Decay estimate and asymptotic behavior of the solutions to dissipative nonlinear Schrödinger equations with t-dependent amplification

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We consider the Cauchy problem of nonlinear Schrödinger with t-dependent amplification. The equation is given by:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \frac{i\gamma}{2(1+t)}u + \lambda\mathcal{N}(u) \\ u(0,x) = u_0, \end{cases} \tag{1}$$

where $\gamma > 0$ represents the amplification coefficient, $(t,x) \in \mathbb{R}\times\mathbb{R}$ and $u = u(t,x)$ is a complex-valued unknown function. For the gauge-invariant nonlinearity $\lambda\mathcal{N}(u)$, α is the nonlinearity exponent, we assume that $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1,\ \lambda_2 \in \mathbb{R},\ \mathcal{N}(u) = |u|^\alpha u$.

In known results for $\gamma = 0$. Hayashi-Naumkin [2] and Hayashi-Kaikina-Naumkin [3] worked on NLS equations under scattering critical and subcritical nonlinearity, but without damping and nonlinear dissipation. They proved that the solution exhibits modified scattering behavior if $\alpha < 2$, and that the L^∞-decay rate is $\|u(t)\|_{L^\infty} = O(t^{-1/2})$. Note that the decay rate is similar to that of the solution to the corresponding linear Schrödinger equation. On NLS with dissipative nonlinearity, i.e., $\lambda_2 < 0$, researchers of the manuscripts [7, 6, 5] proved that, if $\alpha < 2$, the solution asymptotically tends to a free solution modified by a phase shift, and decays like $\|u(t)\|_{L^\infty} = O(t^{-1/\alpha})$.
In the case $\gamma < 0$. Bamri [1] studied a dissipative NLS equation with t-dependent "damping" and derived crucial decay estimates. He established that, if $\alpha < 2/(1-\gamma)$, solutions decay at a rate faster than free solutions. Specifically, it is found that the solution $u(t)$ satisfies $\|u(t)\|_{L^\infty} = O(t^{-1/\alpha})$ as $t \to \infty$.

We extend these studies by investigating a decay estimate of the solution to a dissipative NLS equation with t-dependent amplification, i.e., $\gamma > 0$. Assume that weak dissipative condition described as

$$|\lambda_2| < \frac{\alpha}{2\sqrt{\alpha+1}}|\lambda_1| \quad \left(\text{or, equivalently } \frac{\alpha+2}{\alpha} < \frac{|\lambda|}{|\lambda_2|}\right). \tag{2}$$

The opposite inequality, i.e., $(\alpha+2)/\alpha \geq |\lambda|/|\lambda_2|$ is called the strong dissipative condition, which provides a decay estimate and asymptotic behavior of solutions without size-restriction of initial data [4, 5, 7]. Let $v = (1+t)^{-\gamma/2}u$. Then (1) is equivalent to

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda(1+t)^{\alpha\gamma/2}\mathcal{N}(v). \tag{3}$$

In what follows, we focus on the treatment of (3). On the Schrödinger equations with t-dependent coefficient in nonlinearity, there are several works. However, these works deal with non-dissipative nonlinearities, and obtain $\|v(t)\|_{L^\infty} = O(t^{-1/2})$.

Theorem 1. *Let $0 < \gamma < 1$ and suppose that α satisfies*

$$\frac{1 + 2\gamma + \sqrt{4\gamma^2 - 28\gamma + 33}}{4(1-\gamma)} < \alpha < \frac{2}{1-\gamma}. \tag{4}$$

Assume that $\lambda = \lambda_1 + i\lambda_2$ satisfies

$$\lambda_2 < 0, \qquad \frac{\alpha+2}{\alpha} < \frac{|\lambda|}{|\lambda_2|} < \frac{\alpha(1-\gamma)-1}{2-\alpha(1-\gamma)}. \tag{5}$$

If $\|v_0\|_{H^1} + \|xv_0\|_{L^2} < \varepsilon$ with $\varepsilon > 0$ sufficiently small, then there exists a unique global solution v to (3) such that $v \in C([0,\infty); H^1) \cap C^1([0,\infty); H^{-1})$ and $xv \in C([0;\infty); L^2)$. Furthermore, for some constant $K_0 > 0$, the solution satisfies

$$\|v(t)\|_{L^\infty} \le K_0(1+t)^{-1/\alpha - \gamma/2}. \tag{6}$$

Theorem 2. *Under the assumptions in Theorem 1, there exists some $\hat{\psi} \in C([0,\infty); L^\infty)$ with $\lim_{t\to\infty} \hat{\psi}(t,\cdot) = \hat{\psi}_\infty(\cdot)$ in L^∞ such that the solution to (3) satisfies*

$$v(t,x) = MD \frac{\hat{\psi}(t,x)}{\left\{1 + \frac{2\alpha|\lambda_2|}{2-\alpha(1-\gamma)} t^{1-\frac{\alpha(1-\gamma)}{2}} \left|\hat{\psi}(t,x)\right|^\alpha \right\}^{\frac{-i\lambda}{\alpha|\lambda_2|}}} + o(t^{-1/\alpha-\gamma/2}) \tag{7}$$

as $t \to \infty$ in L^∞.

Proposition 3. *Let γ, α, $\lambda = \lambda_1 + i\lambda_2$ and v_0 satisfy the conditions in Theorem 1 with $\varepsilon > 0$ sufficiently small. Also let $d_{\alpha,\lambda} = \frac{\alpha+2}{2}\lambda_2 + \frac{\alpha}{2}|\lambda|$ and $\beta = d_{\alpha,\lambda}(K + CC_1\varepsilon)^\alpha$ with*

$$K > \left(\frac{2-\alpha(1-\gamma)}{2\alpha|\lambda_2|}\right)^{1/\alpha}.$$

Then there exists a unique global solution v to (3) such that, for some $C > 0$ and $C_1 > 0$,

$$\|v(t)\|_{H^1} + \|Jv(t)\|_{L^2} \le 2C_1\varepsilon(1+t)^\beta, \tag{8}$$

where $t \in [0,\infty)$.

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