

Emergence of an unsaturated Darcy's law and a Richards-type equation via homogenization of the Stokes-Cahn-Hilliard system

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We derive macroscopic unsaturated flow laws from a microscopic Stokes-Cahn-Hilliard system posed in an ε -periodically perforated porous medium $\Omega^\varepsilon \subset \mathbb{R}^3$. The model describes incompressible two-phase flow with diffuse interfaces and incorporates capillary effects through a chemical potential associated with a logarithmic free energy. More precisely, with $I = (0, T)$ for some $T > 0$, we consider a phase-field approach motivated by the following Stokes Cahn Hilliard (SCH) system

$$-\varepsilon^2 \operatorname{div} (\eta(\phi^\varepsilon) Du^\varepsilon) = -\nabla p^\varepsilon + \rho(\phi^\varepsilon) \mathbf{g} - \sigma \phi^\varepsilon \nabla \mu^\varepsilon \quad \text{in } I \times \Omega^\varepsilon, \quad (1)$$

$$\operatorname{div} u^\varepsilon = 0 \quad \text{in } I \times \Omega^\varepsilon, \quad (2)$$

$$\partial_t \phi^\varepsilon + u^\varepsilon \cdot \nabla \phi^\varepsilon = \operatorname{div} (M(\phi^\varepsilon) \nabla \mu^\varepsilon) \quad \text{in } I \times \Omega^\varepsilon, \quad (3)$$

$$\mu^\varepsilon = -\lambda \Delta \phi^\varepsilon + \frac{1}{\lambda} \Psi'(\phi^\varepsilon) \quad \text{in } I \times \Omega^\varepsilon, \quad (4)$$

where $Du = \frac{\nabla u + (\nabla u)^T}{2}$, with the initial and boundary conditions

$$\phi^\varepsilon(0, x) = \phi_0^\varepsilon(x) \quad \text{in } \Omega^\varepsilon, \quad (5)$$

$$u^\varepsilon(t, x) = 0 \quad \text{on } I \times \partial\Omega^\varepsilon, \quad (6)$$

$$\mathbf{n} \cdot \nabla \phi^\varepsilon = 0 \quad \text{on } I \times \partial\Omega^\varepsilon, \quad (7)$$

$$\mathbf{n} \cdot M(\phi^\varepsilon) \nabla \mu^\varepsilon = 0 \quad \text{on } I \times \partial\Omega^\varepsilon. \quad (8)$$

Here, η is the dynamic viscosity, ϕ is the phase field which represents microscopic concentration of one of the fluids with values lying in the interval $[-1, 1]$. u is the volume averaged velocity and p is the pressure. $\rho(\phi) = \frac{1+\phi}{2} \rho_1 + \frac{1-\phi}{2} \rho_2$ is the volume averaged density of the mixture, where ρ_i , $i = 1, 2$, is the specific density of the i -th phase. $\mathbf{g}$ denotes the gravity vector field. $\sigma > 0$ is a constant representing the surface tension coefficient and μ is the chemical potential associated to ϕ . $M(\phi)$ is the mobility coefficient and Ψ is the free energy density. $\lambda > 0$ is a fixed thickness of the interface between the phases. $\mathbf{n}$ denotes the outward unit normal vector field on the relevant boundary. The additional term $\phi \nabla \mu$ in Stokes equation (1) represents the capillary force and the term $u \cdot \nabla \phi$ in (3) describes the transport of the phase field by the Stokes flow. Our main result shows that, in the homogenization limit, the averaged velocity satisfies a

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Darcy–Korteweg law. The effective permeability tensor is determined solely by the pore geometry, while the driving force contains both the pressure gradient and the capillary force. Consequently, the resulting macroscopic law is not, in general, reducible to the classical Darcy–Buckingham form driven by a scalar hydraulic potential. Combining this Darcy–Korteweg law with the homogenized Cahn–Hilliard equation, we obtain a generalized Richards-type system. This is a joint work with Rong LEI at Tohoku University.

Theorem 1 (Main theorem). *Let $(u^\varepsilon, \phi^\varepsilon, \mu^\varepsilon, p^\varepsilon)$ be a weak solution for the Stokes–Cahn–Hilliard system. Then, under suitable assumptions, there exists an extension $(\tilde{u}^\varepsilon, \tilde{\phi}^\varepsilon, \tilde{\mu}^\varepsilon, \tilde{p}^\varepsilon)$ of $(u^\varepsilon, \phi^\varepsilon, \mu^\varepsilon, p^\varepsilon)$ in $I \times \Omega$ such that up to a subsequence of ε denoted by the same symbol, $\tilde{u}^\varepsilon$ converges weakly in $L^2(I \times \Omega)^d$ to $\bar{u}$, $\tilde{\phi}^\varepsilon$ converges strongly in $L^2(I \times \Omega)$ to ϕ , $\tilde{\mu}^\varepsilon$ two-scale converges to μ , and $\tilde{p}^\varepsilon$ two-scale converges to p , where $(\bar{u}, p)$ is a solution of the following Darcy Korteweg law*

$$\begin{aligned} \bar{u}(t, x) &= \frac{1}{\eta(\phi(t, x))} A(\rho(\phi(t, x))\mathbf{g} - \nabla p(t, x) - \sigma\phi(t, x)\nabla\mu(t, x)) \quad \text{in } I \times \Omega, \\ \operatorname{div} \bar{u} &= 0 \quad \text{in } I \times \Omega, \\ \mathbf{n} \cdot \bar{u} &= 0 \quad \text{on } I \times \partial\Omega, \end{aligned} \tag{9}$$

where $\Omega = \Omega^0$, $A = (A_{ij})$ is a symmetric, positive definite matrix defined by its entries

$$A_{ij} = \int_{Y_f} w^i \cdot e_j(y) dy,$$

Y_f is the fluid part of the unit cell $Y = (0, 1]^3$, (w^i, q^i) is the unique weak solution to the cell problem:

$$\begin{cases} -\operatorname{div}_y(Dw^i) + \nabla_y q^i = e_i & \text{in } Y_f, \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f, \\ w^i = 0 & \text{on } \partial Y_s, \\ (w^i, q^i) \text{ } Y\text{-periodic.} \end{cases}$$

Moreover, ϕ is a solution to the following effective Cahn–Hilliard system:

$$\partial_t \phi = \frac{1}{|Y_f|} \operatorname{div} (B(\phi)\nabla\mu - \phi\bar{u}) \quad \text{in } I \times \Omega, \quad \mu = -\frac{\lambda}{|Y_f|} \operatorname{div} (Q\nabla\phi) + \frac{1}{\lambda} \Psi'(\phi) \quad \text{in } I \times \Omega,$$

where $B = (B_{ij})$ is bounded and coercive, and $Q = (Q_{ij})$ is symmetric, bounded, and coercive, given by

$$B_{ij} := \int_{Y_f} M(\phi)(e_j + \nabla_y \chi^j) \cdot e_i dy, \quad Q_{ij} := \int_{Y_f} (e_j + \nabla_y \theta^j) \cdot e_i dy$$

and $\chi^j, \theta^j \in H_{\#}^1(Y_f)/\mathbb{R}$ are the unique weak solutions to the cell problems:

$$\operatorname{div}_y (M(\phi)(e_j + \nabla_y \chi^j)) = 0, \quad \operatorname{div}_y ((e_j + \nabla_y \theta^j)) = 0.$$

Finally, we obtain a Richards type equation,

$$\partial_t \phi = \frac{1}{|Y_f|} \operatorname{div} \left[\left(B(\phi) + \sigma \frac{\phi^2}{\eta(\phi)} A \right) \nabla\mu + \frac{\phi}{\eta(\phi)} A (\nabla p - \rho(\phi)\mathbf{g}) \right] \quad \text{in } I \times \Omega.$$