \pm y)$, $\varphi''_{\pm}(\theta) = y^{-1}\cos\theta$. If $0 < y \leq 1$, $\varphi'_{+}(\theta)$ never vanishes.

3.1 Behavior at the interior of the line segment

We consider the case $z = iy\,(0 < y < 1)$. We have $I_n^+ = O(1/n)$ because φ'_+ never vanishes. This implies, by (2) and (6),

$$S = -\frac{in}{y} I_n = -\frac{in}{2y} I_n^- + O(1). \tag{7}$$

On the other hand, the asymptotic behavior of I_- can be calculated by using the method of stationary phase. The phase function φ_- has two stationary points $\sin^{-1} y$, $\pi - \sin^{-1} y$ on $0 \leq \theta \leq \pi$. We have

$$\begin{aligned}
&\varphi_-(\sin^{-1} y) = -\sqrt{1-y^2}/y - \sin^{-1} y,\\
&\varphi_-(\pi - \sin^{-1} y) = \sqrt{1-y^2}/y + \sin^{-1} y - \pi,\\
&\varphi''_-(\sin^{-1} y) = \sqrt{1-y^2}/y,\ \varphi''_-(\pi - \sin^{-1} y) = -\sqrt{1-y^2}/y,
\end{aligned}$$

Summing up the contribution from the two stationary points (Proposition 2), we obtain

$$I_n^- \sim \begin{cases} 2y\left(\dfrac{2\pi y}{n\sqrt{1-y^2}}\right)^{1/2} \cos\left[n\left(\frac{\sqrt{1-y^2}}{y} + \sin^{-1} y\right) - \dfrac{\pi}{4}\right] & (n : \text{even}),\\ -2iy\left(\dfrac{2\pi y}{n\sqrt{1-y^2}}\right)^{1/2} \sin\left[n\left(\frac{\sqrt{1-y^2}}{y} + \sin^{-1} y\right) - \dfrac{\pi}{4}\right] & (n : \text{odd}). \end{cases}$$

Therefore we obtain, by (3) and (7), the following result.

Theorem 4. *For each y such that $0 < y < 1$, we have, as $n \to \infty$,*

$$\begin{aligned}
&\mathrm{Im}\left\{e^{-in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\,\middle|\,\frac{iy}{2n}\right)\right\}\\
&\sim -\left(\frac{n\pi y}{2\sqrt{1-y^2}}\right)^{1/2} \cos\left[n\left(\frac{\sqrt{1-y^2}}{y} + \sin^{-1} y\right) - \frac{\pi}{4}\right]\ (n : \textit{even}),\\
&\mathrm{Re}\left\{e^{-in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\,\middle|\,\frac{iy}{2n}\right)\right\}\\
&\sim -\left(\frac{n\pi y}{2\sqrt{1-y^2}}\right)^{1/2} \sin\left[n\left(\frac{\sqrt{1-y^2}}{y} + \sin^{-1} y\right) - \frac{\pi}{4}\right]\ (n : \textit{odd}).
\end{aligned}$$

The behavior on $-1 < y < 0$ can be obtained by complex conjugation.

Only either the real or the imaginary part has been calculated here. This is less than satisfactory but at least it has been proved that the asymptotic behavior is different from that in $\mathcal{E}(\mathcal{C})$ (power-times-exponential growth with oscillation) and $\mathcal{I}(\mathcal{C})$ (decay of order $-\alpha = -1$ with no oscillation). In the next section, we will see still another type of behavior at $z = \pm i$. Nothing is known about the remaining part of $\mathcal{C}$.

3.2 Behavior at the end points

Assume $z = i\,(y = 1)$. Since

$$\varphi_-(\pi/2) = -\pi/2,\ \varphi'_-(\pi/2) = \varphi''_-(\pi/2) = 0,\ \varphi'''_-(\pi/2) = -1,$$

we get, by using (4) and (5),

$$I_n^- \sim \frac{\Gamma(1/3)(-i)^n}{\sqrt{3}}\left(\frac{6}{n}\right)^{1/3}.$$

Therefore, we have the following.

Theorem 5. *The asymptotic behavior at $z = i$ as $n \to \infty$ is*

$$\mathrm{Im}\left\{e^{-in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\,\middle|\,\frac{i}{2n}\right)\right\} \sim -\frac{6^{1/3}\Gamma(1/3)(-1)^{n/2}}{4\sqrt{3}y}n^{2/3}\ (n : \textit{even}),$$
$$\mathrm{Re}\left\{e^{-in/y}\,{}_3F_1\left(\begin{matrix} n & -n & 1 \\ & 1/2 & \end{matrix}\,\middle|\,\frac{i}{2n}\right)\right\} \sim \frac{6^{1/3}\Gamma(1/3)(-1)^{(n+1)/2}}{4\sqrt{3}y}n^{2/3}\ (n : \textit{odd}).$$

The behavior at $z = -i$ can be obtained by complex conjugation.

Acknowledgement

This work was partially supported by JSPS KAKENHI Grant Number 26400127.

References

[1] Mark J. Ablowitz and Athanassios S. Fokas, Complex Variables: introduction and applications, Cambridge University Press, 1997.

[2] NIST Digital Library of Mathematical Functions, `https://dlmf.nist.gov/`

[3] A. Erdélyi, Asymptotic expansions, Dover, 1956.

[4] Atul Dixit, Lin Jiu, Victor H. Moll and Christophe Vignat, The finite Fourier transform of classical polynomials, *J. Aust. Math. Soc.* **98** (2015), 145–160.

[5] Thorsten Neuschel, Asymptotics for Ménage polynomials and certain hypergeometric polynomials of type ${}_3F_1$, *J. Approx. Theory*, **164** (2012), 981-1006.

Shotaro Nakai
Graduate School of Science and Technology
Kwansei Gakuin University
Gakuen 2-1 Sanda, Hyogo 669-1337
Japan

Hideshi Yamane
School of Science and Technology
Kwansei Gakuin University
Gakuen 2-1 Sanda, Hyogo 669-1337
Japan
e-mail: yamane@kwansei.ac.jp